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CYCLIC VECTORS IN THE WEIGHTED $VMOA$ SPACE

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ABSTRACT. We try to identify the functions whose polynomial multiples are dense in the logarithmic weighted $VMOA$ space ($VMOA_{\log}$) and prove that if $|f(z)| \geq |g(z)|$ in the unit disk and g is cyclic in $VMOA_{\log}$, then f is cyclic.

1. Introduction. Let $D = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbf{C}$, and let $H(D)$ denote the set of all analytic functions on D. Let E be a Banach space of analytic functions in D such that the shift operator $M_z : f(z) \to zf(z)$ is a continuous map of E into itself. The cyclic vectors in E are those functions f such that the polynomial multiples of f are dense in E. In 1949, Beurling [**3**] characterized the cyclic vectors on the classical Hardy space H^2. The important consequences of this work in function theory has led many investigators to study questions related to cyclicity on other Banach spaces of analytic functions. For example, cyclic vectors in the Dirichlet space were studied by Brown and Shields in [**4**]; Anderson, Brown, Fernandez and Shields characterized the cyclic vectors in the classical Bloch space and the little Bloch space in [**1, 5**]; Ye studied cyclic vectors in $BMOA$ and α-Bloch spaces in [**11, 12**]. For some recent results related to cyclic vectors, see [**6, 13, 14**] and the references therein.

The space of analytic functions with bounded mean oscillation, denoted by $BMOA$, consists of all analytic functions f defined on D, satisfying

$$\|f\|_{BMOA} = \sup_{a\in D} \left(\int_D |f'(z)|^2 (1 - |\varphi_a(z)|^2)\, dm(z) \right)^{1/2} < \infty,$$

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where $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ is the Möbius transformation of D and $dm(z)$ denotes the Lebesgue area measure on D ([**2, 7**]). Let $VMOA$ denote the subspace of $BMOA$ consisting of those $f \in BMOA$ for which

$$\lim_{|a|\to 1^-} \int_D |f'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z) = 0.$$

The cyclic vectors in $BMOA$ ($VMOA$) space were characterized by the first author in [**11**]; the first author proved that for $f \in BMOA$ ($VMOA$), f is a cyclic vector in $BMOA$ ($VMOA$) if and only if f is an outer function.

In this paper we will study cyclic vectors in the logarithmic weighted $VMOA$ space. Let $BMOA_{\log}$ denote the logarithmic weighted $BMOA$ space consisting of $f \in H(D)$ with

$$\|f\|_* = \sup_{a\in D} \log\frac{2}{1-|a|}\left(\int_D |f'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z)\right)^{1/2} < \infty,$$

and $VMOA_{\log}$ the subspace of $BMOA_{\log}$ consisting of $f \in H(D)$ for which

$$\lim_{|a|\to 1^-} \log^2\frac{2}{1-|a|}\int_D |f'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z) = 0.$$

It is easy to check that $BMOA_{\log}$ is a Banach space under the norm

$$\|f\|_{BMOA_{\log}} = |f(0)| + \|f\|_*$$

and $BMOA_{\log} \subset BMOA$, $VMOA_{\log} \subset VMOA$. The space $BMOA_{\log}$ appeared first in [**10**] and was used to characterize the multipliers in $BMOA$ space (see also [**8**]). In [**9**], Siskakis and Zhao proved that $VMOA_{\log}$ is a closed subspace of $BMOA_{\log}$ and coincides with the closure of polynomials under the norm $\|\cdot\|_{BMOA_{\log}}$. Therefore the polynomials are weak* dense in the second dual $(VMOA_{\log})^{**}$. For $f \in VMOA_{\log}$, let $[f]$ be the norm closure in $VMOA_{\log}$ of the polynomial multiples of f; for $f \in (VMOA_{\log})^{**}$, let $[f]_*$ be the weak* closure in $(VMOA_{\log})^{**}$ of the polynomial multiples of f. Thus f is cyclic in $VMOA_{\log}$ if and only if $[f] = VMOA_{\log}$, f is cyclic in $(VMOA_{\log})^{**}$ if and only if $[f]_* = (VMOA_{\log})^{**}$. Note that if f is in $VMOA_{\log}$, then f is norm cyclic in $VMOA_{\log}$ if and only if it is weak* cyclic in $(VMOA_{\log})^{**}$. When we refer to cyclic vectors in $(VMOA_{\log})^{**}$, the weak* is always understood. As main results of this paper, we prove the following theorems.

Theorem 1. *For $f,\ g \in VMOA_{\log}$, if $|f(z)| \geq |g(z)|$ in D and g is cyclic in $VMOA_{\log}$, then f is cyclic in $VMOA_{\log}$.*

Theorem 2. *For $f \in VMOA_{\log}$, if f is cyclic in $VMOA_{\log}$, then f is an outer function.*

In what follows C will stand for positive constants not depending on the functions being considered, but whose value may change from line to line.

2. Some lemmas.

For the proof of Theorem 1 we need some lemmas.

Lemma 1. *For $x \in [0,1)$ and $t \in [0,1]$,*

$$\frac{(1-x)\log^2 2/(1-x)}{(1-tx)\log^2 2/(1-tx)} < 3.$$

Proof. Let $g(x) = (1-x)\log^2 2/(1-x)$ and $x_0 = 1-2/e^2$. We know that $g(x)$ is increasing on $[0,x_0]$ and decreasing on $[x_0,1)$.

If $t > 9/10$, $x > (10/9)x_0$, then $g(x) \leq g(tx)$.

If $t > 9/10$, $x \leq (10/9)x_0$, then

$$\frac{g(x)}{g(tx)} \leq \frac{g(x_0)}{\min\big(g(0), g((10/9)x_0)\big)} \leq \frac{g(x_0)}{\min\big(g(0), g((9/10))\big)} = \frac{8}{e^2\log^2 2} < 3.$$

If $t \leq 9/10$, since $\log^2 2 \leq g(tx) \leq 8/e^2$, we have

$$\frac{g(x)}{g(tx)} \leq \frac{8}{e^2\log^2 2} < 3. \qquad \square$$

Lemma 2. *If* $0 < t < 1$, *then*

$$\int_0^1 \frac{1}{1-rt} \log^2 \frac{1-rt}{1-r}\, dr \le \frac{1}{e} + \frac{32}{e^2}.$$

Proof. Write

$$\begin{aligned}\int_0^1 \frac{1}{1-rt} \log^2 \frac{1-rt}{1-r}\, dr &= \int_0^t \frac{1}{1-rt} \log^2 \frac{1-rt}{1-r}\, dr \\ &\quad + \int_t^1 \frac{1}{1-rt} \log^2 \frac{1-rt}{1-r}\, dr \\ &\triangleq I_1 + I_2.\end{aligned}$$

Since $(\log x)/x \le 1/e$ for $1 \le x < \infty$ and $\log(1+x) \le x$ for $x > 0$, it follows that

$$\frac{1}{1-rt} \log^2 \frac{1-rt}{1-r} \le \frac{1}{e(1-r)} \log \frac{1-rt}{1-r} \le \frac{1}{e(1-r)} \left(\frac{1-rt}{1-r} - 1 \right) < \frac{1-t}{e(1-r)^2}.$$

Then

$$I_1 \le \frac{1-t}{e} \int_0^t \frac{1}{(1-r)^2}\, dr = \frac{t}{e} < \frac{1}{e}.$$

Since $x^{1/2} \log^2 1/x \le (16/e^2)$ for $0 < x \le 1$,

$$\begin{aligned}\frac{1}{1-rt} \log^2 \frac{1-rt}{1-r} &= \left(\frac{1-r}{1-rt} \right)^{1/2} \log^2 \frac{1-rt}{1-r} \frac{1}{(1-r)^{1/2}(1-rt)^{1/2}} \\ &\le \frac{16}{e^2 (1-r)^{1/2} (1-rt)^{1/2}}.\end{aligned}$$

We obtain

$$I_2 \le \frac{16}{e^2} \int_t^1 \frac{dr}{(1-r)^{1/2}(1-rt)^{1/2}} = \frac{16}{e^2} \int_{\sqrt{1+t}}^{+\infty} \frac{2dx}{x^2 - t} \le \frac{32}{e^2}.$$

Thus,

$$\int_0^1 \frac{1}{1-rt} \log^2 \frac{1-rt}{1-r}\, dr < \frac{1}{e} + \frac{32}{e^2}. \qquad \square$$

The logarithmic Bloch space β_L is the space of $f \in H(D)$ with

$$\|f\|_{\beta_L} = \sup_{z\in D}(1-|z|^2)\log\frac{2}{1-|z|^2}|f'(z)| < \infty$$

(see [**14, 15, 16**] for more information on β_L).

Lemma 3. *For $0 < t < 1$, if $f \in \beta_L$, then*

$$|f(z) - f(tz)| \le 2\log\frac{\log(2/1-|z|)}{\log(2/1-|tz|)}\|f\|_{\beta_L}, \quad z \in D.$$

Proof. For $f \in \beta_L$ and $z \in D$, the definition of β_L yields

$$\begin{aligned} |f(z) - f(tz)| &= \left| z\int_t^1 f'(zt)\,dt\right| \\ &\le \|f\|_{\beta_L}\int_t^1 \frac{|z|}{(1-|zt|^2)\log\frac{2}{1-|zt|^2}}\,dt \\ &\le 2\|f\|_{\beta_L}\int_{t|z|}^{|z|}\frac{dx}{(1-x)\log(2/1-x)} \\ &= 2\|f\|_{\beta_L}\left(\log\log\frac{2}{1-|z|} - \log\log\frac{2}{1-|tz|}\right) \\ &= 2\log\frac{\log(2/1-|z|)}{\log(2/1-|tz|)}\|f\|_{\beta_L}. \qquad \square \end{aligned}$$

Lemma 4. $BMOA_{\log} \subset \beta_L$.

Proof. Let $a = re^{i\theta} \in D$ and $r \ge 1/2$. Denote by I_a the arc centered at $e^{i\theta}$ satisfying $|I_a| = 2(1-r)$, and by $S(I_a) = \{re^{i\theta} : 1-|I| \le r < 1, e^{i\theta} \in I_a\}$ the corresponding Carleson box. Let

$$\widetilde{\Delta} = \left\{ z \in D : |z-a| < \frac{|I_a|}{4}\right\}.$$

Then $\widetilde{\Delta} \subset S(I_a)$ and

$$\frac{|I_a|}{4} \le 1 - |z|^2 \le 2|I_a|$$

for all $z \in \widetilde{\Delta}$. For $f \in BMOA_{\log}$, by the subharmonicity of $|f'|^2$, it follows that

$$\begin{aligned}|f'(a)|^2 &\le \frac{16}{\pi|I_a|^2}\int_{\widetilde{\Delta}} |f'(z)|^2\, dm(z)\\ &\le \frac{64}{\pi|I_a|^3}\int_{\widetilde{\Delta}} |f'(z)|^2(1-|z|^2)\, dm(z)\\ &\le \frac{64}{\pi|I_a|^3}\int_{S(I_a)} |f'(z)|^2(1-|z|^2)\, dm(z).\end{aligned}$$

Then

$$\begin{aligned}&(1-|a|^2)\log\frac{2}{1-|a|}|f'(a)|\\ &\qquad \le 8\left(\frac{1}{\pi|I_a|}\log^2\frac{2}{|I_a|}\int_{S(I_a)} |f'(z)|^2(1-|z|^2)\, dm(z)\right)^{1/2}.\end{aligned}$$

Applying Lemma 3.4 of [**8**] yields

$$\|f\|_{\beta_L} \le C\|f\|_*. \qquad \square$$

Lemma 5. *For* $0 \le t \le 1$*, if* $f \in BMOA_{\log}$*, then*

$$|f(z)-f(tz)| \le C\frac{\log(1-|tz|)/(1-|z|)}{\log(2/1-|tz|)}\|f\|_*, \quad z \in D.$$

Proof. From Lemmas 3 and 4,

$$|f(z)-f(tz)| \le C\log\frac{\log(2/1-|z|)}{\log(2/1-|tz|)}\|f\|_*.$$

Since $\log(1+x) \le x$ for $x > 0$, it follows that

$$\begin{aligned}|f(z)-f(tz)| &\le C\left(\frac{\log(2/1-|z|)}{\log(2/1-|tz|)}-1\right)\|f\|_*\\ &= C\frac{\log(1-|tz|)/(1-|z|)}{\log(2/1-|tz|)}\|f\|_*. \qquad \square\end{aligned}$$

Lemma 6. *For $0 < t < 1$, $z \in D$, let $f_t(z) = f(tz)$. If $f \in BMOA_{\log}$, then $f_t \in VMOA_{\log}$ and $\|f_t\|_* \le \|f\|_*$.*

Proof. If $f \in BMOA_{\log}$ and $0 < t < 1$, then f_t is analytic on the closed unit disk; a simple computation shows that $f_t \in VMOA_{\log}$. In addition, by the Poisson formula, we have

$$f_t(z) = \int_0^{2\pi} f(ze^{i\theta}) \frac{1-t^2}{|e^{i\theta}-t|^2} \frac{d\theta}{2\pi}, \quad z \in D.$$

So

$$\begin{aligned}
&\log^2 \frac{2}{1-|a|} \int_D |(f_t)'(z)|^2 (1-|\varphi_a(z)|^2)\, dm(z) \\
&\le \log^2 \frac{2}{1-|a|} \int_D \int_0^{2\pi} |f'(e^{i\theta}z)|^2 \frac{1-t^2}{|e^{i\theta}-t|^2} \frac{d\theta}{2\pi} (1-|\varphi_a(z)|^2)\, dm(z) \\
&= \int_0^{2\pi} \log^2 \frac{2}{1-|a|} \int_D |f'(e^{i\theta}z)|^2 (1-|\varphi_a(z)|^2)\, dm(z) \frac{1-t^2}{|e^{i\theta}-t|^2} \frac{d\theta}{2\pi} \\
&\le \|f\|_*{}^2 \int_0^{2\pi} \frac{1-t^2}{|e^{i\theta}-t|^2} \frac{d\theta}{2\pi} = \|f\|_*{}^2.
\end{aligned}$$

That is,

$$\|f_t\|_* \le \|f\|_*. \qquad \square$$

Lemma 7. *If $g \in H^\infty$, $f \in VMOA_{\log}$ and $fg \in VMOA_{\log}$; then $fg \in [f]_*$.*

Proof. For $0 < t < 1$, $g_t(z) = g(tz)$, let P_n denote the partial sum of the power series of g_t; we can easily show that $P_n f \to g_t f$ (norm) as $n \to \infty$. Then $g_t f \in [f]$. Similarly, $g_t f \in [f]_*$. Applying Proposition 2 of [**4**] implies that $g_t f \to gf$ (weak*) as $t \to 1^-$ if $\sup_t \|g_t f\|_{BMOA_{\log}} < \infty$, so $fg \in [f]_*$. We next show that $\sup_t \|g_t f\|_{BMOA_{\log}} < \infty$.

Since $g \in H^\infty$, we have

$$\begin{aligned}
\|g_t f\|_*^2 &= \sup_{a\in D} \log^2 \frac{2}{1-|a|} \int_D |(g_t(z)f(z))'|^2 (1-|\varphi_a(z)|^2)\, dm(z) \\
&\le 2\|g\|_\infty^2 \|f\|_*{}^2 \\
&\quad + 2\sup_{a\in D} \log^2 \frac{2}{1-|a|} \int_D |f(z) g_t'(z)|^2 (1-|\varphi_a(z)|^2)\, dm(z).
\end{aligned}$$

Write

$$\begin{aligned}
&\log^2 \frac{2}{1-|a|}\int_D |f(z)g_t'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z)\\
&\quad\le 2\log^2 \frac{2}{1-|a|}\int_D |f(z)-f_t(z)|^2|g_t'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z)\\
&\qquad+2\log^2 \frac{2}{1-|a|}\int_D |f_t(z)g_t'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z)\\
&\quad\triangleq 2I+2II.
\end{aligned}$$

Let $|z|=r$. Using Lemmas 1, 2 and 5, we obtain

$$\begin{aligned}
I &\le C\|f\|_*{}^2\log^2\frac{2}{1-|a|}\int_D \frac{\log^2\frac{1-|tz|}{1-|z|}}{\log^2\frac{2}{1-|tz|}}\frac{\|g\|_\infty^2}{(1-|tz|)^2}\\
&\qquad\qquad\times\frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}\,dm(z)\\
&\le C\|f\|_*{}^2\|g\|_\infty{}^2\log^2\frac{2}{1-|a|}\int_0^1\frac{\log^2\frac{1-rt}{1-r}}{\log^2\frac{2}{1-rt}}\frac{1}{(1-rt)^2}\\
&\qquad\qquad\times\int_0^{2\pi}\frac{(1-|a|^2)(1-r^2)}{|1-\overline{a}re^{i\theta}|^2}\,d\theta\,dr\\
&\le C\|f\|_*{}^2\|g\|_\infty{}^2\int_0^1\frac{(1-|a|^2)\log^2\frac{2}{1-|a|}}{(1-|ar|^2)\log^2\frac{2}{1-|ar|}}\frac{(1-r^2)\log^2\frac{2}{1-|ar|}}{(1-rt)\log^2\frac{2}{1-rt}}\\
&\qquad\qquad\times\frac{1}{1-rt}\log^2\frac{1-rt}{1-r}\,dr\\
&\le C\|f\|_*{}^2\|g\|_\infty{}^2\int_0^1\frac{(1-r)\log^2\frac{2}{1-r}}{(1-rt)\log^2\frac{2}{1-rt}}\frac{1}{1-rt}\log^2\frac{1-rt}{1-r}\,dr\\
&\le C\|f\|_*{}^2\|g\|_\infty{}^2\int_0^1\frac{1}{1-rt}\log^2\frac{1-rt}{1-r}\,dr\\
&\le C\|f\|_*{}^2\|g\|_\infty{}^2<\infty.
\end{aligned}$$

From Lemma 6,

$$\begin{aligned} II &\leq 2\log^2 \frac{2}{1-|a|}\int_D |(fg)_t'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z) \\ &\quad + 2\log^2 \frac{2}{1-|a|}\int_D |f_t'(z)|^2|g_t(z)|^2(1-|\varphi_a(z)|^2)\,dm(z) \\ &\leq 2\|(fg)_t\|_*^2 + 2\|g\|_\infty{}^2\|f_t\|_*^2 \\ &\leq 2\|fg\|_*^2 + 2\|g\|_\infty{}^2\|f\|_*{}^2 < \infty. \end{aligned}$$

Thus,

$$\sup_t \|g_t f\|_{BMOA_{\log}} = |f(0)g(0)| + \sup_t \|g_t f\|_* < \infty.$$

The proof of Lemma 7 is finished. □

3. Proofs of Theorems.

Proof of Theorem 1. By Lemma 7, we have $(g/f)f = g \in [f]_*$. Since g is cyclic in $(VMOA_{\log})^{**}$, f is cyclic in $(VMOA_{\log})^{**}$. Hence f is cyclic in $VMOA_{\log}$. □

Theorem 1 gives an affirmative answer to Question 3 in [**4**] for the space $VMOA_{\log}$.

Corollary 1. *For $f \in VMOA_{\log}$, if $|f(z)| \geq C > 0$ in D, then f is cyclic in $VMOA_{\log}$.*

Proof of Theorem 2. If f is a cyclic vector in $VMOA_{\log}$, by Proposition 6 of [**4**], then f is cyclic in $VMOA$. From Theorem 2 of [**11**], f is an outer function. □

Conjecture 2. *$BMOA_{\log}$ with the norm $\|\cdot\|_{BMOA_{\log}}$ is isometric to $(VMOA_{\log})^{**}$.*

If the conjecture is true, then Theorems 1 and 2 hold for the space $BMOA_{\log}$.

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