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AN EQUIVALENT CONDITION FOR THE FULL HAUSDORFF MEASURE OF A SUBSET OF THE SET OF FINITE TYPE

JIANDONG YIN AND ZUOLING ZHOU

ABSTRACT. In this paper, the Hausdorff dimensions of subsets of the set of finite type with positive Parry measure are gotten firstly. Then by this result, an equivalent condition for subsets of the set of finite type with the full Hausdorff measure is given.

1. Introduction and preliminaries. It is well known that the theory of Hausdorff measure and Hausdorff dimension is the basis of fractal geometry, so how to compute or estimate the Hausdorff measures and Hausdorff dimensions of the fractal sets is an important problem. In general, to compute or estimate the Hausdorff measures and Hausdorff dimensions of fractals is very difficult, and to compute the Hausdorff measures of fractals is more difficult. Hence, up to now, there are few concrete results about the computation of Hausdorff measure even for some simple fractals (see [**6**–**9**]). Furthermore, the results about the Hausdorff measures of subsets of the set of finite type determined by $(0,1)$-matrix are more rare.

In the present paper, we will investigate the Hausdorff dimensions and Hausdorff measures about subsets of the set of finite type determined by an irreducible $(0,1)$-matrix. Firstly, the Hausdorff dimensions about subsets of the set of finite type with positive Parry measure are obtained. Then by this result, we give an equivalent condition for subsets of the set of finite type with full Hausdorff measure.

Let (X,d) be a metric space, $E\subset X$. For $U\subset X$, denote by $|U|$ the diameter of U, i.e., $|U|=\sup\{d(x,y):x,y\in U\}$. If $E\subset\bigcup_{i\geq 0}U_i$ and

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for all $i, 0 < |U_i| \leq \delta(\delta > 0)$, then $\{U_i\}_{i=0}^{\infty}$ is called a δ-covering of E. Let $s \geq 0$, for $\delta > 0$, and write

(1.1)
$$H_\delta^s(E) = \inf\left\{ \sum_{i=0}^{\infty} |U_i|^s, \{U_i, i \geq 0\} \text{ is a countable } \delta\text{-covering of } E \right\}.$$

Let $\delta \to 0$, write $H^s(E) = \lim_{\delta\to 0} H_\delta^s(E)$; $H^s(E)$ is called the s-dimensional Hausdorff measure of E. Furthermore, there exists a unique nonnegative number s satisfying

$$H^t(E) = \begin{cases} 0 & t > s \\ \infty & t < s, \end{cases}$$

s is called the Hausdorff dimension of E. Denote by $\dim_H(\cdot)$ and $H^s(\cdot)$ the Hausdorff dimension and the s-dimensional Hausdorff measure, respectively.

Put $S = \{0, 1, \dots, k-1\}$ $(k \geq 2)$ with discrete topology. The one-sided symbolic space generated by S is denoted as

$$\Sigma_k = \{x = (x_0, x_1, \dots) \mid x_i \in S, \text{ for all } i \geq 0\}. \tag{1.2}$$

Under the product topology, Σ_k is a compact metric space with the second axiom of countability. Now we define a metric d which is compatible with the product topology on Σ_k as follows: for all $x = (x_0, x_1, \dots)$, $y = (y_0, y_1, \dots) \in \Sigma_k$,

$$d(x, y) = \begin{cases} 0 & x = y \\ \frac{1}{k^N} & x \neq y,\ N = \min\{n : x_n \neq y_n\} \end{cases}.$$

Let $A = (a_{ij})_{0\leq i,j\leq k-1}$ be a $k \times k$-$(0,1)$ matrix, and suppose its every row, as well as every column, has at least one 1. Such a matrix is called irreducible if, for any i, j, there is some $n > 0$ such that $a_{ij}^{(n)} > 0$, where $a_{ij}^{(n)}$ is the (i,j)-element of A^n. Matrix $A = (a_{ij})_{0\leq i,j\leq k-1}$ is called aperiodic if there exists an $n > 0$ such that $a_{ij}^{(n)} > 0$ for all i, j. Let

(1.3)
$$\Sigma_A = \{x = (x_0, x_1, \dots, x_n, \dots) \in \Sigma_k, a_{x_i x_{i+1}} = 1, \text{ for all } i \geq 0\}.$$

Then Σ_A is a compact subset of Σ_k, and Σ_A is called the set of finite type determined by the matrix A. Set

(1.4)
$$[i_0, i_1, \ldots, i_{n-1}]_A = \{x \in \Sigma_A | x_0 = i_0, x_1 = i_1, \ldots, x_{n-1} = i_{n-1}\}.$$

Then we call it a relative cylinder of Σ_A with length n. The relative cylinder is both open and closed in Σ_A, and all of the relative cylinders form a subbase under the relative topology of Σ_A (see [**1**–**3, 5**]).

We say that $A = (a_{ij})_{0 \le i,j \le k-1}$ satisfies the property M, if for any given relative cylinder $[i_0, i_1, \ldots, i_{n-1}]_A (n > 0)$, we have

$$\text{diam}^s([i_0, i_1, \ldots, i_{n-1}]_A) = \sum_{i_n \in S} \text{diam}^s([i_0, i_1, \ldots, i_{n-1} i_n]_A), \tag{1.5}$$

where $s = \dim_H(\Sigma_A)$ and diam $(\cdot)$ denotes the diameter.

2. Lemmas.

Lemma 2.1 [**5**] (Perron-Frobenius theorem). *Let* $A = (a_{ij})_{0 \le i,j \le k-1}$ *be a* $k \times k - (0,1)$ *matrix,* $k \ge 2$. *Then*

(1) *There is a nonnegative eigenvalue* $\rho(A)$ *such that no eigenvalue of* A *with absolute values greater than* $\rho(A)$ *and* $\rho(A)$ *is called the spectral radius of* A;

(2) $\min\limits_i(\sum_{j=0}^{k-1} a_{ij}) \le \rho(A) \le \max\limits_i(\sum_{j=0}^{k-1} a_{ij})$;

(3) *Corresponding to* $\rho(A)$, *there exist nonnegative row eigenvector* $u = (u_0, u_1, \ldots, u_{k-1})$ *and nonnegative column eigenvector* $v = (v_0, v_1, \ldots, v_{k-1})^T$;

(4) *When* A *is an irreducible matrix, the row eigenvector* u *and column eigenvector* v *are strictly positive and* $\rho(A)$ *is the unique eigenvalue with this property.*

Let A be an irreducible $k \times k - (0,1)$ matrix. Then $\rho(A) > 0$. $u = (u_0, u_1, \ldots, u_{k-1})$ and $v = (v_0, v_1, \ldots, v_{k-1})^T$ are the row eigenvector and column eigenvector corresponding to $\rho(A)$ with $uv = 1$. Let $p_i = u_i v_i, 0 \le i \le k-1$. Then $\mathbf{P} = (p_0, p_1, \ldots p_{k-1})$ is a probability vector. Put

$$p_{ij} = \frac{a_{ij} v_j}{\rho(A) v_i}, 0 \le i, j \le k-1. \tag{2.1}$$

$\mathbf{P} = (p_{ij})_{0\le i,j\le k-1}$ is a $k \times k$ stochastic matrix. Now a measure on the relative cylinder of Σ_A is defined as follows:

$$m([i_0, i_1, \dots, i_{n-1}]_A) = p_{i_0} p_{i_0 i_1} \cdots p_{i_{n-2} i_{n-1}} > 0, \text{ for all } n > 0. \tag{2.2}$$

m can be extended to the σ-algebra $\mathfrak{B}(\Sigma_A)$ of all the Borel subsets of Σ_A to be a probability measure, we call it the Parry measure.

Lemma 2.2 [4]. *Let $A = (a_{ij})_{0\le i,j\le k-1}$ be a $k \times k - (0,1)$ matrix, $k \ge 2$. Then*

$$s = \dim_H(\Sigma_A) = \frac{\log \rho(A)}{\log k}. \tag{2.3}$$

Lemma 2.3. *Let Σ_A be the set of finite type determined by a $k \times k - (0,1)$ matrix $A = (a_{ij})_{0\le i,j\le k-1}$. We can use the countable covering consisting of the relative cylinders of Σ_A to calculate its Hausdorff measure.*

Proof. Let $U \subset \Sigma_A$. Obviously $|U| = |\overline{U}|$. Therefore, there exist $x, y \in \overline{U}$, such that $d(x, y) = |\overline{U}|$. By the definition of d, there is an $n \ge 0$, such that

$$d(x, y) = \frac{1}{k^n}, \text{ i.e. } x_i = y_i, \quad i = 0, 1, \dots, n-1, \ x_n \ne y_n \tag{2.4}$$

and for any $z \in U$, $d(x, z) \le d(x, y)$. So it is easy to prove that $z \in [x_0, x_1, \dots, x_{n-1}]_A$, that is, $U \subset \overline{U} \subset [x_0, x_1, \dots, x_{n-1}]_A$. It means that for any subset of Σ_A, there exists a relative cylinder of Σ_A such that the former is included in the latter and they have the same diameter. So Lemma 2.3 holds. □

We point out: for any subset C of Σ_A, we can also use the countable covering consisting of the relative cylinders of Σ_A to calculate its Hasudorff measure. By a proof similar to that of Lemma 2.3, we can prove it.

Lemma 2.4. *Let $A = (a_{ij})_{0\le i,j\le k-1}$ be an irreducible $k \times k - (0,1)$ matrix and Σ_A the set of finite types determined by A. m is the Parry* *measure. For any* $C \subset \Sigma_A$*, if* C *is measurable and* $m(C) = 1$*. Then, for any relative cylinder* $[i_0, i_1, \dots, i_n]_A$*, we have* $[i_0, i_1, \dots, i_n] \cap C \neq \varnothing$*.*

Proof. Suppose that there is some relative cylinder $[j_0, j_1, \dots, j_l]_A$ $(l \geq 0)$, such that

$$[j_0, j_1, \dots, j_l]_A \cap C = \varnothing. \tag{2.5}$$

Since $[j_0, j_1, \dots, j_l]_A \cup C \subseteq \Sigma_A$, by the monotonic property of a measure and Lemma 2.1, we obtain

$$m(\Sigma_A) \geq m(C) + m([j_0, j_1, \cdots, j_l]_A) = 1 + p_{j_0} p_{j_0 j_1} \cdots p_{j_{l-1} j_l} > 1. \tag{2.6}$$

But m is a probability measure on Σ_A, i.e., $m(\Sigma_A) = 1$, so it is contradicting. □

3. Main results.

Theorem 3.1. *Let* $A = (a_{ij})_{0 \leq i,j \leq k-1}$ *be an irreducible* $k \times k - (0,1)$ *matrix and* Σ_A *the set of finite type determined by* A*.* m *is the Parry measure. For any* $C \subset \Sigma_A$*, if* C *is measurable and* $m(C) > 0$*. Then*

$$\dim_H(C) = \dim_H(\Sigma_A) = \frac{\log \rho(A)}{\log k}. \tag{3.1}$$

Proof. If $C \subset \Sigma_A$ and $m(C) > 0$, then $0 < m(C) \leq 1$. So m is a mass distribution on C. Suppose $\varnothing \neq U \subset C$. Then there exist $x, y \in \overline{U}$, $x = (x_0, x_1, \dots)$, $y = (y_0, y_1, \dots) \in \Sigma_A$, such that $d(x,y) = |\overline{U}| = |U|$. From the definition of d, there is some $n \geq 0$, such that $d(x,y) = \frac{1}{k^n}$, so that $U \subset \overline{U} \subset [x_0, \dots, x_{n-1}]_A$. Furthermore,

$$\begin{aligned} m(U) &\leq m([x_0, \dots, x_{n-1}]_A) \leq p_{x_0} p_{x_0 x_1} \cdots p_{x_{n-2} x_{n-1}} \\ &= u_{x_0} v_{x_0} \frac{a_{x_0 x_1} v_{x_1}}{\rho(A) v_{x_0}} \cdots \frac{a_{x_{n-2} x_{n-1}} v_{x_{n-1}}}{\rho(A) v_{x_{n-2}}} \\ &\leq \frac{\max\{u_i v_j | 0 \leq i, j \leq n-1\}}{\rho(A)^{n-1}}. \end{aligned}$$

Let $s = \frac{\log \rho(A)}{\log k}$. Then $|U|^s = (\frac{1}{k^n})^s = \frac{1}{\rho(A)^n}$. Hence for any $U \subset C$ with $|U| \leq \frac{1}{k^n}$, we get

$$m(U) \leq \rho(A) \max\{u_i v_j : 0 \leq i, j \leq n-1\} |U|^s. \tag{3.2}$$

By the mass distribution principle and Lemma 2.1, we have

$$H^s(C) \geq \frac{1}{\rho(A) \max\{u_i v_j : 0 \leq i, j \leq n-1\}} > 0. \tag{3.3}$$

So $\dim_H(C) \geq s = \frac{\log \rho(A)}{\log k}$. But $C \subset \Sigma_A$, therefore

$$\dim_H(C) \leq \dim_H(\Sigma_A) = \frac{\log \rho(A)}{\log k}$$

and

$$\dim_H(C) = \dim_H(\Sigma_A) = \frac{\log \rho(A)}{\log k}. \tag{3.4}$$

□

Theorem 3.2. *Let $A = (a_{ij})_{0 \leq i,j \leq k-1}$ be an irreducible $k \times k$-$(0,1)$ matrix with the property M. Then, for any relative cylinder $[i_0, i_1, \ldots, i_p]_A \subset \Sigma_A (p \geq 0)$, we have*

$$H^s([i_0, i_1, \ldots, i_p]_A) = \operatorname{diam}^s([i_0, i_1, \ldots, i_p]_A), \text{ where } s = \frac{\log \rho(A)}{\log k}.$$

Proof. Let $[i_0, i_1, \ldots, i_p]_A \subset \Sigma_A$ $(p \geq 0)$ be a relative cylinder. From Lemma 2.1 and Theorem 3.1, we know that $m([i_0, i_1, \ldots, i_p]_A) > 0$ and $\dim_H([i_0, i_1, \ldots, i_p]_A) = s = \frac{\log \rho(A)}{\log k}$. Let $\alpha = \{U_i\}_{i=0}^{\infty}$ be any open covering of $[i_0, i_1, \ldots, i_p]$. By Lemma 2.3, we can assume that α consists of the relative cylinders of Σ_A. Because $[i_0, i_1, \ldots, i_p]$ is compact, we can also assume that α is a finite covering of $[i_0, i_1, \ldots, i_p]_A$. Let $\delta > 0$, and for any i, $|U_i| \leq \delta$. By the property M, for any $n > 0$, we have

$$\begin{aligned} \operatorname{diam}^s([i_0, i_1, \ldots, i_p]_A) &= \sum_{0 \leq i_{p+1}, \ldots, i_{p+n} < k} \operatorname{diam}^s([i_0, i_1, \ldots, i_p, i_{p+1}, \ldots, i_{p+n}]_A) \\ &\leq \sum_{\alpha} |U_i|^s. \end{aligned} \tag{3.5}$$

Hence, from the definition of Hausdorff measure, we obtain

$$\mathrm{diam}^s([i_0, i_1, \dots, i_p]_A) \leq H^s_\delta([i_0, i_1, \dots, i_p]_A). \tag{3.6}$$

In (3.6), letting $\delta \to 0$, we get

$$\mathrm{diam}^s([i_0, i_1, \dots, i_p]_A) \leq H^s([i_0, i_1, \dots, i_p]_A). \tag{3.7}$$

Moreover, since $A = (a_{ij})_{0 \leq i,j \leq k-1}$ satisfies property M, by the definition of Hausdorff measure and Lemma 2.3, for any $\delta > 0$ and $n \geq 1$, we have

$$\begin{aligned} &H^s_\delta([i_0, i_1, \dots, i_p]_A) \\ &\qquad \leq \sum_{0 \leq i_{p+1}, \dots, i_{p+n} < k} \mathrm{diam}^s([i_0, i_1, \dots, i_p, i_{p+1}, \dots, i_{p+n}]_A) \\ &\qquad = \mathrm{diam}^s([i_0, i_1, \dots, i_p]_A). \end{aligned}$$

Therefore,

$$H^s([i_0, i_1, \dots, i_p]_A) \leq \mathrm{diam}^s([i_0, i_1, \dots, i_p]_A). \tag{3.8}$$

From (3.7)and (3.8), we obtain

$$H^s([i_0, i_1, \dots, i_p]_A) = \mathrm{diam}^s([i_0, i_1, \dots, i_p]_A). \tag{3.9}$$ □

Theorem 3.3. *Let $A = (a_{ij})_{0 \leq i,j \leq k-1}$ be an irreducible $k \times k$-$(0,1)$ matrix with property M and Σ_A the set of finite type determined by A. m is the Parry measure. $C \subset \Sigma_A$, C is measurable. Then $H^s(C) = H^s(\Sigma_A)$ if and only if $m(C) = 1$, where $s = \frac{\log \rho(A)}{\log k}$.*

Proof. Firstly, we prove the necessary condition as follows.

If there is a measurable set $C \subset \Sigma_A$, such that $H^s(C) = H^s(\Sigma_A)$, but $0 < m(C) < 1$. Let $B = \Sigma_A - C$ (where $\Sigma_A - C$ denotes the difference set of Σ_A and C). Since C is measurable, B is measurable and

$$0 < m(B) = m(\Sigma_A - C) = m(\Sigma_A) - m(C) < 1. \tag{3.10}$$

So there exists a relative cylinder $[i_0, i_1, \dots, i_q]_A (q > 0)$, such that $[i_0, i_1, \dots, i_q]_A \subset B$. By Theorem 3.1 and Theorem 3.2, we have

$$\dim_H([i_0, i_1, \dots, i_q]) = s = \frac{\log \rho(A)}{\log k} \tag{3.11}$$

and

$$H^s([i_0, i_1, \dots, i_q]) = \operatorname{diam}^s([i_0, i_1, \dots, i_q]) > 0. \tag{3.12}$$

So

(3.13)
$$H^s(C) = H^s(\Sigma_A) - H^s(B) \le H^s(\Sigma_A) - H^s([i_0, i_1, \dots, i_q]) < H^s(\Sigma_A),$$

which contradicts $H^s(C) = H^s(\Sigma_A)$. Hence, $m(C) = 1$.

Next we will show the sufficient condition.

Suppose $\Sigma_A - C \neq \varnothing$. By Lemma 2.3, we can only use the countable covering consisting of the relative cylinders of Σ_A to calculate the Hausdorff measure of C. Let $\{U_i\}_{i=0}^{\infty}$ be a countable covering of C consisting of the relative cylinders of Σ_A. Suppose that, for any i, $|U_i| \le \delta (0 < \delta \le 1)$. We can assume that, by the definition of Hausdorff measure, for any i, U_i is a relative cylinder as $[i_0, \dots, i_k]_A, k \ge 1$ and $U_i \cap U_j = \varnothing (i \neq j)$. By Lemma 2.4, for any $x = (x_0, x_1, \dots, x_m, \dots) \in \Sigma_A - C$ and any positive integer q, we have $[x_0, x_1, \dots, x_q]_A \cap C \neq \varnothing$. So there exists some positive integer p, such that $U_p \in \{U_i\}_{i=0}^{\infty}$ and $[x_0, x_1, \dots, x_q]_A \cap U_p \neq \varnothing$. Suppose that $p \le q$ (in fact, if $p > q$, from Lemma 2.4 and $U_i \cap U_j = \varnothing (i \neq j)$, there is another relative cylinder $[x_0, \dots, x_r]_A (r > p > q)$ including x, such that $[x_0, \dots, x_r]_A \cap U_p \neq \varnothing$. In this case, we replace q with r), so that $x \in [x_0, \dots, x_q]_A \subset U_p$ and $x \in \bigcup_{i=1}^{\infty} U_i$. The arbitrariness of x implies $(\Sigma_A - C) \subset \bigcup_{i=0}^{\infty} U_i$. This means that all the $\delta-$coverings of C, which consist of the relative cylinders of Σ_A, are also coverings of Σ_A. By the definition of Hausdorff measure, we obtain $\Sigma_i |U_i|^s \ge H^s_\delta(\Sigma_A)$. Also the arbitrariness of $\{U_i\}_{i=0}^{\infty}$ implies that $H^s_\delta(C) \ge H^s_\delta(\Sigma_A)$. Let $\delta \to 0$. We have $H^s(C) \ge H^s(\Sigma_A)$. Obviously, $H^s(C) \le H^s(\Sigma_A)$; hence, $H^s(C) = H^s(\Sigma_A)$. □

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Department of Mathematics, Nanchang University, Nanchang 330031, PR China
Email address: yjdaxf@163.com

Lingnan College, Zhongshan University, Guangzhou 510275, PR China
Email address: lnszzl@mail.sysu.edu.cn