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POINTWISE CONVERGENCE OF QUASICONTINUOUS MAPPINGS AND BAIRE SPACES

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ABSTRACT. The notion of quasicontinuity was perhaps the first time used by Baire in [**1**]. Using the Choquet game for Baire spaces X we give a complete answer to the question when the pointwise limit of the sequence of real-valued quasicontinuous functions defined on X is also quasicontinuous. Moreover, in the class of metrizable spaces and in the class of quasi-regular T_1 topological spaces with locally countable π-base ([**27**]) we give a characterization of Baire spaces by the above mentioned fact.

1. Introduction. In what follows let X, Y be Hausdorff topological spaces and R the space of real numbers with the usual metric.

In the paper [**17**] Kempisty introduced a notion similar to continuity for real-valued functions defined in R. For general topological spaces this notion can be given the following equivalent formulation.

Definition 1.1. A function $f : X \to Y$ is called quasicontinuous at $x \in X$ if for every open set $V \subset Y, f(x) \in V$ and open set $U \subset X$, $x \in U$ there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

The notion of quasicontinuity was perhaps the first time used by Baire in [**1**] in the study of points of continuity of separately continuous functions. As Baire indicated in his paper [**1**] the condition of quasicontinuity has been suggested by Vito Volterra.

There is a rich literature concerning the study of quasicontinuity (see, for instance [**7, 18, 20, 26, 28**]).

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The notion of quasicontinuity recently turned out to be instrumental in the proof that some semitopological groups are actually topological ones (see [**5, 6**]), in the proof of some generalizations of Michael's selection theorem (see [**10**]) and also in characterizations of minimal usco maps and densely continuous forms via their selections (see [**13, 14, 22**]).

Quasicontinuity of real-valued separately continuous functions of two variables was studied very frequently in connection with the existence of points of joint continuity for such functions (see [**21, 25, 28–30**]).

Continuity points of quasicontinuous mappings were studied in many papers; see for example [**3, 15, 18, 19**].

2. Pointwise convergence of quasicontinuous mappings. Of course it is very easy to verify that the pointwise limit of a sequence of even continuous functions need not be quasicontinuous.

However it is known that the pointwise limit of an equicontinuous sequence of functions is continuous. Of course equicontinuity is too strong; it is not necessary to guarantee continuity of the pointwise limit of a sequence of continuous functions.

In [**2**] necessary and sufficient conditions for continuity of the pointwise limit of a net of continuous functions are given.

Bledsoe in his paper [**3**] studied the pointwise limit of a sequence of quasicontinuous mappings with values in a metric space. He proved that if $\{f_n : n \in \omega\}$ is a sequence of quasicontinuous mappings defined on a topological space X with values in a metric space Y pointwise convergent to $f : X \to Y$, then the set $D(f)$ of discontinuity points of f is of the first Baire category in X.

The same result was rediscovered later by Giles and Bartlett in their paper [**10**].

For simplicity we will work only with real-valued functions.

Definition 2.1. Let $\{f_n : n \in \omega\}$ be a sequence of real-valued functions defined on a topological space X. We say that the sequence $\{f_n : n \in \omega\}$ is equi-quasicontinuous at $x \in X$ if for every $\varepsilon > 0$ and every open neighborhood U of x there is an $n_0 \in \omega$ and a nonempty open set $W \subset U$ such that

$$|f_n(z) - f_n(x)| < \varepsilon$$

for every $z \in W$ and for every $n \geq n_0$.

We say that $\{f_n : n \in \omega\}$ is equi-quasicontinuous if it is equi-quasicontinuous at every $x \in X$.

Remark. Notice that in a natural way we can define the equi-quasicontinuity also for a net of functions.

Of course every equicontinuous sequence is also equi-quasicontinuous and there are easy examples of equi-quasicontinuous sequences which are not equicontinuous.

Proposition 2.2. *Let $\{f_n : n \in \omega\}$ be a sequence of real-valued functions defined on a topological space X pointwise convergent to a real-valued function f defined on X. If $\{f_n : n \in \omega\}$ is equi-quasicontinuous at $x \in X$, then f is quasicontinuous at x.*

Proof. Let $\varepsilon > 0$, and let U be an open set with $x \in U$. There is an $n_0 \in \omega$ and a nonempty open set $W \subset U$ such that $|f_n(x) - f_n(z)| < \varepsilon/3$ for every $n \geq n_0$ and every $z \in W$.

Let $w \in W$. The pointwise convergence of $\{f_n : n \in \omega\}$ to f implies that there is an $n_1 \geq n_0$ such that

$$|f_n(w) - f(w)| < \varepsilon/3 \text{ and } |f_n(x) - f(x)| < \varepsilon/3 \text{ for every } n \geq n_1.$$

Then we have:

$$\begin{aligned} |f(x) - f(w)| &\leq |f(x) - f_{n_1}(x)| \\ &\quad + |f_{n_1}(x) - f_{n_1}(w)| \\ &\quad + |f_{n_1}(w) - f(w)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3. \end{aligned}$$ □

A topological space is a Baire space, provided countable collections of dense open subsets have a dense intersection (equivalently, nonempty open subsets are of second Baire category).

We will use Oxtoby's characterization of Baire spaces. In [**16**] we can find the following theorem:

Definition 2.3 ([**16**]). Let X be a nonempty topological space. The Choquet game G_X of X is defined as follows: Players I and II take turns in playing nonempty open subsets of X

$$\begin{array}{lll} \text{I} & U_0 & U_1 \\ & & \quad\cdots \\ \text{II} & V_0 & V_1 \end{array}$$

so that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \cdots$. We say that II wins this run of the game if $\bigcap_n V_n (= \bigcap_n U_n) \neq \emptyset$. (Thus I wins if $\bigcap_n U_n (= \bigcap_n V_n) = \emptyset$.)

A strategy for I in this game is a "rule" that tells him how to play, for each n, his nth move U_n, given II's previous moves $V_0, \dots, V_{n-1}$. Formally, this is defined as follows: Let T be the tree of legal positions in the Choquet game G_X, i.e., T consists of all finite sequences $(W_0, \dots, W_n)$, where W_i are nonempty open subsets of X and $W_0 \supseteq W_1 \supseteq \cdots \supseteq W_n$. A strategy for I in G_X is a subtree $\sigma \subseteq T$ such that

i) σ is nonempty;

ii) if $(U_0, V_0, \dots, U_n) \in \sigma$, then for all open nonempty $V_n \subseteq U_n$, $(U_0, V_0, \dots, U_n, V_n) \in \sigma$;

iii) if $(U_0, V_0, \dots, U_{n-1}, V_{n-1}) \in \sigma$, then for a unique U_n, $(U_0, V_0, \dots, U_{n-1}, V_{n-1}, U_n) \in \sigma$.

Intuitively, the strategy σ works as follows: I starts playing U_0 where $(U_0) \in \sigma$ (and this is unique by iii)); II then plays any nonempty open $V_0 \subseteq U_0$; by ii) $(U_0, V_0) \in \sigma$. Then I responds by playing the unique nonempty open $U_1 \subseteq V_0$ such that $(U_0, V_0, U_1) \in \sigma$, etc.

A position $(W_0, \dots, W_n) \in T$ is compatible with σ if $(W_0, \dots, W_n) \in \sigma$. A run of the game $(U_0, V_0, U_1, V_1, \dots)$ is compatible with σ if for every $n \in \omega$ we have

$$(U_0, V_0, \dots, U_{n-1}, V_{n-1}, U_n) \in \sigma \text{ and } (U_0, V_0, \dots, U_n, V_n) \in \sigma.$$

The strategy σ is a winning strategy for I if he wins every compatible with σ run $(U_0, V_0, \dots)$ (i.e., if $(U_0, V_0, \dots)$ is a run compatible with σ then $\bigcap_n U_n (= \bigcap_n V_n) = \emptyset$).

The corresponding notions of strategy and winning strategy for II are defined mutatis mutandis.

Theorem 2.4 [**16**]. *A nonempty topological space X is a Baire space if and only if player* I *has no winning strategy in the Choquet game G_X.*

Theorem 2.5. *Let X be a Baire space. Let $\{f_n : n \in \omega\}$ be a sequence of real-valued quasicontinuous functions defined on X pointwise convergent to a function $f : X \to R$. Then the following are equivalent*:

(1) *f is quasicontinuous*;

(2) *$\{f_n : n \in \omega\}$ is equi-quasicontinuous.*

Proof. (2) $\Rightarrow$ (1) is established by Proposition 2.2. Now we prove (1) $\Rightarrow$ (2). Suppose, by way of contradiction, that $\{f_n : n \in \omega\}$ is not equi-quasicontinuous at x_0. Then there are $\varepsilon > 0$ and an open neighborhood U of x_0 such that for every $n \in \omega$ and every nonempty open subset $W \subset U$ there is a $k > n$ and a point $w \in W$ such that

$$|f_k(w) - f_k(x_0)| > \varepsilon.$$

For every $n \in \omega$ put $A_n = \{w \in U :$ there is $k > n, |f_k(w) - f_k(x_0)| > \varepsilon\}$. By the above, A_n is dense in U for every $n \in \omega$.

The quasicontinuity of f at x_0 implies that there is a nonempty open set $O \subset U$ such that

$$|f(x) - f(x_0)| < \varepsilon/4 \quad \text{for every } x \in O.$$

Now we will define a strategy σ for the first player I:

To define $U_0 \in \sigma$, realize that the set A_0 is dense in U; thus also in O. There are $k_0 > 0$ and $x(k_0) \in O$ such that

$$|f_{k_0}(x(k_0)) - f_{k_0}(x_0)| > \varepsilon,$$

and there is a nonempty open set $O(k_0) \subset O$ such that

$$|f_{k_0}(x(k_0)) - f_{k_0}(z)| < \varepsilon/4$$

for every $z \in O(k_0)$. Put $U_0 = O(k_0)$.

Now let V_0 be a nonempty open subset of U_0. We will define U_1:

There are $k_1 > k_0, k_1 > 1$ and $x(k_1) \in V_0$ such that

$$|f_{k_1}(x(k_1)) - f_{k_1}(x_0)| > \varepsilon$$

and there is a nonempty open subset $O(k_1)$ of V_0 such that

$$|f_{k_1}(x(k_1)) - f_{k_1}(z)| < \varepsilon/4$$

for every $z \in O(k_1)$. Put $U_1 = O(k_1)$.

Suppose now that $(U_0, V_0, \dots, U_{n-1}, V_{n-1}) \in \sigma$, where $U_i = O(k_i)$, $k_0 < k_1 < \cdots < k_{n-1}$ and $k_i > i$ for every $i \leq n-1$. We will define U_n. There are $k_n > k_{n-1}, k_n > n$ and $x(k_n) \in V_{n-1}$ such that

$$|f_{k_n}(x(k_n)) - f_{k_n}(x_0)| > \varepsilon$$

and there is a nonempty open subset $O(k_n)$ of V_{n-1} such that

$$|f_{k_n}(x(k_n)) - f_{k_n}(z)| < \varepsilon/4$$

for every $z \in O(k_n)$. Put $U_n = O(k_n)$.

Since X is a Baire space, by Theorem 2.4 there is no winning strategy for the first player I. Thus, for an appropriate choice of $V_0, V_1, V_2, \dots, \bigcap_n U_n \neq \varnothing$. Let $b \in \bigcap_n U_n$.

The pointwise convergence of $\{f_n : n \in \omega\}$ to f implies that there is an $n \in \omega$ such that for every $k \geq n$

$$|f_k(b) - f(b)| < \varepsilon/4 \quad \text{and} \quad |f_k(x_0) - f(x_0)| < \varepsilon/4.$$

Now $k_n > n$. Consider $|f_{k_n}(x(k_n)) - f_{k_n}(x_0)|$. We have

$$|f_{k_n}(x(k_n)) - f_{k_n}(x_0)| \leq |f_{k_n}(x(k_n)) - f_{k_n}(b)| + |f_{k_n}(b) - f_{k_n}(x_0)|.$$

Since $b \in O(k_n)$ we have that $|f_{k_n}(x(k_n)) - f_{k_n}(b)| < \varepsilon/4$. Now

$$\begin{aligned}|f_{k_n}(b) - f_{k_n}(x_0)| &\leq |f_{k_n}(b) - f(b)| + |f(b) - f(x_0)| + |f(x_0) - f_{k_n}(x_0)| \\ &< 3\varepsilon/4.\end{aligned}$$

Thus $|f_{k_n}(x(k_n)) - f_{k_n}(x_0)| < \varepsilon$, a contradiction. □

A collection $\mathcal{G}$ of nonempty open subsets of a topological space X is called a π-base ([**23, 24, 31**]) (also known as a pseudo-base) for X provided that every nonempty open subset of X contains some member of $\mathcal{G}$.

There is a characterization of quasicontinuity in terms of π-bases:

Proposition 2.6. *Let X, Y be topological spaces, and let $f : X \to Y$ be a function. Then f is quasicontinuous if and only if there exists a π-base τ for the topology of X such that, whenever U is open in X and V is open in Y with $f(U) \cap V \neq \varnothing$, there exists a $T \in \tau$ with $T \subset U$ such that $f(T) \subset V$.*

Proof. Suppose first that $f : X \to Y$ is quasicontinuous. Put

$$\tau = \{\mathrm{Int}\,(f^{-1}(V) \cap U) : f^{-1}(V) \cap U \neq \varnothing, V \text{ open in } Y, U \text{ open in } X\}.$$

To prove that τ is a π-base for the topology of X, let U be a nonempty open set in X. Take $x \in U$. Let V be an open set in Y such that $f(x) \in V$. The quasicontinuity of f at x implies that there is a nonempty open set H such that $H \subset U$ and $f(H) \subset V$. Then $H \subset f^{-1}(V) \cap U$. Thus $T = Int(f^{-1}(V) \cap U) \neq \varnothing$ and $T \subset U$. Thus τ is a π-base for the topology of X. It is easy to verify that τ satisfies the condition of the Proposition.

Suppose now that $f : X \to Y$ is such a function that there is a π-base τ for the topology of X such that whenever U is open in X and V is open in Y with $f(U) \cap V \neq \varnothing$ and there exists a $T \in \tau$ with $T \subset U$ such that $f(T) \subset V$. We show that f is quasicontinuous. Let $x \in X$. Let U be an open neighborhood of x and V an open neighborhood of $f(x)$. By the assumption there is a $T \in \tau$ such that $T \subset U$ and $f(T) \subset V$. Since T is a nonempty open set in X, the quasicontinuity of f at x is proved. ◻

Theorem 2.7. *Let (X, d) be a metric space. Then the following are equivalent*:

(1) *X is Baire*;

(2) *If $\{f_n : n \in \omega\}$ is a sequence of real-valued quasicontinuous functions defined on X pointwise convergent to a quasicontinuous function $f : X \to R$, then $\{f_n : n \in \omega\}$ is equi-quasicontinuous.*

To prove Theorem 2.7 we will use the following lemma which was proved in [**8**]:

Lemma ([**8**]). *Let (X, d) be a metric space without isolated points. Let D be a nonempty closed nowhere dense subset of X. Then there is a quasicontinuous function $f : X \to [0, 1]$ such that f is continuous at each point $x \notin D$, $f(z) = 0$ for every $z \in D$ and in every neighborhood V of $z \in D$ there are $x, y \in V \cap D^c$ with $f(x) = 0$ and $f(y) = 1$.*

Proof of Theorem 2.7. Only (2) $\Rightarrow$ (1) needs some explanation. Suppose X is not a Baire space. Let G be a nonempty open set in X which is of the first Baire category. Let $\{K_n : n \in \omega\}$ be a sequence of nowhere dense subsets of G such that $G = \cup\{K_n : n \in \omega\}$. For every $n \in \omega$ put $L_n = \cup\{\overline{K_i} : i \leq n\} \cup (\overline{G} \setminus G)$. Then $\overline{G} = \cup\{L_n : n \in \omega\}$. Of course every L_n is a closed nowhere dense set in $\overline{G}$.

Let $n \in \omega$. We will apply the above lemma on $\overline{G}$ and L_n. Of course $\overline{G}$ has no isolated point. There is a quasicontinuous function $f_n^* : \overline{G} \to [0, 1]$ such that $f_n^*(z) = 0$ for every $z \in L_n$ and for every $z \in L_n$ and every neighborhood V of $z \in L_n$ there is a $y \in V$ with $f_n^*(y) = 1$. Let $f_n : X \to [0, 1]$ be a function defined as follows: $f_n(x) = f_n^*(x)$ for every $x \in \overline{G}$ and $f_n(x) = 0$ otherwise. Of course also f_n is quasicontinuous.

It is easy to verify that the sequence $\{f_n : n \in \omega\}$ pointwise converges to the function identically equal to 0. However the sequence $\{f_n : n \in \omega\}$ is not equi-quasicontinuous. Let $x \in G$ be arbitrary. We will show that $\{f_n : n \in \omega\}$ is not equi-quasicontinuous at x. In fact, we claim that with $\varepsilon = 1/2$, for every $n \in \omega$, for every nonempty open set $V \subset G$ there are $k > n$, $z_k \in V$ with

$$|f_k(z_k) - f_k(x)| > \varepsilon.$$

Let $n \in \omega$ and $V \subset G$ be a nonempty open set. Let $z \in V$ be arbitrary. There is a $k > n$ with $x, z \in L_k$. Thus $f_k(x) = 0$ and also $f_k(z) = 0$. There is a $z_k \in V$ with $f_k(z_k) = 1$. □

It is easy to verify that the above theorem also works for a quasi-regular T_1-topological space X with a π-base, elements of which are metrizable. A topological space is quasi-regular ([**23**]) if every nonempty open set contains a closed subset with nonempty interior.

A notion which will be helpful for us further is one of locally countable π-base (see [**27**] and also [**32**]). A π-base is a locally countable π-base if each member of it contains only countably many members of the π-base.

Theorem 2.8. *Let X be a quasi-regular T_1 topological space which has a locally countable π-base. The following are equivalent*:

(1) *X is Baire*;

(2) *If $\{f_n : n \in \omega\}$ is a sequence of real-valued quasicontinuous functions defined on X pointwise convergent to a quasicontinuous function $f : X \to R$, then $\{f_n : n \in \omega\}$ is equi-quasicontinuous.*

Proof. Only (2) $\Rightarrow$ (1) needs some explanation. Suppose X is not a Baire space. Let U be a nonempty open set in X which is of the first Baire category. Let τ be a locally countable π-base of X. Let $G \in \tau$ be a nonempty subset of U. Then of course also G is of the first Baire category.

Let $\{V_n : n \in \omega\} \subset \tau$ be a countable π-base of G, and let $\{K_n : n \in \omega\}$ be a sequence of nowhere dense subsets of G such that $G = \bigcup\{K_n : n \in \omega\}$.

For every $n \in \omega$ we will define a function f_n as follows:

$\bigcup\{K_i : i \leq n\}$ is nowhere dense in G (so also in X); i.e., $V_n \setminus \bigcup\{\overline{K_i} : i \leq n\} \neq \varnothing$. Moreover the quasi-regularity of X implies that there is an open set H_n such that

$$H_n \subset \overline{H_n} \subset V_n \setminus \bigcup\{\overline{K_i} : i \leq n\}.$$

Define the function f_n as follows: $f_n(x) = 1$ for $x \in \overline{H_n}$ and $f_n(x) = 0$ otherwise. It is easy to verify that f_n is quasicontinuous for every $n \in \omega$ and that the sequence $\{f_n : n \in \omega\}$ pointwise converges to the function f identically equal to 0.

We will show that the sequence $\{f_n : n \in \omega\}$ is not equi-quasicontinuous. Let $x \in G$ be arbitrary. We will show that $\{f_n : n \in \omega\}$ is not equi-quasicontinuous at x. In fact, we claim that with $\varepsilon = 1/2$, for every $n \in \omega$, for every nonempty open set $V \subset G$ there are $k > n$, $z_k \in V$ with

$$|f_k(z_k) - f_k(x)| > \varepsilon.$$

Let $n \in \omega$ and $V \subset G$ be a nonempty open set. There is $m \in \omega$ with $V_m \subset V$. Since G is of the first Baire category it has no isolated points; thus, every open subset of G contains infinitely many points and infinitely many elements of the sequence $\{V_n : n \in \omega\}$. Let $n_0 \in \omega$ be such that $x \in K_{n_0}$. There are $k \in \omega$, $k > \max\{n, n_0\}$ with $V_k \subset V_m$. Then we have

$$|f_k(x_k) - f_k(x)| = 1 > \varepsilon.$$

Thus $\{f_n : n \in \omega\}$ is not equi-quasicontinuous at x, a contradiction with (2). □

In [**4**] the authors proved that property (2) in previous theorems does not characterize Baire spaces in the class of T_1 topological spaces.

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