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ON BIHOLOMORPHIC MAPPINGS IN COMPLEX BANACH SPACES

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ABSTRACT. Let X be a complex Banach space with norm $\|\cdot\|$, B the unit ball in X. In this paper, we introduce a class of holomorphic mappings $\mathcal{M}_g$ on B. Let $f(x)$ be a normalized locally biholomorphic mapping on B such that $(Df(x))^{-1}f(x) \in \mathcal{M}_g$. We obtain the growth and covering theorems, as well as coefficient estimates for $f(x)$. Especially, as corollaries, we unify and generalize many known results. Moreover, in view of proofs of corollaries, we can see the relations among the subclasses of starlike mappings.

1. Introduction. Let X be a complex Banach space with norm $\|\cdot\|$, X^* the dual space of X, B the unit ball in X, B^n the Euclidean unit ball in $\mathbf{C}^n$, D the unit disk in $\mathbf{C}$ and D^n the unit polydisc in $\mathbf{C}^n$. Let ∂B be the boundary of B, $\overline{B}$ the closure of B and $\partial_0 D^n$ the Šilov boundary (i.e., the boundary at which the maximum modulus of the holomorphic function can be attained) of D^n. Let $H(B)$ be the set of all holomorphic mappings from B into X, $H(B,B)$ the set of all holomorphic mappings from B into B. As is known to us, if $f \in H(B)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y-x)^n),$$

for all y in some neighborhood of $x \in B$, where $D^n f(x)$ is the nth-Fréchet derivative of f at x, and for $n \geq 1$,

$$D^n f(x)((y-x)^n) = D^n f(x)\underbrace{(y-x, \dots, y-x)}_{n}.$$

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A holomorphic mapping $f : B \to X$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. If $f : B \to X$ is a holomorphic mapping, we say that f is normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator X into X.

For each $x \in X \setminus \{0\}$, we define $T(x) = \{T_x \in X^* : \|T_x\| \leq 1, T_x(x) = \|x\|\}$. According to the Hahn-Banach theorem, $T(x)$ is nonempty. Notice that for $x \in X$ and any $\alpha(\neq 0) \in \mathbf{C}$, when T_x is chosen and fixed, then $\|(|\alpha|/\alpha)T_x\| = \|T_x\| \leq 1$, and $(|\alpha|/\alpha)T_x(\alpha x) = (|\alpha|/\alpha)\alpha T_x(x) = |\alpha|\|x\| = \|\alpha x\|$, so we can denote $T_{\alpha x} = (|\alpha/\alpha)T_x$.

Since Barnard, FitzGerald and Gong [**1**] established firstly the growth and covering theorems for normalized biholomorphic starlike mappings from B^n into $\mathbf{C}^n$, Zhang and Dong [**16**] and Liu and Ren [**13**] obtained the generalization on other domains such as B and bounded starlike circular domains in $\mathbf{C}^n$, respectively. After that, many mathematicians investigated the growth and covering theorems for the subclasses of starlike mappings. Liu and Lu [**12**] studied the growth and covering theorems for normalized biholomorphic starlike mappings of order α (see Definition 2) on the bounded starlike circular domains in $\mathbf{C}^n$; in addition, Hamada et al. [**8**] obtained corresponding results on B. Using the method of Loewner chains, Chuaqui [**2**], Hamada [**7**] and Graham et al. [**5**] obtained respectively the growth and covering theorems for normalized biholomorphic strongly starlike mappings (see Definition 3) on B^n, the bounded starlike circular domains in $\mathbf{C}^n$ and the unit ball in $\mathbf{C}^n$ with respect to an arbitrary norm; Feng and Lu [**4**] got the growth and covering theorems for normalized biholomorphic almost starlike mappings of order α (see Definition 4) on the bounded starlike circular domains in $\mathbf{C}^n$ and for the case of B, Feng [**3**] also gave the corresponding results; Kohr and Liczberski [**11**] and Feng [**3**] obtained the growth and covering theorems for normalized biholomorphic strongly starlike mappings of order α (see Definition 5) on B^n and B, respectively.

In this paper, we introduce a class of holomorphic mappings $\mathcal{M}_g$ on B. Let $f(x)$ be a normalized locally biholomorphic mapping on B such that $(Df(x))^{-1}f(x) \in \mathcal{M}_g$. We obtain the growth and covering theorems, as well as coefficient estimates for $f(x)$. Especially, as corollaries, we unify and generalize the results mentioned above.

Moreover, from the proofs of corollaries, we can see the relations among the subclasses of starlike mappings.

Firstly, we recall a class of mappings $\mathcal{M}$ which plays the role of the Carathéodory class in several complex variables.

$$\mathcal{M} = \{h \in H(B) : h(0) = 0, \quad Dh(0) = I, \\ \operatorname{Re}[T_x(h(x))] > 0, \quad x \in B\backslash\{0\},\ T_x \in T(x)\}.$$

Now, we introduce the following class $\mathcal{M}_g$ on B in X, which was first introduced by Graham, Hamada and Kohr [**6**] in the case $X = \mathbf{C}^n$ with an arbitrary norm, and by Kohr [**10**] in the case $X = \mathbf{C}^n$ with the Euclidean norm.

Definition 1. Let $g \in H(D)$ be a biholomorphic function such that $g(0) = 1$, $g(\overline{\xi}) = \overline{g(\xi)}$, for $\xi \in D$, $\operatorname{Re} eg(\xi) > 0$ on $\xi \in D$, and assume g satisfies the following conditions for $r \in (0,1)$:

$$\begin{cases} \min_{|\xi|=r} \operatorname{Re} g(\xi) = \min\{g(r), g(-r)\} \\ \max_{|\xi|=r} \operatorname{Re} g(\xi) = \max\{g(r), g(-r)\}. \end{cases} \tag{1}$$

We define $\mathcal{M}_g$ to be the class of mappings given by

$$\mathcal{M}_g = \left\{p \in H(B) : p(0) = 0,\ Dp(0) = I,\ \frac{1}{\|x\|} T_x(p(x)) \in g(D), \\ x \in B\backslash\{0\}, T_x \in T(x)\right\}.$$

Clearly, if $g(\xi) = 1 + \xi/1 - \xi$, $\xi \in D$, then $\mathcal{M}_g$ becomes the class $\mathcal{M}$. Especially, if $X = \mathbf{C}^n$, $B = D^n$, then

$$\mathcal{M}_g = \left\{p \in H(D^n) : p(0) = 0,\ Dp(0) = I,\ \frac{p_j(z)}{z_j} \in g(D),\ z \in D^n\backslash\{0\}\right\},$$

where $p(z) = (p_1(z), \dots, p_n(z))'$ is a column vector in $\mathbf{C}^n$, $|z_j| = \|z\| = \max_{1\le k\le n}\{|z_k|\}$.

A normalized biholomorphic mapping $f : B \to X$ is said to be starlike if $f(B)$ is a starlike domain with respect to the origin. As is well known to us, the following classes of mappings are also subclasses of starlike mappings.

Definition 2 [**8**]. Suppose $0 < \alpha < 1$. A normalized locally biholomorphic mapping $f : B \to X$ is said to be a starlike mapping of order α if

$$\left|\frac{1}{\|x\|}T_x[(Df(x))^{-1}f(x)] - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \quad \text{for all } x \in B \setminus \{0\}.$$

If $X = \mathbf{C}^n$, $B = D^n$, then it is obvious that the above condition is equivalent to

$$\left|\frac{g_j(z)}{z_j} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \quad \text{for all } z \in D^n \setminus \{0\},$$

where $g(z) = (g_1(z), \dots, g_n(z))' = (Df(z))^{-1}f(z)$ is a column vector in $\mathbf{C}^n$, $|z_j| = \|z\| = \max_{1\le k\le n}\{|z_k|\}$.

Definition 3 [**1**]. Suppose $0 < c < 1$. A normalized locally biholomorphic mapping $f : B \to X$ is said to be a strongly starlike mapping if

$$\left|\frac{1}{\|x\|}T_x[(Df(x))^{-1}f(x)] - \frac{1+c^2}{1-c^2}\right| < \frac{2c}{1-c^2}, \quad \text{for all } x \in B\backslash\{0\}.$$

If $X = \mathbf{C}^n$, $B = D^n$, then it is clear that the above condition is equivalent to

$$\left|\frac{g_j(z)}{z_j} - \frac{1+c^2}{1-c^2}\right| < \frac{2c}{1-c^2}, \quad \text{for all } z \in D^n\backslash\{0\},$$

where $g(z) = (g_1(z), \dots, g_n(z))' = (Df(z))^{-1}f(z)$ is a column vector in $\mathbf{C}^n$, $|z_j| = \|z\| = \max_{1\le k\le n}\{|z_k|\}$.

When $X = \mathbf{C}^n$, $B = B^n$, the above definition was firstly introduced by Chuaqui [**2**].

Definition 4 [**3**]. Suppose $0 \le \alpha < 1$. A normalized locally biholomorphic mapping $f : B \to X$ is said to be an almost starlike mapping of order α if

$$\operatorname{Re}\{T_x[(Df(x))^{-1}f(x)]\} \ge \alpha\|x\|, \quad \text{for all } x \in B \setminus \{0\},\ T_x \in T(x).$$

If $X = \mathbf{C}^n$, $B = D^n$, then it is clear that the above condition is equivalent to

$$\operatorname{Re} \frac{g_j(z)}{z_j} \geq \alpha, \quad \text{for all } z \in D^n \backslash \{0\},$$

where $g(z) = (g_1(z), \dots, g_n(z))' = (Df(z))^{-1} f(z)$ is a column vector in $\mathbf{C}^n$, $|z_j| = \|z\| = \max_{1 \leq k \leq n}\{|z_k|\}$.

Definition 5. Suppose $0 < \alpha \leq 1$. A normalized locally biholomorphic mapping $f : B \to X$ is said to be a strongly starlike mapping of order α if

$$\left| \arg \frac{1}{\|x\|} T_x[(Df(x))^{-1} f(x)] \right| < \frac{\pi}{2}\alpha, \quad \text{for all } x \in B \backslash \{0\}.$$

If $X = \mathbf{C}^n$, $B = D^n$, then it is clear that the above condition is equivalent to

$$\left| \arg \frac{g_j(z)}{z_j} \right| < \frac{\pi}{2}\alpha, \quad \text{for all } z \in D^n \backslash \{0\},$$

where $g(z) = (g_1(z), \dots, g_n(z))' = (Df(z))^{-1} f(z)$ is a column vector in $\mathbf{C}^n$, $|z_j| = \|z\| = \max_{1 \leq k \leq n}\{|z_k|\}$.

When $X = \mathbf{C}^n$, $B = B^n$, Kohr and Liczberski [**11**] firstly introduced the above definition.

Definition 6. Suppose $f, g \in H(D)$. If there exists a function $\varphi \in H(D, D)$, $\varphi(0) = 0$ such that $f = g \circ \varphi$, then we say that f is subordinate to g (written $f \prec g$).

2. Some lemmas. In order to prove the desired results, we first give some lemmas.

Lemma 1 [**9**]. *Suppose $x(t) : [0, 1] \to X$ is differentiable at the point s which belongs to $[0, 1]$, and $\|x(t)\|$ is differentiable at the point s with respect to t. Then*

$$\operatorname{Re}\left[T_{x(t)}\left(\frac{dx(t)}{dt} \right) \right] \bigg|_{t=s} = \frac{d(\|x(t)\|)}{dt} \bigg|_{t=s}.$$

Lemma 2 [**16**]. *Suppose f is a starlike mapping on B, $x \in B \setminus \{0\}$, $x(t) = f^{-1}(tf(x))$ $(0 \leq t \leq 1)$. Thens*

(a) $\|x(t)\|$ *is strictly increasing on* $[0,1]$ *with respect to* t;

(b) $\|f(x)\| = \lim_{t \to 0} \|x(t)\|/t, dx(t)/dt = (1/t)[Df(x(t))]^{-1} f(x(t))$, $t \in (0,1)$.

Lemma 3. *If $h \in \mathcal{M}_g$, then*

$$\begin{aligned} \|x\| \cdot \min\{g(\|x\|), g(-\|x\|)\} &\leq \operatorname{Re} T_x(h(x)) \\ &\leq \|x\| \cdot \max\{g(\|x\|), g(-\|x\|)\} \end{aligned} \tag{2}$$

for all $x \in B$.

Proof. Fix $x \in B \backslash \{0\}$, and denote $x_0 = x/\|x\|$. Let $p : D \to \mathbf{C}$ be given by

$$p(\xi) = \begin{cases} (1/\xi) T_x(h(\xi x_0)) & \xi \neq 0, \\ 1 & \xi = 0. \end{cases}$$

Then $p \in H(D)$, $p(0) = g(0) = 1$ and, since $h \in \mathcal{M}_g$, we deduce that

$$\begin{aligned} p(\xi) &= \frac{1}{\xi} T_x(h(\xi x_0)) = \frac{1}{\xi} T_{x_0}(h(\xi x_0)) \\ &= \frac{1}{\|\xi x_0\|} T_{\xi x_0}(h(\xi x_0)) \in g(D), \quad \xi \in D. \end{aligned}$$

Therefore, $p \prec g$, and from the subordination principle it follows that $p(rD) \subseteq g(rD)$, $r \in (0,1)$, where $rD = \{z \in \mathbf{C} : |z| < r\}$. On the other hand, combining the maximum and minimum principles for harmonic functions with (1), we deduce that

$$\min\{g(|\xi|), g(-|\xi|)\} \leq \operatorname{Re} p(\xi) \leq \max\{g(|\xi|), g(-|\xi|)\}, \quad \xi \in D.$$

Setting $\xi = \|x\|$ in the above relation, we obtain (2), as desired. $\square$

3. Main results and their proofs.

Theorem 1. *Let $f : B \to X$ be a normalized locally biholomorphic mapping. If $(Df(x))^{-1}f(x) \in \mathcal{M}_g$, then*

$$\begin{aligned}\|x\| \exp \int_0^{\|x\|} \left[\frac{1}{\max\{g(y), g(-y)\}} - 1\right] \frac{dy}{y} &\le \|f(x)\| \\ &\le \|x\| \exp \int_0^{\|x\|} \left[\frac{1}{\min\{g(y), g(-y)\}} - 1\right] \frac{dy}{y}, \quad x \in B.\end{aligned}$$

Proof. Since $(Df(x))^{-1}f(x) \in \mathcal{M}_g$, we deduce from Lemma 3 that

$$\begin{aligned}\|x\| \cdot \min\{g(\|x\|), g(-\|x\|)\} &\le \operatorname{Re} T_x(Df(x))^{-1}f(x) \\ &\le \|x\| \cdot \max\{g(\|x\|), g(-\|x\|)\}\end{aligned} \tag{3}$$

for all $x \in B$. Clearly, in view of the assumption of Theorem 1, we conclude that f belongs to either starlike mappings class or its subclasses. Fix $x \in B \setminus \{0\}$ and let $x(t) = f^{-1}(tf(x))(0 \le t \le 1)$. According to (a) of Lemma 2, we obtain that $\|x(t)\|$ is strictly increasing on $[0,1]$. Hence, $\|x(t)\|$ is differentiable on $[0,1]$, almost everywhere. From Lemmas 1, 2 (b) and (3), we deduce that for $t \in (0,1]$

$$\|x(t)\| \cdot \min\{g(\|x(t)\|), g(-\|x(t)\|)\} \tag{4}$$

$$\le t\frac{d\|x(t)\|}{dt} \le \|x(t)\| \cdot \max\{g(\|x(t)\|), g(-\|x(t)\|)\}, \tag{5}$$

and we may rewrite (4) and (5) as

$$\begin{aligned}&\frac{1}{\|x(t)\| \cdot \max\{g(\|x(t)\|), g(-\|x(t)\|)\}} \frac{d\|x(t)\|}{dt} \\ &\qquad \le \frac{1}{t} \le \frac{1}{\|x(t)\| \cdot \min\{g(\|x(t)\|), g(-\|x(t)\|)\}} \frac{d\|x(t)\|}{dt}.\end{aligned}$$

Integrating both sides of the above inequalities with respect to t and making a change of variable, we obtain

$$\begin{aligned}\int_{\|x(\varepsilon)\|}^{\|x\|} &\frac{dy}{y\max\{g(y),g(-y)\}}\\ &= \int_{\varepsilon}^{1} \frac{1}{\|x(t)\|\max\{g(\|x(t)\|,g(-\|x(t)\|))\}}\frac{d\|x(t)\|}{dt}\,dt\\ &\le \int_{\varepsilon}^{1}\frac{1}{t}dt,\end{aligned}$$

and

$$\begin{aligned}\int_{\|x(\varepsilon)\|}^{\|x\|} &\frac{dy}{y\min\{g(y),g(-y)\}}\\ &= \int_{\varepsilon}^{1} \frac{1}{\|x(t)\|\min\{g(\|x(t)\|,g(-\|x(t)\|))\}}\frac{d\|x(t)\|}{dt}\,dt\\ &\ge \int_{\varepsilon}^{1}\frac{1}{t}dt,\end{aligned}$$

where $0<\varepsilon<1$. It is elementary to verify that

$$\log\frac{\|x(\varepsilon)\|}{\varepsilon} \ge \int_{\|x(\varepsilon)\|}^{\|x\|}\left[\frac{1}{\max\{g(y),g(-y)\}}-1\right]\frac{dy}{y}+\log\|x\|, \tag{6}$$

and

$$\log\frac{\|x(\varepsilon)\|}{\varepsilon} \le \int_{\|x(\varepsilon)\|}^{\|x\|}\left[\frac{1}{\min\{g(y),g(-y)\}}-1\right]\frac{dy}{y}+\log\|x\|. \tag{7}$$

If we now let $\varepsilon\to 0+$ in the above inequalities (6), (7) and use Lemma 2 (b), we have

$$\begin{aligned}\|x\|\exp\int_0^{\|x\|}&\left[\frac{1}{\max\{g(y),g(-y)\}}-1\right]\frac{dy}{y}\le\|f(x)\|\\ &\le \|x\|\exp\int_0^{\|x\|}\left[\frac{1}{\min\{g(y),g(-y)\}}-1\right]\frac{dy}{y},\quad x\in B,\end{aligned}$$

as claimed. This completes the proof of Theorem 1. □

Remark. When B is the unit ball in $\mathbf{C}^n$ with respect to an arbitrary norm, Theorem 1 was obtained by Graham et al. [5]. However, their method of proof is different from that of Theorem 1.

As corollaries to Theorem 1, we have the following growth and covering results for starlike mappings and its subclasses.

Corollary 1. *If f is a starlike mapping on B, then*

$$\frac{\|x\|}{(1+\|x\|)^2} \le \|f(x)\| \le \frac{\|x\|}{(1-\|x\|)^2}, \quad x \in B. \tag{8}$$

Consequently, $f(B) \supset (1/4)B$.

Proof. Let $g(\xi) = (1+\xi)/(1-\xi)$, $\xi \in D$. Clearly, g is a biholomorphic function on D that satisfies the assumptions of Definition 1. If $(Df(x))^{-1}f(x) \in \mathcal{M}_g$, then $\operatorname{Re} T_x[(Df(x))^{-1}f(x)] \ge 0$, $x \in B$, i.e., f is a starlike mapping on B. Using Theorem 1, we obtain (8). This completes the proof. $\square$

Remark 1. Corollary 1 was obtained by Zhang and Dong [**16**] and was obtained by Barnard et al. [**1**] in the case $X = \mathbf{C}^n$ with the Euclidean structure.

Corollary 2. *If f is a starlike mapping of order α $(0 < \alpha \le 1)$ on B, then*

$$\frac{\|x\|}{(1+\|x\|)^{2(1-\alpha)}} \le \|f(x)\| \le \frac{\|x\|}{(1-\|x\|)^{2(1-\alpha)}}, \quad x \in B. \tag{9}$$

Consequently, $f(B) \supset 1/(2^{2(1-\alpha)})B$.

Proof. Let $g(\xi) = (1-\xi)/(1+(1-2\alpha)\xi)$, $\xi \in D$. It is elementary to verify that g is a biholomorphic function on D and satisfies the hypotheses of Definition 1. If $(Df(x))^{-1}f(x) \in \mathcal{M}_g$, then $|1/\|x\|T_x[(Df(x))^{-1}f(x)] - 1/2\alpha| < 1/2\alpha$, $x \in B \setminus \{0\}$, i.e., f is a starlike mapping of order α on B. By Theorem 1, we obtain (9). This completes the proof. $\square$

Remark 2. Corollary 2 was obtained by Hamada et al. [**8**] using the analytical characterization of starlike mappings of order α.

Corollary 3. *If f is a strongly starlike mapping on B, then*

$$\frac{\|x\|}{(1+c\|x\|)^2} \leq \|f(x)\| \leq \frac{\|x\|}{(1-c\|x\|)^2}, \quad x \in B. \tag{10}$$

Consequently, $f(B) \supset 1/((1+c)^2)B$.

Proof. Let $g(\xi) = (1+c\xi)/(1-c\xi)$, $\xi \in D$. Obviously, we deduce that g satisfies the hypotheses of Definition 1. If $(Df(x))^{-1}f(x) \in \mathcal{M}_g$, then $|1/\|x\|T_x[(Df(x))^{-1}f(x)] - (1+c^2)/(1-c^2)| < 2c/(1-c^2)$, i.e., f is a strongly starlike mapping on B. From Theorem 1, we obtain (10). This completes the proof. □

Remark 3. The estimations of Corollary 3 are sharp. In order to see that these estimates are sharp on B, it suffices to consider

$$f(x) = \frac{x}{(1-cT_u(x))^2}, \quad 0 < c < 1,\ x \in B,\ u \in \partial B. \tag{11}$$

It is clear that f is a normalized holomorphic mapping on B. Straightforward calculation shows that

$$Df(x) = \frac{2cxT_u(\cdot)}{(1-cT_u(x))^3} + \frac{1}{(1-cT_u(x))^2}I,$$
$$(Df(x))^{-1} = (1-cT_u(x))^2\left(I - \frac{2cxT_u(\cdot)}{1+cT_u(x)}\right),$$

where $xT_u(\cdot) : X \to X$ is a continuous linear functional. So f is a normalized locally biholomorphic mapping on B, and we have

$$\frac{1}{\|x\|}T_x[(Df(x))^{-1}f(x)] = \frac{1-cT_u(x)}{1+cT_u(x)},$$
$$0 < c < 1, \quad x \in B \setminus \{0\}, \quad u \in \partial B,$$

that is,

$$\left|\frac{1}{\|x\|}T_x[(Df(x))^{-1}f(x)] - \frac{1+c^2}{1-c^2}\right| < \frac{2c}{1-c^2}, \quad \text{for all } x \in B\setminus\{0\}.$$

Hence, f is a strongly starlike mapping on B. Take $x = ru$ or $x = -ru (0 \le r < 1)$, $u \in \partial B$; we deduce that the estimations of Corollary 3 are sharp.

Corollary 4. *If f is an almost starlike mapping of order α $(0 \le \alpha < 1)$ on B, then*

(12)
$$\frac{\|x\|}{(1+(1-2\alpha)\|x\|)^{[2(1-\alpha)]/(1-2\alpha)}} \le \|f(x)\| \le \frac{\|x\|}{(1-(1-2\alpha)\|x\|)^{[2(1-\alpha)]/(1-2\alpha)}},$$

(13)
$$f(B) \supset \frac{1}{(2(1-\alpha))^{[2(1-\alpha)]/(1-2\alpha)}} B,$$

for $x \in B$, $0 \le \alpha < 1$, $\alpha \ne 1/2$ and

(14)
$$\|x\| \exp(-\|x\|) \le \|f(x)\| \le \|x\| \exp(\|x\|), \quad f(B) \supset \frac{1}{e} B,$$

for $x \in B$, $\alpha = 1/2$.

Proof. Let $g(\xi) = [1 + (1-2\alpha)\xi]/(1-\xi)$, $\xi \in D$. It is not difficult to check that g is a biholomorphic function on D and satisfies the hypotheses of Definition 1. If $(Df(x))^{-1} f(x) \in \mathcal{M}_g$, then $\operatorname{Re} T_x[(Df(x))^{-1} f(x)] \ge \alpha \|x\|$, $x \in B$, i.e., f is an almost starlike mapping of order α on B. By Theorem 1, we obtain (12) and (14). This completes the proof. $\square$

Remark 4. Using a similar method as in the Remark of Corollary 3, we deduce that

$$f(x) = \begin{cases} x/[(1-(1-2\alpha)T_u(x))^{[2(1-\alpha)]/(1-2\alpha)}] & \alpha \in [0,1),\ \alpha \ne 1/2,\ x \in B, \\ x e^{T_u(x)} & \alpha = 1/2,\ x \in B, \end{cases}$$

where $u \in \partial B$ is an almost starlike mapping of order α on B. Taking $x = ru$ or $-ru (0 \le r < 1)$, $u \in \partial B$, we conclude that the estimations of Corollary 4 are sharp.

Corollary 5. *If f is a strongly starlike mapping of order α $(0 < \alpha \leq 1)$ on B, then*

(15)

$$\|x\| \exp \int_0^{\|x\|} \left[\left(\frac{1-y}{1+y} \right)^{\alpha} - 1 \right] \frac{dy}{y} \leq \|f(x)\| \leq \|x\| \exp \int_0^{\|x\|} \left[\left(\frac{1+y}{1-y} \right)^{\alpha} - 1 \right] \frac{dy}{y}$$

for $x \in B$. Consequently, $f(B) \supset r(\alpha)B$, where $r(\alpha) = \exp \int_0^1 [((1-y)/(1+y))^{\alpha} - 1](dy/y)$.

Proof. Let $g(\xi) = ((1+\xi)/(1-\xi))^{\alpha}$, $\xi \in D$. The branch of the power function is chosen such that $((1+\xi)/(1-\xi))^{\alpha}|_{\xi=0} = 1$. It is easy to see that g satisfies the hypotheses of Definition 1. If $(Df(x))^{-1}f(x) \in \mathcal{M}_g$, then $|\arg 1/\|x\|T_x[(Df(x))^{-1}f(x)]| < (\pi/2)\alpha$, for all $x \in B\backslash\{0\}$, i.e., f is a strongly starlike mapping of order α on B. In view of Theorem 1, we obtain (15). This completes the proof. $\square$

Remark 5. When $X = \mathbf{C}^n, B = B^n$, Corollary 5 is the corresponding result of [**10**]. In addition, the estimations of Corollary 5 are sharp. It is easy to check that

$$f(x) = x \exp \int_0^{T_u(x)} \left[\left(\frac{1-t}{1+t} \right)^{\alpha} - 1 \right] \frac{dt}{t},$$
$$0 < \alpha \leq 1, \quad x \in B, \quad u \in \partial B,$$

is a strongly starlike mapping of order α on B. Taking $x = ru$ or $-ru (0 \leq r < 1)$, $u \in \partial B$, we deduce that the equalities of (15) hold.

Let $f(x)$ be a normalized locally biholomorphic mapping on B such that $(Df(x))^{-1}f(x) \in \mathcal{M}_g$. We shall obtain some estimates for the second order Fréchet derivative of $f(x)$ on the unit ball of a complex Banach space and on the polydisc in $\mathbf{C}^n$, respectively. It seems to be more difficult to estimate the higher order Fréchet derivatives by this method.

Theorem 2. *Let $f : B \to X$ be a normalized locally biholomorphic mapping. If $(Df(x))^{-1}f(x) \in \mathcal{M}_g$, then*

$$\frac{|T_x(D^2f(0)(x^2))|}{2!} \leq |g'(0)|\|x\|^2, \quad x \in X, \quad T_x \in T(x).$$

Proof. Fix $x \in X\backslash\{0\}$, and denote $x_0 = x/\|x\|$ and $h(x) = (Df(x))^{-1}f(x)$. Let $p : D \to \mathbf{C}$ be given by

$$p(\xi) = \begin{cases} (1/\xi)T_x(h(\xi x_0)) & \xi \neq 0, \\ 1 & \xi = 0. \end{cases}$$

Using similar arguments as in the proof of Lemma 3, we deduce that $p \prec g$, which implies $|p'(0)| \leq |g'(0)|$ by the subordination principle. Noticing that

$$p(\xi) = 1 + \frac{T_x(D^2h(0)(x_0^2))}{2!}\xi + \cdots, \quad \xi \in D,$$

we obtain

$$\left|\frac{T_x(D^2h(0)(x^2))}{2!}\right| \leq |g'(0)|\|x\|^2, \quad x \in X. \tag{16}$$

Since $f(x) = Df(x)h(x)$, we have

$$\begin{aligned} x + &\frac{D^2f(0)(x^2)}{2!} + \cdots \\ &= (I + D^2f(0)(x,\cdot) + \cdots)\Big(Dh(0)x + \frac{D^2h(0)(x^2)}{2!} + \cdots\Big). \end{aligned}$$

Comparing the second order terms two sides of the above equality, we obtain

$$\frac{D^2f(0)(x^2)}{2!} = -\frac{D^2h(0)(x^2)}{2!}. \tag{17}$$

Thus we deduce from (16) and (17) that

$$\frac{|T_x(D^2f(0)(x^2))|}{2!} \leq |g'(0)|\|x\|^2, \quad x \in X, \quad T_x \in T(x).$$

This completes the proof. $\square$

Theorem 3. *Let $f : D^n \to \mathbf{C}^n$ be a normalized locally biholomorphic mapping. If $(Df(z))^{-1}f(z) \in \mathcal{M}_g$, then*

$$\frac{\|D^2f(0)(z^2)\|}{2!} \leq |g'(0)|\|z\|^2, \quad z \in \mathbf{C}^n.$$

Proof. Fix $z \in \mathbf{Z}^n\backslash\{0\}$, and denote $z_0 = z/\|z\|$ and $h(z) = (Df(z))^{-1}f(z)$. Let $p : D \to \mathbf{C}$ be given by

$$p_j(\xi) = \begin{cases} (h_j(\xi z_0)\|z\|)/\xi z_j & \xi \neq 0, \\ 1 & \xi = 0, \end{cases}$$

where $|z_j| = \|z\| = \max_{1\leq k\leq n}\{|z_k|\}$. Then $p_j \in H(D)$, $p_j(0) = g(0) = 1$, and since $h \in \mathcal{M}_g$, we deduce that $h_j(\xi) \in g(D)$, $\xi \in D$. Therefore we deduce that $h_j \prec g$, which implies $|p'(0)| \leq |g'(0)|$ by the subordination principle. Noticing that $p_j(\xi) = 1+\frac{D^2h_j(0)(z_0)^2\|z\|}{2!z_j}\xi+\cdots$, with the similar method and reasoning as in the proof of Theorem 2, we obtain

$$\frac{|D^2f_j(0)(z_0)^2|}{2!} \leq |g'(0)|.$$

If $z_0 \in \partial_0 D^n$, then we have

$$\frac{|D^2f_j(0)(z_0)^2|}{2!} \leq |g'(0)|, \quad j = 1, 2, \ldots, n.$$

Also since $D^2f_j(0)(z^2)$ is a holomorphic function on $\overline{D}^n$, in view of the maximum modulus theorem of holomorphic function on the unit polydisc, we obtain

$$\frac{|D^2f_j(0)(z_0)^2|}{2!} \leq |g'(0)|, \quad z_0 \in \partial D^n,\ j = 1, 2, \ldots, n;$$

that is,

$$\frac{|D^2f_j(0)(z^2)|}{2!} \leq |g'(0)|\|z\|^2, \quad z \in \mathbf{C}^n,\ j = 1, 2, \ldots, n.$$

So

$$\frac{\|D^2f(0)(z^2)\|}{2!} \leq |g'(0)|\|z\|^2, \quad z \in \mathbf{C}^n.$$

This completes the proof. □

Using Theorems 2 and 3, we can obtain the following results. Now we only state them and the proofs are omitted.

Corollary 6. *If f is a starlike mapping on B, then*

$$\frac{|T_x(D^2f(0)(x^2))|}{2!} \leq 2\|x\|^2, \quad x \in X,\ T_x \in T(x).$$

If $X = \mathbf{C}^n, B = D^n$, then

$$\frac{\|D^2f(0)(z^2)\|}{2!} \leq 2\|z\|^2, \quad z \in \mathbf{C}^n.$$

Remark 6. When B is the unit ball in $\mathbf{C}^n$ with respect to an arbitrary norm, Corollary 6 was obtained by Graham et al. [**5**] using the method of Loewner chains.

Corollary 7. *If f is a starlike mapping of order α $(0 < \alpha < 1)$ on B, then*

$$\frac{|T_x(D^2f(0)(x^2))|}{2!} \leq 2(1-\alpha)\|x\|^2, \quad x \in X, T_x \in T(x).$$

If $X = \mathbf{C}^n$, $B = D^n$, then

$$\frac{\|D^2f(0)(z^2)\|}{2!} \leq 2(1-\alpha)\|z\|^2, \quad z \in \mathbf{C}^n.$$

Corollary 8. *If f is an almost starlike mapping of order α $(0 \leq \alpha < 1)$ on B, then*

$$\frac{|T_x(D^2f(0)(x^2))|}{2!} \leq 2(1-\alpha)\|x\|^2, \quad x \in X,\ T_x \in T(x).$$

If $X = \mathbf{C}^n$, $B = D^n$, then

$$\frac{\|D^2f(0)(z^2)\|}{2!} \leq 2(1-\alpha)\|z\|^2, \quad z \in \mathbf{C}^n.$$

Remark 7. Recently, Corollaries 7 and 8 have been proved by Liu and Liu [**14**], where the authors make use of the analytical characterizations of starlike mappings of order α and almost starlike mappings of order α, respectively. In the same paper, the authors also showed that Corollaries 7 and 8 are sharp.

Corollary 9. *If f is a strongly starlike mapping of order α $(0 < \alpha \leq 1)$ on B, then*

$$\frac{|T_x(D^2 f(0)(x^2))|}{2!} \leq 2\alpha \|x\|^2, \quad x \in X, \ T_x \in T(x). \tag{18}$$

If $X = \mathbf{C}^n$, $B = D^n$, then

$$\frac{\|D^2 f(0)(z^2)\|}{2!} \leq 2\alpha \|z\|^2, \quad z \in \mathbf{C}^n. \tag{19}$$

Remark. The estimations of Corollary 9 are sharp. From the Remark of Corollary 5, we know that

$$f(x) = x \exp \int_0^{T_u(x)} \left[\left(\frac{1+t}{1-t} \right)^\alpha - 1 \right] \frac{dt}{t},$$
$$0 < \alpha \leq 1, \quad x \in B, \quad u \in \partial B,$$

is a strongly starlike mapping of order α. Taking $x = ru (0 \leq r < 1)$, $u \in \partial B$ on B, we deduce that the equality of (18) holds.

When $X = \mathbf{C}^n$, $B = D^n$, it is easy to check that

$$f(z) = \left(z_1 \exp \int_0^{z_1} \left[\left(\frac{1+t}{1-t} \right)^\alpha - 1 \right] \frac{dt}{t}, z_2, \dots, z_n \right)',$$
$$0 < \alpha \leq 1, \quad z \in D^n$$

is a strongly starlike mapping of order α on D^n. Take $z_1 = r (0 \leq r < 1)$, $z_j = 0$, $j = 2, 3, \dots, n$; the equality of (19) holds.

Corollary 10. *If f is a strongly starlike mapping on B, then*

$$\frac{|T_x(D^2 f(0)(x^2))|}{2!} \leq 2c \|x\|^2, \quad 0 < c < 1, \ x \in X, \ T_x \in T(x). \tag{20}$$

If $X = \mathbf{C}^n, B = D^n$, *then*

$$\frac{\|D^2 f(0)(z^2)\|}{2!} \le 2c\|z\|^2, \quad 0 < c < 1, \ z \in \mathbf{C}^n. \tag{21}$$

Remark. The estimations of Corollary 10 are sharp. From the Remark of Corollary 3, we know that

$$f(x) = \frac{x}{(1 - cT_u(x))^2}, \quad 0 < c < 1, \ x \in B, \ u \in \partial B,$$

is a strongly starlike mapping on B. Taking $x = ru(0 \le r < 1)$, the equality of (20) holds.

When $X = \mathbf{C}^n$, $B = D^n$, it is not difficult to verify that

$$f(z) = \left(\frac{z_1}{(1 - cz_1)^2}, z_2, \dots, z_n\right)', \quad 0 < c < 1, \ z \in D^n$$

is a strongly starlike mapping on D^n. Taking $z_1 = r(0 \le r < 1)$, $z_j = 0$, $j = 2, 3, \dots, n$, the equality of (21) holds.

Remark. Let $f(x)$ be a normalized locally biholomorphic mapping on B such that $(Df(x))^{-1} f(x) \in \mathcal{M}_g$ and $x = 0$ is the zero of order $k + 1 (k \in \mathbf{N})$ of $f(x) - x$. In another paper, we also investigate the growth and covering theorems, as well as coefficient estimates for $f(x)$, and obtain the corresponding results.

REFERENCES

1. R.W. Barnard, C.H. FitzGerald and S. Gong, *The growth and* $(1/4)$*-theorems for starlike mappings in* $\mathbf{C}^n$, Pacific J. Math. **150** (1991), 13–22.

2. M. Chuaqui, *Applications of subordination chains to starlike mappings in* $\mathbf{C}^n$, Pacific J. Math. **168** (1995), 33–48.

3. Shu-Xia Feng, *Some classes of holomorphic mappings in several complex variables*, University of Science and Technology of China, Doctoral thesis, 2004.

4. S.X. Feng and K.P. Lu, *The growth theorem for almost starlike mappings of order* α *on bounded starlike circular domains*, Chin. Quart. J. Math. **15** (2000), 50–56.

5. I. Graham, H. Hamada and G. Kohr, *Parametric representation of univalent mappings in several complex variables*, Canadian J. Math. **54** (2002), 324–351.

6. I. Graham and G. Kohr, *Geometric function theory in one and higher dimensions*, Marcel Dekker, New York, 2003.

7. H. Hamada, *Starlike mappings on bounded balanced domains with $\mathcal{C}^1$-plurisubharmonic defining functions*, Pacific J. Math. **194** (2000), 359–371.

8. H. Hamada, G. Kohr and P. Liczberski, *Starlike mappings of order α on the unit ball in complex Banach spaces*, Glas. Mat. **36** (2001), 39–48.

9. T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan. **19** (1967), 508–520.

10. G. Kohr, *Using the method of Löwner chains to introduce some subclasses of biholomorphic mappings in* $\mathbf{C}^n$, Rev. Roum. Math. Pures Appl. **46** (2001), 743–760.

11. G. Kohr and P. Liczberski, *On strongly starlikeness of order alpha in several complex variables*, Glasnik Matem. **33** (1998), 185–198.

12. H. Liu and K.P. Lu, *Two subclasses for starlike mappings in* $\mathbf{C}^n$, Chinese Ann. Math. **21** (2000), 533–546 (in Chinese).

13. T.S. Liu and G.B. Ren, *The growth theorem for starlike mappings on bounded starlike circular domains*, Chinese Ann. Math. **19** (1998), 401–408.

14. X.S. Liu and T.S. Liu, *A refinement for coefficient estimations of biholomorphic mappings*, Sci. China. **35** (2005), 361–374 (in Chinese).

15. Xiao-Song Liu, *Properties of some subclasses of biholomorphic mappings in geometric function theory of several complex variables*, University of Science and Technology of China, Doctoral thesis, 2004.

16. W.J. Zhang and D.Z. Dong, *The growth and* (1/4)*-theorems for starlike mappings in a Banach space*, Chinese Ann. Math. **13** (1992), 417–423 (in Chinese).

17. W.J. Zhang and T.S. Liu, *The growth and covering theorems for quasi-convex mappings on the unit ball in complex Banach spaces*, Sci. China. **45** (2002), 1538–1547.

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