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ON THE NUMBERS OF FACES OF LOW-DIMENSIONAL REGULAR TRIANGULATIONS AND SHELLABLE BALLS

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ABSTRACT. We investigate the conjectured sufficiency of a condition for h-vectors $(1, h_1, h_2, \dots, h_d, 0)$ of regular d-dimensional triangulations. (The condition is already shown to be necessary in [**2**]). We first prove that the condition is sufficient when $h_1 \geq h_2 \geq \cdots \geq h_d$. We then derive some new shellings of squeezed spheres and use them to prove that the condition is sufficient when $d = 3$. Finally, in the case $d = 4$, we construct shellable 4-balls with the desired h-vectors, showing them to be realizable as regular triangulations when $h_4 = 0$ or $h_4 = h_1$.

1. Introduction.

1.1. Polytopes and the g-theorem. The g-theorem [**2, 13**] characterizes the f-vectors of simplicial (and hence also simple) convex polytopes. One corollary is a necessary condition for the f-vectors of simple unbounded polyhedra [**2**], which are the duals of regular triangulations [**15**]. In this paper we investigate the sufficiency of this condition, verifying it in several cases.

We begin with some definitions; more details can be found, for example, in [**4, 15**]. A *convex polyhedron* is an intersection of finitely many closed halfspaces in $\mathbf{R}^d$. A bounded convex polyhedron is called a *convex polytope*. The *f-vector* of a d-dimensional polyhedron (d-polyhedron) P is $f(P) = (f_0(P), \dots, f_{d-1}(P))$, where $f_j(P)$ denotes the number of j-faces of P. We also take $f_{-1}(P) = f_d(P) = 1$. A d-polytope is *simplicial* if every face is a simplex, and *simple* if every vertex (0-face) is contained in exactly d edges (1-faces), equivalently, in exactly d facets ($(d-1)$-faces). Simple polytopes are precisely the duals of simplicial polytopes.

The *h-vector* of a simplicial d-polytope P (or of any $(d-1)$-dimensional simplicial complex) is $h(P) = (h_0(P), \dots, h_d(P))$, where

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$$h_i(P) = \sum_{j=0}^{i} (-1)^{j-i} \binom{d-j}{d-i} f_{j-1}(P), \quad i = 0, \ldots, d. \tag{1}$$

We also define $h_i(P) = 0$ if $i < 0$ or $i > d$. These equations are invertible:

$$f_{j-1}(P) = \sum_{i=0}^{j} \binom{d-i}{d-j} h_i(P),\ j = 0, \ldots, d. \tag{2}$$

The *g-vector* of P is $g(P) = (g_0(P), \ldots, g_{\lfloor d/2 \rfloor}(P))$, where $g_0(P) = h_0(P)$ and $g_i(P) = h_i(P) - h_{i-1}(P)$, $i = 1, \ldots, \lfloor d/2 \rfloor$.

For positive integers a and i, the *i-canonical representation of a* is the unique expression of a in the form

$$a = \binom{b_i}{i} + \binom{b_{i-1}}{i-1} + \cdots + \binom{b_j}{j},$$

where $b_i > b_{i-1} > \cdots > b_j \geq j \geq 1$. From this, define

$$a^{\langle i \rangle} = \binom{b_i + 1}{i+1} + \binom{b_{i-1}+1}{i} + \cdots + \binom{b_j + 1}{j+1},$$

and we also take $a^{\langle 0 \rangle} = 0$. A vector $(a_0, \ldots, a_n)$ is called an *M-vector* if it consists of nonnegative integers satisfying $a_0 = 1$ and $a_{i+1} \leq a_i^{\langle i \rangle}$, $i = 1, \ldots, n-1$.

Theorem 1 (g-theorem).*A vector of nonnegative integers $(h_0, \ldots, h_d)$ is the h-vector of a simplicial convex d-polytope if and only if*

1. $h_i = h_{d-i}$, $i = 0, \ldots, d$ (*the Dehn-Sommerville equations*), *and*
2. $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$ *is an M-vector.*

This characterizes the f-vectors of simplicial polytopes and, by duality, f-vectors of simple polytopes.

1.2. Regular triangulations and simple polytopes. Consider a simplicial convex d-polytope P and a vertex v of P. Deleting v from the boundary complex of P yields the *antistar of* v *in* ∂P, the simplicial $(d-1)$-complex $(\partial P) - v$. By applying a projective transformation that sends the vertex v onto the hyperplane at infinity, and projecting $(\partial P) - v$ onto a hyperplane orthogonal to the direction of projection, it is evident that antistars are combinatorially equivalent to *regular triangulations* [**15**] and vice versa.

Let P^* be a simple polytope dual to P and F the facet of P^* corresponding to v. By applying a projective transformation that sends F onto the hyperplane at infinity, it is also evident that simple unbounded polyhedra are combinatorially dual to antistars, and vice versa.

A *patch* of a simplicial d-polytope P is a nonempty proper subset $\mathcal{C} = \{F_1, \ldots, F_k\}$ of facets of P such that there is a line shelling [**15**] beginning with these facets. We will usually identify the patch with its naturally associated simplicial $(d-1)$-complex. Equivalently, there is a polytope P' projectively equivalent to P and a point $v' \notin P'$ such that the set of facets $\mathcal{C}'$ of P' corresponding to $\mathcal{C}$ is precisely the set of facets that v' is beyond. Also equivalently, there is a polytope P'' projectively equivalent to P and a point $v'' \notin P''$ such that the set of facets $\mathcal{C}''$ of P'' corresponding to $\mathcal{C}$ is precisely the set of facets that v'' is beneath. (See [**4**] for the definitions of beneath and beyond.) It is evident, then, that the complementary sets of facets to patches are themselves patches, and that patches are combinatorially equivalent to antistars, for in the latter equivalence we may take $Q = \operatorname{conv}(P'' \cup \{v''\})$ and then form the antistar $(\partial Q) - v''$.

Now assume that P contains the origin in its interior and P^* is its polar dual [**4, 15**]. Then $\mathcal{C}$ is a patch of P if and only if there is a hyperplane H such that the associated set of vertices $\mathcal{V}$ of P^* and the complementary set of vertices $(\operatorname{vert} P^*) \setminus \mathcal{V}$ lie in opposite open halfspaces associated with H. If H^- and H^+ are the two closed halfspaces associated with H and, say, $\mathcal{V} \subset H^+$, then we will call $P^* \cap H^-$ the *truncated simple polytope* associated with $\mathcal{V}$.

1.3. Computing h-vectors of regular triangulations. We recall that the h-vector of a shellable simplicial $(d-1)$-complex can be computed from a shelling order $F_1, \ldots, F_m$ of its facets in the following way [**15**]. For facet F_j there is a unique, minimal face

$G_j = R(F_j)$ of F_j such that G_j is not in the simplicial complex induced by $F_1 \cup \cdots \cup F_{j-1}$. Let $\sigma(F_j) = \text{card}\, G_j$. Then $h_i = \text{card}\,\{j : \sigma(F_j) = i\}$. In this way we can determine the h-vectors of antistars/patches/regular triangulations.

The following is a useful lemma.

Lemma 1. *Assume $\mathcal{C}$ is a patch of a simplicial d-polytope P, and $\mathcal{C}'$ is the complementary patch consisting of those facets of P not in $\mathcal{C}$. Then $h(\mathcal{C}') + h^r(\mathcal{C}) = h(P)$, where $h^r(\mathcal{C})$ is the reverse of the vector $h(\mathcal{C})$.*

Proof. This follows quickly from the equations $h(P) = h(\mathcal{C}') + h(\mathcal{C}^\circ)$ and $h(\mathcal{C}^\circ) = h^r(\mathcal{C})$ (see [**10**]), where $\mathcal{C}^\circ$ denotes the collection of nonboundary faces of $\mathcal{C}$. □

It is well known that the dual to a line shelling of a simplicial polytope P is a sweeping of its dual P^* by a hyperplane: Choose a vector $a \in \mathbf{R}^d$ such that $a \cdot v_i \neq a \cdot v_j$ for all pairs of vertices v_i, v_j of P^*, $i \neq j$. For each edge $v_i v_j$ of P^*, orient it from v_i to v_j if $a \cdot v_i < a \cdot v_j$. Then the h-vector of P is given by $h_i = \text{card}\,\{j : v_j \text{ has indegree } i\}$, $i = 0, \ldots, d$. (See, for example, [**3**].) Similarly we can compute the h-vector for a patch by restricting attention only to those vertices and edges lying in H^-.

One corollary [**2**] of the g-theorem is the following:

Corollary 1. *If $(h_0, \ldots, h_d)$ is the h-vector of an antistar of a simplicial d-polytope (equivalently of a patch of a simplicial d-polytope, or of a regular $(d-1)$-triangulation), then $(\star)$ holds*:

Condition $(\star)$: *The vector $(h_0 - h_{d+k}, h_1 - h_{d+k-1}, \ldots, h_m - h_{d+k-m})$ is an M-vector for all integers $k = 0, \ldots, d+1$, $m = \lfloor (d+k-1)/2 \rfloor$, taking $h_i = 0$ if $i > d$.*

This condition is in fact necessary for the f-vectors of arbitrary three- and four-dimensional balls. The g-vectors of 2-spheres are trivially M-vectors. Kalai [**6**] showed that $g_2 \geq 0$ for arbitrary triangulated d-manifolds without boundary, $d \geq 2$, from which it follows that the

g-vectors of 3- and 4-spheres are also M-vectors. The arguments in [**1**] show that if $(1, h_1, h_2, h_3, 0)$ is the h-vector of a 3-ball (and hence itself an M-vector) then $(1, h_1 - h_3)$ is the g-vector of a 2-sphere and $(1, h_1, h_2 - h_3)$ is the g-vector of a 3-sphere, so both are M-vectors. Similarly, if $(1, h_1, h_2, h_3, h_4, 0)$ is the h-vector of a 4-ball (and hence itself an M-vector) then $(1, h_1 - h_4, h_2 - h_3)$ is the g-vector of a 3-sphere, and $(1, h_1, h_2 - h_4)$ is the g-vector of a 4-sphere, so both are M-vectors. Finally, $(1, h_1, h_2, h_3 - h_4)$ is an M-vector since $h_3 - h_4 \geq 0$, which follows from Theorem 4.30 of [**14**] (thanks to Ed Swartz for bringing this to the authors' attention).

In this paper we investigate the conjectured sufficiency [**2**] of this condition:

Conjecture 1. *A vector of nonnegative integers $(h_0, \ldots, h_d)$ is the h-vector of a regular $(d-1)-$triangulation if and only if Condition $(\star)$ holds.*

2. h-Vectors nonincreasing after h_1.

We begin by proving sufficiency in the following case:

Theorem 2. *If $h = (1, h_1, \ldots, h_d)$ is a vector of nonnegative integers satisfying $h_1 \geq h_2 \geq \cdots \geq h_{d-1} \geq h_d = 0$ (with $d \geq 1$), then h satisfies Condition $(\star)$ and $h = h(\Delta)$ for some regular triangulation Δ.*

Using the inequality assumptions and the fact that $h_i \leq h_i^{\langle i \rangle}$, it is easy to verify that h satisfies Condition $(\star)$. We will construct an appropriate truncated simple polytope. We begin with a d-simplex $P^0 \subset \mathbf{R}^d$ having vertices $x_0^0 < \cdots < x_d^0$, where the ordering of the vertices (and the orientation of the edges) is with respect to increasing last coordinate. Note that vertex x_i^0 has indegree i, $i = 0, \ldots, d$. Assume that the last coordinate of x_0^0 is 0, the last coordinates of $x_1^0, \ldots, x_{d-1}^0$ are very close to 0, and the last coordinate of x_d^0 is 1. We will perform a sequence of truncations to achieve a simple polytope such that the indegree sequence of the vertices having last coordinate less than 1/2 is the vector h. A vertex with last coordinate less than (greater than) 1/2 will be called *low* (*high*). Note that the indegree sequence of the low vertices of P^0 is $(1, \ldots, 1, 0)$.

At any stage we will assume that the current polytope P^s (1) has vertices $x_0^0, x_1^0, \ldots, x_{d-1}^0, x_1^1, x_2^1, \ldots, x_{d-1}^1, x_1^2, x_2^2, \ldots, x_{d-1}^2, \ldots, x_1^s, x_2^s, \ldots, x_{d-1}^s, x_d^s$; (2) each vertex x_i^k has indegree i; (3) vertex x_d^s has neighbors $x_0^0 < x_1^s < x_2^s < \cdots < x_{d-1}^s$, (4) vertices $x_1^s, \ldots, x_k^s$ are low and vertices $x_{k+1}^s, \ldots, x_d^s$ are high for some $k = 1, \ldots, d-1$, and (5) the vertices $x_1^s, x_2^s, \ldots, x_{d-1}^s, x_d^s$ form a facet.

For any $\ell = 1, \ldots, k$ we define a truncation of type ℓ as follows: Choose points $x_1^{s+1}, \ldots, x_{d-1}^{s+1}$ on edges $\overline{x_d^s x_1^s}, \ldots, \overline{x_d^s x_{d-1}^s}$, respectively, and point x_d^{s+1} on edge $\overline{x_d^s x_0^0}$ such that $x_1^{s+1} < \cdots < x_d^{s+1}$; $x_1^{s+1}, \ldots, x_\ell^{s+1}$ are low; and $x_{\ell+1}^{s+1}, \ldots, x_d^{s+1}$ are high. Construct the polytope P^{s+1} by truncating the vertex x_d^s using the hyperplane determined by the points $x_1^{s+1}, \ldots, x_d^{s+1}$. It is now straightforward to check that conditions (1) through (5) hold for P^{s+1}. Further, the indegree sequence of the low vertices increases by 1 for indegrees $1, \ldots, \ell$ and by 0 for indegrees $\ell + 1, \ldots, d$.

To realize the given vector $(h_0, \ldots, h_d)$ we now start with P^0 and perform $h_{d-1} - 1$ truncations of type $d-1$, $h_{d-2} - h_{d-1}$ truncations of type $d-2$, $h_{d-3} - h_{d-2}$ truncations of type $d-3, \ldots$, and $h_1 - h_2$ truncations of type 1. Then in the final polytope $P^{h_1 - 1}$ the low vertices will have the indegree sequence $(h_0, \ldots, h_d)$.

By duality, we conclude there is a regular $(d-1)$-triangulation Δ with $h = h(\Delta)$. □

3. New shelling orders for squeezed spheres. In this section we describe an approach to constructing patches. There are two basic steps: (1) find a shellable subcollection of facets of a simplicial polytope that induces a simplicial complex Δ with the desired h-vector; (2) show that there is a point z beyond precisely those facets in Δ (or else beyond precisely the complementary set of facets). This implies that Δ is a patch. So in (1) the desired patch is created combinatorially, and then in (2) it is realized geometrically.

We will begin by describing several shelling orders of squeezed spheres and explore h-vectors that can be achieved by partial shellings of low-dimensional squeezed spheres. See [**5, 9**] for more information on squeezed spheres. To describe the orders we need a few definitions.

Let $[n]^{(d)}$ denote the set of all subsets of $\{1, 2, \ldots, n\}$ of cardinality d. Consider $S \in [n]^{(d)}$. If $1, 2, \ldots, k \in S$, $k < d$, and $k + 1 \notin S$, then

$\{1, \dots, k\}$ is the *left* set of S; if $1 \notin S$ then the left set of S is empty. If $j, j+1, \dots, d \in S$, $j > 1$, and $j-1 \notin S$, then $\{j, \dots, d\}$ is the *right* set of S; if $d \notin S$, then the right set of S is empty. If $j > 1$, $k < d$, $j, j+1, \dots, k \in S$, $j-1 \notin S$, and $k+1 \notin S$, then $\{j, \dots, k\}$ is a *middle* set of S.

For any collection F of subsets of $\{1, \dots, n\}$ define cone$(F, 0)$ to be $\{S \cup \{0\} : S \in F\}$.

Recall the definition of *reverse lexicographic order* on $[n]^{(d)}$ (the set of all subsets of $\{1, 2, \dots, n\}$ of cardinality d) by specifying that $F <_{RL} G$ if there is some k such that $k \in G \setminus F$, and $i \in F \Leftrightarrow i \in G$ for all $i > k$.

Consider some element $j \in F \subseteq [n] = \{1, 2, \dots, n\}$. Define $\ell(F, j) = \max\{i \in \mathbf{Z} : i < j, i \notin F\}$ and $L(F, j) = (F \backslash \{j\}) \cup \{\ell(F, j)\}$. Thus $L(F, j)$ is the set resulting from "pushing" j and its immediate predecessors in F one step to the left. Define $r(F, j) = \min\{i \in \mathbf{Z} : i > j, i \notin F\}$ and $R(F, j) = (F \backslash \{j\}) \cup \{r(F, j)\}$. So $R(F, j)$ is the set resulting from "pushing" j and its immediate successors in F one step to the right. Let d be a positive odd integer, and let n be a positive integer. Take $F_d(n)$ to be the collection of all members of $[n]^{(d+1)}$ having only even cardinality left, middle, and right sets. So sets in $F_d(n)$, consist of unions of $(d+1)/2$, disjoint pairs of adjacent elements. For $F \in F_d(n)$, when we write $F = \{a_1, a_2, \dots, a_{d+1}\}$ we implicitly assume that the elements have been indexed so that $a_1 < \cdots < a_{d+1}$. For an even positive integer d, put $F_d(n) = $ cone$(F_{d-1}(n), 0)$. For positive d consider a nonempty subcollection I of $F_d(n)$ that is an *initial set* of $F_d(n)$ with respect to the partial order $\leq_p$. Equivalently, for $F \in I$, $L(F, j) \in I$ whenever $L(F, j) \in F_d(n)$.

The simplicial d-complex $B(I) = \{G : G \subseteq F \text{ for some } F \in I\}$ with facets in I is a d-ball. These simplicial complexes are called *squeezed balls*. Kalai observed that squeezed d-balls $B(I)$ for even d are just those simplicial complexes of the form cone$(B(J), 0)$, where J is an initial set of $F_{d-1}(n)$. The boundary of a squeezed d-ball $B(I)$, $S(I) = \partial B(I)$, is topologically a $(d-1)$-sphere and is called a *squeezed sphere*. The facets of $S(I)$ are those subsets of $B(I)$ of cardinality d that are contained in exactly one facet of $B(I)$. Squeezed balls are shellable and the reverse lexicographic ordering of the facets is a shelling order [**5**].

By examining this shelling, one can readily characterize the facets of $S(I)$ [**9**]. Suppose d is odd and $B(I)$ is a squeezed d-ball. Let

$F = \{a_1, \dots, a_{d+1}\} \in I$ be a facet of $B(I)$ and $a_i \in F$. Then $F - a_i$ is a facet of $S(I)$ if and only if (1) i is even and a_i is in the left set of F, or (2) i is odd and $R(F, a_i) \notin I$. Suppose d is even and $B(I) = \operatorname{cone}(B(J), 0)$ is a squeezed d-ball. Then a subset $F \in B(I)$ of cardinality d is a facet of $S(I)$ if and only if (1) F is a facet of $B(J)$, or (2) $F = G \cup \{0\}$ where G is a facet of $S(J)$.

3.1. Shelling order #1. Assume $B(I)$ is a squeezed d-ball for odd d. Let F, $G \in I$, where $F = \{a_1, \dots, a_{d+1}\}$ and $G = \{b_1, \dots, b_{d+1}\}$. Suppose $F - a_k$ and $G - b_\ell$ are facets of $S(I)$. Set $r = \min\{j : a_i = b_i \text{ for all } i \geq j\}$ (taking $r = d+2$ if $a_{d+1} \neq b_{d+1}$). Define $(G - b_\ell) <_{s_1} (F - a_k)$ if one of the following conditions holds.

1. k is odd and ℓ is even.
2. k and ℓ are both even and $G <_{RL} F$.
3. k and ℓ are both even, $G = F$, and $\ell > k$.
4. k and ℓ are both odd, $k = \ell \geq r$, and $G <_{RL} F$.
5. k and ℓ are both odd and $\ell > k \geq r$.
6. k and ℓ are both odd and $\ell \geq r > k$.
7. k and ℓ are both odd, k, $\ell < r$, and $F <_{RL} G$.

Theorem 3 (Lee [**9**]). *If d is odd and $S(I)$ is a squeezed $(d-1)$-sphere, then $<_{s_1}$ is a shelling order of its facets.*

For $F = \{a_1, \dots, a_{d+1}\}$ and sphere facet $F - a_k = F'$, F' is in the *first half* of the facets of $S(I)$ if k is even and is in the *second half* of the facets of $S(I)$ if k is odd. Note that there is a distinct separation between the first and second half of the facets of $S(I)$ using shelling order #1. As a corollary, squeezed $(d-1)$-spheres for d even are also shellable [**9**].

The *dependence relation on facets* of a squeezed sphere with respect to a shelling order is the partial order that is the transitive closure of the following: H_j *depends* on H_i if H_i and H_j share a subfacet and H_j comes later than H_i in the shelling order. We will write $\sigma(H_j) = i$ if the facet H_j contributes to the increase of h_i when H_j is shelled, i.e., when H_j is added to the simplicial complex induced by $H_1, \dots, H_{j-1}$.

So $\sigma(H_j) = \operatorname{card} R(H_j)$. We will consider *strong partial shellings* of $S(I)$ that preserve the values of $\sigma(H_j)$ and the minimal new faces $R(H_j)$. These are equivalent initial segments with respect to the above dependence relation (i.e., closed downward under this partial order).

There are h-vectors that satisfy Condition $(\star)$ that cannot be obtained using strong partial shellings of shelling order #1 [**11**]. For instance, if we assume $h_1 \leq 3$, we can obtain all h-vectors satisfying Condition $(\star)$ except the 20 listed in Figure 1.

(1,2,1,1,0,0)	(1,2,1,1,1,0)	(1,2,2,1,1,0)
(1,2,3,3,0,0)	(1,3,1,1,0,0)	(1,3,1,1,1,0)
(1,3,2,1,0,0)	(1,3,2,1,1,0)	(1,3,2,2,0,0)
(1,3,2,2,1,0)	(1,3,2,2,2,0)	(1,3,3,1,1,0)
(1,3,3,2,2,0)	(1,3,4,1,1,0)	(1,3,4,4,0,0)
(1,3,5,2,1,0)	(1,3,5,5,0,0)	(1,3,6,4,0,0)
(1,3,6,5,0,0)	(1,3,6,6,0,0)	

FIGURE 1. Unobtainable from shelling order #1.

For example, $h = (1, 2, 1, 1, 0, 0)$ satisfies Condition $(\star)$ but cannot be achieved using shelling order #1. We would need to choose the first facet of a suitable squeezed sphere for h_0, the second and fourth facets for h_1, and the third facet for h_2. We cannot choose the fifth facet since that would give us two facets contributing to h_2 because of its dependence relations. Since we do not choose the fifth facet then all other facets depending on it are eliminated, which includes all other facets in the sphere. So it becomes impossible to complete the partial shelling of the ball with h-vector $(1, 2, 1, 1, 0, 0)$. This h-vector cannot be achieved by a strong partial shelling of shelling order #1 for any squeezed sphere.

3.2. Shelling order #2. We now define a second ordering of the facets of squeezed spheres (shelling order #2) and prove it is a shelling order. It turns out that more h-vectors satisfying Condition $(\star)$ (but not all) can be achieved with strong partial shellings of shelling order #2.

To obtain shelling order #2, we rearrange shelling order #1. We take the second half of shelling order #1 and make it the first half of shelling order #2, then take the first half of shelling order #1 and reverse it and make it the second half of shelling order #2.

Define $(G - b_\ell) <_{s_2} (F - a_k)$ if one of the following conditions holds.

1. k is even and ℓ is odd.
2. k and ℓ are both even and $F <_{RL} G$.
3. k and ℓ are both even and $G = F$, and $\ell < k$.
4. k and ℓ are both odd, $k = \ell \geq r$, and $G <_{RL} F$.
5. k and ℓ are both odd and $\ell > k \geq r$.
6. k and ℓ are both odd and $\ell \geq r > k$.
7. k and ℓ are both odd, k, $\ell < r$, and $F <_{RL} G$.

Theorem 4 (Shelling order #2). *If d is odd and $S(I)$ is a squeezed $(d-1)$-sphere, then $<_{s_2}$ is a shelling order of its facets.*

Proof. Let $H_1, \ldots, H_r, H_{r+1}, \ldots, H_s$ be shelling order #1, where $H_1, \ldots, H_r$ is the first half and $H_{r+1}, \ldots, H_s$ is the second half. We want to show that $H_{r+1}, \ldots, H_s, H_r, \ldots, H_1$ is also a shelling order. Shellings of spheres are reversible; therefore, we know $H_s, \ldots, H_{r+1}$, $H_r, \ldots, H_1$ is a shelling order. In particular, once $H_s, \ldots, H_{r+1}$ have been shelled, we know $H_r, \ldots, H_1$ can be shelled onto $H_s \cup \cdots \cup H_{r+1}$. Therefore, once we know $H_{r+1}, \ldots, H_s$ is the beginning of a shelling of $S(I)$, we know it can be completed by shelling $H_r, \ldots, H_1$. So it remains to prove that $H_{r+1}, \ldots, H_s$ is a (beginning of a) shelling of $S(I)$.

This proof mimics the proof in [**9**]. It suffices to show that for each facet of $S(I)$ there is a subset X with the properties (a) for each $a_i \in X$ there is a preceding facet (under $<_{s_2}$) of $S(I)$ containing $X - a_i$ and (b) X by itself is not contained in any preceding facet.

Assume that $F = \{a_1, \ldots, a_{d+1}\} \in I$ and $F - a_k$ is a facet of $S(I)$.

It will be sufficient to prove such an X exists when k is odd (since k even refers to facets $H_r, \ldots, H_1$ which we know will complete the shelling). Note that $F - a_k$ has exactly one odd middle or right set (the elements of F immediately following a_k). Let $X = X_1 \cup X_2$ where $X_1 = \{a_i : i < k$ and i is even and a_i is not in its leftmost position$\}$, and $X_2 = \{a_i : i > k$ and i is odd$\}$. Choose any $a_i \in X$. In each of the following cases we will find a facet H of $S(I)$ that precedes $F - a_k$ and contains $(F - a_k) - a_i$.

Case 1. Assume $a_i \in X_1$. Note that this means that a_i is not in its leftmost set. Let $F_1 = L(F, a_i)$, $a = \ell(F, a_i)$, $F_2 = R(F_1, a_k)$, and $F_3 = R(F, a_k)$. Note that $F_3 \notin I$ since $F - a_k$ is a facet of $S(I)$. If $F_2 \notin I$, then take $H = F_1 - a_k$, which is a preceding facet by condition 4. If, on the other hand, $F_2 \in I$, then $R(F_2, a) = F_3 \notin I$, so $H = F_2 - a$ is a preceding facet by condition 7.

Case 2. Assume $a_i \in X_2$. Let $F_1 = R(F, a_i)$, $F_2 = R(F, a_k)$, and $F_3 = R(F_1, a_k)$. Note that $F_2 \notin I$ since $F - a_k$ is a facet of $S(I)$. Hence, F_3 is also not in I, since $F_2 \leq_p F_3$. If $F_1 \notin I$, then take $H = F - a_i$, which is a preceding facet by condition 5. If, on the other hand, $F_1 \in I$, then take $H = F_1 - a_k$, which is a preceding facet by condition 7.

So at this point we have verified that X satisfies (a). Now assume $G - b_\ell$ is a preceding facet of $S(I)$ that contains X. Then $(G - b_\ell) <_{s_2} (F - a_k)$ via conditions 4, 5, 6, or 7, since k is odd. The removal of X from G creates $|X|$ odd sets, one for each element in a pair that is removed. The positioning of the elements of $G \setminus X$ is severely restricted since G has to be in $F_d(n)$. So when you add the elements of $G \setminus X$ to X you must make sure that your result has even cardinality left, middle, and right sets.

Condition 4 requires that G must agree with F from a_k onward since $k = \ell \geq r$. If you try to add elements to complete your odd sets but in a different way than $F - a_k$ you end up with $G >_{RL} F$, a contradiction since $G <_{RL} F$. So completing G, other than becoming F, is impossible if G is to contain X.

Condition 5 also requires that G must agree with F from a_k onward since $\ell > k \geq r$. However, then ℓ is odd and $\ell > k$ implies that $b_\ell \in X_2$, and therefore is in X. This is a contradiction, since in the beginning we assume that $G - b_\ell$ is a preceding facet that contains X, and hence b_ℓ cannot be in X.

Condition 6 requires G to disagree with F at some point on or after a_k but to agree with F from b_ℓ onward since $\ell \geq r > k$. However, then we once again have ℓ is odd and $\ell > k$ and therefore $b_\ell \in X$, which is again a contradiction.

Condition 7 requires $F <_{RL} G$ and that G disagrees with F on or after a_k since $k, \ell < r$. Let $F' = R(F, a_k)$, since $F - a_k$ is a facet of $S(I)$ then $F' \notin B(I)$. This implies that all facets lexicographically greater than F' are also not in $B(I)$. Therefore $G <_{RL} F'$ since we want G to be a facet of $B(I)$. So we need to find a G such that $F <_{RL} G <_{RL} F'$. We know that G contains X, if we then add elements to G to make $G <_{RL} F'$ then it must agree with F from a_k onward, which is a contradiction to our condition. Therefore X satisfies (b), completing the proof. □

Although shelling order #2 is an improvement on shelling order #1, strong partial shellings achieving some of the h-vectors missed by shelling order #1, one still cannot use strong partial shellings to obtain every potential h-vector that satisfies Condition ($\star$) [**11**]. In higher dimensions, when $h_3 > h_2$ the h-vector cannot be constructed using strong partial shellings of shelling order #2. For example, the h-vector $(1, 3, 3, 4, 0, 0, 0, 0)$ satisfies Condition ($\star$) but is not obtainable using shelling order #2.

3.3. Shelling orders #3 and #4. Although shelling order #2 is an improvement, there are still potential h-vectors satisfying Condition ($\star$) that cannot be obtained from strong partial shellings. We now look at the reverse of the first two shelling orders and see if we can find an improvement on shelling order #2. Shelling order #3 is the reverse of shelling order #2. This is a shelling order since reversing a shelling order of a sphere produces another shelling order [**8**].

When we reverse shelling order #2 and create shelling order #3, we are able to construct some h-vectors for which $h_3 > h_2$, such as $(1, 3, 3, 4, 0, 0, 0, 0)$. However, this shelling order creates problems with higher indices h_i and we can no longer achieve h-vectors such as $(1, 2, 3, 2, 2, 2, 1, 0)$ which shelling order #2 achieves [**11**]. We hazard a conjecture:

Conjecture 2. *Every potential h-vector that satisfies Condition ($\star$) can be achieved by a strong partial shelling using either shelling order #2 or shelling order #3.*

Shelling order #4 is the reverse of shelling order #1. Similar to shelling order #1 it fails to achieve several h-vectors that satisfy Condition ($\star$), such as $(1, 2, 1, 1, 0, 0)$ [**11**].

3.4. Shelling orders when d is even. Recall that squeezed d-balls $B(I)$ for even d are just those simplicial complexes of the form cone $(B(J), 0)$, where J is an initial set of $F_{d-1}(n)$ and a subset $F \in B(I)$ of cardinality d is a facet of $S(I)$ if and only if (1) F is a facet of $B(J)$, or (2) $F = G \cup \{0\}$ where G is a facet of $S(J)$.

So we have to shell facets of the ball, $B(J)$, and the facets of the boundary of the ball, $S(J)$. Now the ball has a reverse lexicographic shelling and we know from previous sections that the boundary of the ball (which is a sphere) has four different shelling orders for odd d. We know that we can shell the ball first and then the cone of the boundary next as in [**8**]. Here are some possible shelling orders:

1. Order A: shell the ball and then use shelling order #1 on boundary.
2. Order B: shell the ball and then use shelling order #2 on boundary.
3. Order C: shell the ball and then use shelling order #3 on boundary.
4. Order D: shell the ball and then use shelling order #4 on boundary.

We also have the reverses of these.

In [**11**] the various orders are examined to see which strong partial orders appeared to achieve most of the potential h-vectors that satisfied Condition $(\star)$. Just as shelling orders #2 and #3 were better in the odd case, there were similar results in the even case–shelling orders C and its reverse appeared to be more useful. Shelling order B ran a close second.

4. Dimension three. In this section we construct a regular 3-triangulation Δ such that $h(\Delta) = h$ for a given vector $h = (1, a, b, c, 0)$ satisfying Condition $(\star)$ by first constructing a simplicial 4-polytope P. We then find a line shelling of P such that an initial segment of the line shelling is a certain set of facets $\mathcal{B}$. We then let $\mathcal{B}'$ be the facets of P not in $\mathcal{B}$. These facets in $\mathcal{B}'$ will comprise a patch with the desired h-vector. This is then equivalent to finding the required regular 3-triangulation.

Theorem 5. *Assume $h = (1, a, b, c, 0)$ is a vector of nonnegative integers, such that h satisfies Condition $(\star)$. Then h is the h-vector of some regular 3-triangulation.*

First observe that if $b \leq a$ then we can apply Theorem 2 since Condition $(\star)$ implies $c \leq b$ and hence h is non-increasing after h_1. Hence, from now on we assume that $b > a$. We now proceed to prove Theorem 5 by a sequence of results.

Lemma 2. *Let $g_0 = 1$, $g_1 = a - 1$ and $g_2 = b - a$. Then $g = (g_0, g_1, g_2)$ is an M-vector.*

Proof. Since $a > 0$ and $b > a$, our values for the g-vector are nonnegative. We only need to check that $g_2 \leq g_1^{\langle 1 \rangle}$. By definition $a^{\langle 1 \rangle} = (1/2)(a^2 + a)$, thus $b \leq (1/2)(a^2 + a)$ since h satisfies Condition $(\star)$. By subtracting a from both sides, we get $g_2 = b - a \leq 1/2(a^2 - a) = g_1^{\langle 1 \rangle}$. Hence $g_2 \leq g_1^{\langle 1 \rangle}$. Therefore $g = (g_0, g_1, g_2)$ is an M-vector.

Lemma 3. *There is a squeezed 4-ball $B(J)$ with h-vector $(g_0, g_1, g_2, 0, 0, 0)$.*

Proof. Let $n = a + 3$. The vertices are $0, 1, \dots, n$. Similar to [**1**], we list those facets that contain 0 and have even cardinality right end set. Except for vertex 0, the vertices of the facets fall naturally into pairs. For $i = 0, 1, 2$, let V^i be the set of facets such that exactly i of these pairs are not in their "leftmost position." For example, V^1 are the facets that contain $\{0, 1, 2\}$ but not 3 since the pair $\{3, 4\}$ is not in its "leftmost position." For all i choose the first g_i sets in V^i, using reverse lexicographic order. These are the facets of a squeezed ball $B(J)$ and they are shellable in reverse lexicographic order. If a chosen F is in V^i then it contributes to $h_i(B(J))$ during the shelling, hence $h(B(J)) = (g_0, g_1, g_2, 0, 0, 0)$. □

In the following we will use the notation $ijk\ell$ to denote the set $\{i, j, k, \ell\}$.

Lemma 4. *The boundary of $B(J)$ is isomorphic to the boundary of a 4-dimensional g-theorem polytope P with $h(P) = (1, a, b, a, 1)$, and includes all the facets in $\mathcal{C} = \{0123, 0134, \dots, 01(n-1)n\}$ where $n = a + 3$.*

Proof. Billera and Lee proved that the boundary of $B(J)$ is isomorphic to the boundary of a 4-dimensional g-theorem polytope P with h-vector $(1, a, b, a, 1)$ [**1**]. For a positive integer $k > 1$, we want to show that the subfacet $01k(k+1)$ is contained in precisely one facet of $B(J)$, and therefore in the boundary of $B(J)$. When $k=2$ we have the subfacet 0123. Since $g_0 = 1$, there is only one facet of $B(J)$ containing 0123, namely, 01234. When $k > 2$ we have the subfacet $01k(k+1)$. Let G be a facet of $B(J)$ containing $01k(k+1)$. Then there is a fifth element ℓ of G to be added to $01k(k+1)$. Since $k > 2$ and 1 must be paired with some vertex in G, we must have $\ell = 2$. Therefore $012k(k+1)$ is the unique facet of $B(J)$ containing $01k(k+1)$. Hence the boundary of $B(J)$ is isomorphic to the boundary of a 4-dimensional g-theorem polytope P, and includes all the facets in $\mathcal{C} = \{0123, 0134, \ldots, 01(n-1)n\}$. $\square$

Lemma 5. *The facets of $\mathcal{C}$ form a shellable ball with h-vector $h(\mathcal{C}) = (1, a, 0, 0, 0)$, with the facets of $\mathcal{C}$ ordered as in Lemma 4.*

Proof. Since $\mathcal{C}$ contains $0123, \ldots, 01(n-1)n$, and $n+1$ is the number of vertices, then we have 1 contribution to h_0 during the shelling, namely by 0123, and then the rest of the facets contributing to h_1. Thus the vertices apart from the first four result in $(n+1) - 4 = n - 3$ contributions to h_1. Since $a = n - 3$, we have the h-vector of $\mathcal{C}$ equal to $(1, a, 0, 0, 0)$. $\square$

Lemma 6. *There exists a point z in $\mathbf{R}^4$ beyond precisely the facets of P that are in $\mathcal{C}$.*

Proof. For a face F of P let $\mathcal{F}_F$ be the set of facets of P containing F. Consider the chain of faces of P: $01 \subset 012 \subset 0123$. Let $\mathcal{C}' = \mathcal{F}_{01} \setminus (\mathcal{F}_{012} \setminus \mathcal{F}_{0123})$. (Note $\mathcal{F}_{01} \supset \mathcal{F}_{012} \supset \mathcal{F}_{0123}$.)

We will show that $\mathcal{C} = \mathcal{C}'$. Let $F = 01k(k+1) \in \mathcal{C}$. If $k = 2$, then $F = 0123$ and $F \in \mathcal{C}'$ since $F \in \mathcal{F}_{01}$, $\mathcal{F}_{012}$ and $\mathcal{F}_{0123}$. If $k > 2$, then $F \in \mathcal{F}_{01}$ but $F \notin \mathcal{F}_{012}$, hence $F \in \mathcal{C}'$. So $\mathcal{C} \subseteq \mathcal{C}'$.

Let $F \in \mathcal{C}'$. If $F = 0123$, then $F \in \mathcal{C}$. Otherwise F contains 01 and is of the form $01k\ell$ with $k < \ell$. If the vertices k and ℓ are consecutive, then F is in $\mathcal{C}$. Therefore assume the remaining two elements are not consecutive elements; thus, $2 \le k < \ell \le n$, and $k \ne \ell - 1$. Recall that the facets of the squeezed 3-ball $B(J)$ contain the vertex 0 and two pairs of consecutive vertices. But $F \in \mathcal{C}'$ and $F \ne 0123$ implies F does not contain 012. Thus $k > 2$. But then there is no way to insert a fifth vertex in $01k\ell$ to create two pairs of consecutive vertices containing 1, k and ℓ. So $\mathcal{C}' \subseteq \mathcal{C}$, and therefore $\mathcal{C} = \mathcal{C}'$. The existence of z now follows by the sewing construction of Shemer [**12**]. □

If $h = (h_0, \ldots, h_d)$, let $h^r = (h_d, \ldots, h_0)$, the reverse of the h-vector. We can now prove our main theorem.

Proof of Theorem 5. We first construct P and $\mathcal{C}$ as in the previous lemmas. Then there is a line shelling of P in which the facets of $\mathcal{C}$ appear first. So as the facets of $\mathcal{C}$ are shelled we obtain a sequence of patches $\mathcal{B}_0, \ldots, \mathcal{B}_a$ with respective h-vectors $(1, 0, 0, 0, 0)$, $(1, 1, 0, 0, 0), \ldots, (1, a, 0, 0, 0)$. In particular, $\mathcal{B}_{a-c}$ is a patch, with h-vector $(1, a-c, 0, 0, 0)$. Then the facets of P not in $\mathcal{B}_{a-c}$ also form a patch $\mathcal{B}'$, and by Lemma 1 it has h-vector $h(P) - h^r(\mathcal{B}_{a-c}) = (1, a, b, a, 1) - (0, 0, 0, a-c, 1) = (1, a, b, c, 0)$. Therefore there is a regular three-dimensional triangulation Δ such that $h(\Delta) = (1, a, b, c, 0)$. □

5. Dimension four. In this section we consider nonnegative vectors $h = (1, a, b, c, d, 0)$ satisfying Condition $(\star)$. We begin by constructing a 5-polytope P. We then find a shelling of P such that a certain set of facets $\mathcal{Q}$ is a strong partial shelling of the facets of P. We let $\mathcal{Q}'$ be the facets of P not in Q, and show that the simplicial complex determined by these facets in $\mathcal{Q}'$ is a shellable 4-ball with the desired h-vector. In the two cases $d = 0$ and $d = a$ we are able to show that the set of facets $\mathcal{Q}'$ is a patch and therefore equivalent to a regular 4-triangulation with the desired h-vector.

5.1. Construction of shellable balls.

Theorem 6. *Assume $h = (1, a, b, c, d, 0)$ is a vector of nonnegative integers such that $b > a$ and h satisfies Condition $(\star)$. Then h is the h-vector of some shellable 4-ball.*

Note. If $b \leq a$ then we can apply Theorem 2 since h is non-increasing after h_1 and Condition $(\star)$ implies $d \leq c \leq b$.

Let $h' = (1, a, b, b, a, 1)$ and $h'' = (1, a-d, b-c, 0, 0, 0)$. Observe that h'' is an M-vector since h satisfies Condition $(\star)$. For example, if $h = (1, 5, 7, 2, 1, 0)$ then $h' = (1, 5, 7, 7, 5, 1)$ and $h'' = (1, 4, 5, 0, 0, 0)$.

Lemma 7. *Let $g_0 = 1$, $g_1 = a - 1$, and $g_2 = b - a$. Then $g = (g_0, g_1, g_2)$ is an M-vector.*

The proof is similar to that of Lemma 2. In our example, $(1, 4, 2)$ is an M-vector.

Lemma 8. *There exists a squeezed 5-ball $B(J)$ with h-vector $(g_0, g_1, g_2, 0, 0, 0, 0)$.*

Proof. Let $n = a + 5$. The vertices are $1, \ldots, n$. Similar to [**1**] we list those facets that contain 12 and have even cardinality right end set. Let V^i be the set of facets such that exactly i of these pairs are not in their "leftmost position." For all i choose the first g_i sets in V^i, using reverse lexicographic order. These are the facets of a squeezed ball $B(J)$ and they are shellable in reverse lexicographic order. If a chosen F is in V^i, then it contributes to $h_i(B(J))$ during the shelling, hence $h(B(J)) = (g_0, g_1, g_2, 0, 0, 0, 0)$. □

Carrying out this process for $h = (1, 5, 7, 2, 1, 0)$, we get the squeezed 5-ball with h-vector $(1, 4, 2, 0, 0, 0, 0)$ and the following facets. The superscripts indicate which vertices should be deleted and in which order to obtain the facets of the boundary of the ball ordered by shelling order #1.

1	2	3	4	5	6	7	8	9	10
1^{26}	2^{3}	3	4^{2}	5	6^{1}				
1^{25}	2^{5}	3	4^{4}		6	7			
1^{24}	2^{6}		4^{23}	5	6	7			
1^{22}	2^{8}	3	4^{7}			7	8		
1^{21}	2^{9}		4^{20}	5		7^{19}	8		
1^{18}	2^{11}	3^{17}	4^{10}				8	9	
1^{16}	2^{13}	3^{15}	4^{12}					9^{14}	10

Lemma 9. *There exists a five-dimensional g-theorem polytope P with h-vector h' whose boundary is isomorphic to the boundary of the ball $B(J)$.*

This follows from the arguments in Billera and Lee [**1**]. In our example, the facets of P with h-vector $(1, 5, 7, 7, 5, 1)$ are given below, listed in shelling order #1. For each facet F, the marked face is $R(F)$.

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5					
2	1	2	3		5	6*				
3	1		3	4*	5	6*				
4	1	2	3			6	7*			
5	1		3	4*		6	7*			
6	1			4	5*	6	7*			
7	1	2	3				7	8*		
8	1		3	4*			7	8*		
9	1			4	5*		7	8*		
10	1	2	3					8	9*	
11	1		3	4*				8	9*	
12	1	2	3						9	10*
13	1		3	4*					9	10*
14	1	2*	3	4*						10*
15	1	2*		4*					9*	10
16		2*	3*	4*					9*	10
17	1	2*		4*				8*	9	
18		2*	3*	4*				8*	9	
19	1	2*		4	5*			8*		
20	1	2*			5*		7*	8		
21		2*		4*	5		7*	8		
22		2*	3*	4*			7*	8		
23	1	2*			5*	6*	7*			
24		2*		4*	5	6*	7			
25		2*	3*	4*		6*	7			
26		2*	3*	4*	5*	6*				

Lemma 10. *There exists a shellable collection $\mathcal{Q}$ of facets of P such that $h(\mathcal{Q}) = h''$ and $\mathcal{Q}$ is the beginning of a shelling of P.*

Proof. Note that $h''_1 \leq h'_1$ and $h''_2 \leq h'_2$. Recall from Section 3 that there are first and second halves of shelling order #1 for P. Each facet F in the first half has $\sigma(F) = 0$, 1, or 2; and each facet F in the second half has $\sigma(F) = 3$, 4, or 5. We shell the polytope P as follows: We first choose the facet 12345, we then choose the first $h''_1 = a - d$ facets of the form $123k(k+1)$ for which $k > 4$, and finally we choose the first $h''_2 = b - c$ facets of the form $1k(k+1)\ell(\ell+1)$ for which $k > 2$, $\ell > k+1$. All of the chosen facets are in the first half of the shelling order. In our example, we would choose the facets 1 through 10.

We claim this results in the collection of facets $\mathcal{Q}$ with h-vector $(1, a-d, b-c, 0, 0, 0)$. Consider the facets in the first half of the shelling of P. Say a facet F is of type i if $\sigma(F) = i$, and thus F contributes to h_i in the shelling of P. Observe that, from the way the facets of P are derived from the facets of $B(J)$, each type 2 facet of P of the form $134p(p+1)$ is paired with and dependent (in the sense of the shelling order) upon the type 1 facet of P of the form $123p(p+1)$. The remaining type 2 facets depend only on earlier type 2 facets. So when choosing facets of $\mathcal{Q}$, we must check that we choose all type 1 facets upon which any chosen type 2 facet depends. But this holds because from the first half of the shelling we are removing d facets with $\sigma(F) = 1$ and c facets with $\sigma(F) = 2$, and Condition $(\star)$ implies $d \leq c$. Thus we never remove more type 2 facets than type 1 facets.

Therefore selecting facet 12345, the first $a - d$ facets of P of type 1, and the first $b - c$ facets of P of type 2, results in a strong partial shelling with h-vector $(1, a-d, b-c, 0, 0, 0)$.

Now we have to complete the shelling to P. So we will then choose the remaining facets of the form $123k(k+1)$ when $k > 4$. Then we choose the remaining facets of the form $1k(k+1)\ell(\ell+1)$ when $k > 2$, $\ell > k+1$. This completes the first half of the shelling order. We can then shell the second half using shelling order #1. □

Proof of Theorem 6. Assume $h = (1, a, b, c, d, 0)$ is a vector of nonnegative integers such that $b \geq a$ and h satisfies Condition $(\star)$. Construct the polytope P with h-vector $h' = (1, a, b, b, a, 1)$. Find the collection of facets $\mathcal{Q}$ as above with h-vector h'', that is, the beginning of a shelling order of P. Let the set of facets $\mathcal{Q}'$ be the facets of P not in $\mathcal{Q}$. The set $\mathcal{Q}'$ is shellable (reverse the shelling of P) and has h-vector $h(P) - h^r(\mathcal{Q}) = (1, a, b, b, a, 1) - (0, 0, 0, b - c, a - d, 1) = (1, a, b, c, d, 0)$. This set of facets $\mathcal{Q}'$ is our desired shellable ball. $\square$

5.2. Realization as patches. We now know that any h-vector $(1, a, b, c, d, 0)$ with $b > a$ satisfying Condition $(\star)$ is combinatorially realizable as a shellable ball. In this section we prove the realizability of the shellable ball as a patch in cases $d = 0$ and $d = a$.

Recall if $b \leq a$ then we can apply the proof from Theorem 2 since h is non-increasing after h_1 and since Condition $(\star)$ implies $d \leq c \leq b$. Therefore, assume that $b > a$. Note Condition $(\star)$ also implies that $a \geq d$, since the vector $(1, a - d, b - c)$ is an M-vector.

For both cases, we need the patches that result in the h-vectors $(1, a, b, 0, 0, 0)$ and $(1, a, 0, 0, 0, 0)$; we call these, respectively, $\mathcal{P}_1$ and $\mathcal{P}_2$.

Let P be the g-theorem polytope with h-vector $(1, a, b, b, a, 1)$. (We know $(1, a, b, b, a, 1)$ satisfies the g-theorem conditions from Lemma 2 and therefore P is constructible.)

Let $\mathcal{P}_1$ be the set of facets of P with an odd left set. Recall that, for a face F of P, $\mathcal{F}_F$ is the set of facets containing F. Let $\mathcal{C}_1 = \mathcal{F}_1 \setminus (\mathcal{F}_{12} \setminus (\mathcal{F}_{123} \setminus (\mathcal{F}_{1234} \setminus \mathcal{F}_{12345}))) = (\mathcal{F}_1 \setminus \mathcal{F}_{12}) \cup (\mathcal{F}_{123} \setminus \mathcal{F}_{1234}) \cup \mathcal{F}_{12345}$. (Note $1 \subset 12 \subset 123 \subset 1234 \subset 12345$ so $\mathcal{F}_1 \supset \mathcal{F}_{12} \supset \mathcal{F}_{123} \supset \mathcal{F}_{1234} \supset \mathcal{F}_{12345}$.)

Lemma 11. *The set of facets in $\mathcal{P}_1$ equals the set of facets in $\mathcal{C}_1$.*

Proof. Recall each facet of P contains 5 vertices. We need to show two things: first that any facet with even (possibly empty) left set is not in $\mathcal{C}_1$, and second that every facet with an odd left set is in $\mathcal{C}_1$.

Case 1. Assume F is a facet of P with empty left set. Since F has no left set then F does not contain the vertex 1, so $F \notin \mathcal{F}_1$, and therefore $F \notin \mathcal{C}_1$. Now assume F is a facet with an even left set, so $F \in \mathcal{F}_{12}$ or $F \in \mathcal{F}_{1234}$. If $F \in \mathcal{F}_{12}$, but $F \notin \mathcal{F}_{123}$, then $F \notin \mathcal{C}_1$. If $F \in \mathcal{F}_{1234}$, but $F \notin \mathcal{F}_{12345}$, then $F \notin \mathcal{C}_1$. So we have shown that $\mathcal{C}_1 \subseteq \mathcal{P}_1$.

Case 2. Now assume G is a facet of P with an odd left set. We need to show that G is in $\mathcal{C}_1$. Since each facet has only five vertices there are only three choices for odd left sets, namely, 1, 123, and 12345. If G has odd left set 1, then $G \in \mathcal{F}_1 \setminus \mathcal{F}_{12}$ and therefore $G \in \mathcal{C}_1$. If G has odd left set 123, then $G \in \mathcal{F}_{123} \setminus \mathcal{F}_{1234}$ and therefore $G \in \mathcal{C}_1$. If G has odd left set 12345, then $G \in \mathcal{F}_{12345}$ and hence $G \in \mathcal{C}_1$. So we have now shown that $\mathcal{P}_1 \subseteq \mathcal{C}_1$. Therefore $\mathcal{P}_1 = \mathcal{C}_1$. □

Lemma 12. *The set of facets in $\mathcal{P}_1$ is shellable in the induced order and results in the h-vector $(1, a, b, 0, 0, 0)$.*

Proof. Since $\mathcal{P}_1$ is a patch this result follows directly from [**9**]. □

Let $\mathcal{P}_2$ be the set of facets of P with an odd left set of cardinality greater than 1. Let $\mathcal{C}_2 = \mathcal{F}_{123} \setminus (\mathcal{F}_{1234} \setminus \mathcal{F}_{12345}) = (\mathcal{F}_{123} \setminus \mathcal{F}_{1234}) \cup \mathcal{F}_{12345}$. (Note $123 \subset 1234 \subset 12345$ so $\mathcal{F}_{123} \supset \mathcal{F}_{1234} \supset \mathcal{F}_{12345}$.)

Lemma 13. *The set of facets in $\mathcal{P}_2$ equals the set of facets in $\mathcal{C}_2$.*

Proof. This proof is similar to the proof of Lemma 11 except the cases when the left set is 1 are excluded. □

Lemma 14. *The set of facets in $\mathcal{P}_2$ results in the h-vector $(1, a, 0, 0, 0, 0)$, where $a = n - 5$.*

Proof. It is easy to see that there are $a+1$ facets in $\mathcal{P}_2$ and each facet F in $\mathcal{P}_2$ after the first has $\sigma(F) = 1$. □

The next two lemmas follow immediately from Shemer's sewing construction [**12**].

Lemma 15. *There exists a point x in $\mathbf{R}^5$ beyond precisely the facets of P that are in $\mathcal{C}_1$.*

Lemma 16. *There exists a point y in $\mathbf{R}^5$ beyond precisely the facets of P that are in $\mathcal{C}_2$.*

So we now have our desired patches and points to begin the details of each case.

Case 1: $d = 0$.

Theorem 7. *Assume $h = (1, a, b, c, 0, 0)$ is a vector of nonnegative integers, such that $b > a$ and h satisfies Condition $(\star)$. Then h is the h-vector of some regular 4-triangulation.*

Proof. We will begin by constructing the polytope P with h-vector $(1, a, b, b, a, 1)$ which we know satisfies the g-theorem conditions from Lemma 2. Recall that we have a set of facets $\mathcal{P}_1$ with h-vector $(1, a, b, 0, 0, 0)$ and a set of facets $\mathcal{P}_2$ with h-vector $(1, a, 0, 0, 0, 0)$ with points x and y as in the previous lemmas. Note that $\mathcal{P}_2 \subseteq \mathcal{P}_1$. There is a beginning of a line shelling of P, moving along a line from $\operatorname{int} P$ to y, in which the facets of $\mathcal{P}_2$ appear first. Then one can extend the shelling by moving from y to x, during which the remaining facets of $\mathcal{P}_1$ appear. So as the facets are shelled we obtain a sequence of patches $\mathcal{B}_0, \dots, \mathcal{B}_{a+b}$ with respective h-vectors $(1, 0, 0, 0, 0, 0)$, $(1, 1, 0, 0, 0, 0), \dots, (1, a, 0, 0, 0, 0)$, $(1, a, 1, 0, 0, 0), \dots,$ $(1, a, b, 0, 0, 0)$. In particular, $\mathcal{B}_{a+b-c}$ is a patch with h-vector $(1, a, b - c, 0, 0, 0)$. Then (using Lemma 1) the facets of P not in $\mathcal{B}_{a+b-c}$ also form a patch $\mathcal{B}'$ with h-vector $h(P) - h^r(\mathcal{B}_{a+b-c}) = (1, a, b, b, a, 1) - (0, 0, 0, b - c, a, 1) = (1, a, b, c, 0, 0)$. Therefore, there is a regular 4-triangulation Δ such that $h(\Delta) = (1, a, b, c, 0, 0)$.

Case 2: $d = a = n - 5$. This case is similar to Case 1 except $d = a$. This results in the h-vector $(1, a, b, c, a, 0)$. Since the h-vector satisfies Condition $(\star)$ this implies that $b = c$, and therefore the desired h-vector is $(1, a, b, b, a, 0)$.

Theorem 8. *Assume $h = (1, a, b, b, a, 0)$ is a vector of nonnegative integers, such that $b > a$ and h satisfies Condition $(\star)$. Then h is the h-vector for some regular 4-triangulation.*

Proof of Theorem 8. We construct the g-theorem polytope P with h-vector $(1, a, b, b, a, 1)$ and delete any one facet. The result is a patch B' with h-vector $h(P) - (0, 0, 0, 0, 0, 1) = (1, a, b, b, a, 1) - (0, 0, 0, 0, 0, 1) = (1, a, b, b, a, 0)$. Therefore there is a regular 4-triangulation Δ such that $h(\Delta) = h$. □

Given the confirmation in the above (admittedly still limited) cases, we may hope that Conjecture 1 holds in dimension four for the remaining cases. Of course, a verification of the sufficiency of Condition $(\star)$ in all dimensions would be welcome.

Note added in proof. Samuel Kolins has disproved Conjecture 1 in dimensions five and higher [**7**].

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