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EXISTENCE OF SOLUTIONS OF SECOND-ORDER PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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ABSTRACT. In this paper we establish the existence of mild solutions for a class of abstract second-order partial neutral functional differential equations with infinite delay in a Banach space.

1. Introduction. In this work we study the existence of mild solutions for a class of abstract second-order neutral functional differential equations with infinite delay. Throughout this paper, X denotes a Banach space endowed with a norm $\|\cdot\|$ and A is the infinitesimal generator of a strongly continuous cosine function $C(\cdot)$ of bounded linear operators on the Banach space X. We will be concerned with equations of the form

$$\frac{d^2}{dt^2}\left(x(t)-g(t,x_t)\right) = Ax(t)+f(t,x_t,x'(t)), \quad t \in I=[0,a], \tag{1.1}$$

$$x_0=\varphi \in \mathcal{B}, \tag{1.2}$$
$$x'(0)=z \in X,$$

where $x(t) \in X$, the history $x_t : (-\infty,0] \to X$, $\theta \mapsto x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, and f and g are appropriate functions.

Motivated by the fact that ordinary neutral functional differential equations (abbreviated, NFDE) arise in many areas of applied mathematics, this type of equation has received considerable attention in recent years. The literature concerning first- and second-order ordinary

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neutral functional differential equations is very extensive. We refer the reader to the books Hale and Lunel [**21**], Lakshmikantham, et al. [**31**], Kolmanovskii and Myshkis [**29**], Gopalsamy [**18**], Bainov and Mishev [**9**], and to the papers [**6, 8, 30, 33, 34, 39, 40, 43**–**46**] related to the subject of this paper. On the other hand, first-order partial neutral functional differential equations have also been studied by several authors, including Adimy [**1**–**5**], Hale [**20**], Wu [**43, 44**], Benchohra [**13**], Chen [**15**] and Ezzinbi [**16**] for finite delay, and Hernández and Henríquez [**22, 23**], Bouzahir [**14**], Hernández [**25**] and Nagel [**36**] for the unbounded delay case.

In addition, some abstract second-order neutral Cauchy problems have been considered recently in the literature, see for instance [**11, 12, 32**]. However, in these works the authors assume that the operator $C(t)$ is compact for $t > 0$. In this case, it follows from [**42**, page 557] that the underlying space X must be finite dimensional. A neutral differential equation of abstract type similar to those considered in [**11, 12**] is studied in [**24**]. We refer to this paper for a list of references with concrete examples. In [**24**], the authors develop a technical framework to establish the existence of mild solutions for a class of abstract second-order neutral functional integro-differential equations with finite delay of the form

$$\begin{aligned} \frac{d}{dt}[x'(t) - g(t, x_t)] \\ = Ax(t) + \int_0^t F\left(t, s, x_s, x'(s), \int_0^s f(s, \tau, x_\tau, x'(\tau))\, d\tau\right) ds, \end{aligned} \tag{1.4}$$

$$x_0 = \varphi \in \mathcal{B}, \qquad x'(0) = z \in X, \tag{1.5}$$

for $t \in I = [0, a]$, in a Banach space X. Here A is the infinitesimal generator of an arbitrary strongly continuous cosine function of bounded linear operators on X; the history $x_t : [-r, 0] \to X$, $x_t(\theta) = x(t + \theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, and F, f and g are functions that satisfy appropriate conditions.

In this work we extend that theory to study the existence and qualitative properties of solutions for the general class of second order abstract neutral functional differential equations with infinite delay described by (1.1)–(1.3).

Next, we review some fundamental facts needed to establish our results. For the theory of cosine functions of operators, we refer the reader to [**17, 41, 42**]. Next, we recall some concepts and properties regarding second-order abstract Cauchy problems. We denote by $S(\cdot)$ the sine function associated with the cosine function $C(\cdot)$, which is defined by $S(t)x = \int_0^t C(s)x\,ds$, for $x \in X$ and $t \in \mathbf{R}$. Moreover, we denote by N and $\widetilde{N}$ some positive constants so that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \widetilde{N}$, for every $t \in I$.

In what follows, $[D(A)]$ represents the domain of A endowed with the graph norm given by $\|x\|_A = \|x\| + \|Ax\|$, for $x \in D(A)$, and E stands for the space consisting of the vectors $x \in X$ for which $C(\cdot)x$ is of class C^1 on $\mathbf{R}$. We know from Kisińsky [**27**] that E endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0 \leq t \leq 1} \|AS(t)x\|, \quad x \in E$$

is a Banach space. The operator valued function $\mathcal{H}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. It follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$, as $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is locally integrable, then $y(t) = \int_0^t S(t-s)x(s)\,ds$ defines an E-valued continuous function. This is a consequence of the fact that

$$\int_0^t \mathcal{H}(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\,ds \\ \int_0^t C(t-s)x(s)\,ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function. Moreover, it follows from the definition of the norm in E that a function $u : I \to E$ is continuous if, and only if, u is continuous with respect to the norm in X and the set of functions $\{AS(t)u(\cdot) : 0 \leq t \leq 1\}$ is an equicontinuous subset of $C(I, X)$.

The existence of solutions of the second-order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad t \in I, \tag{1.6}$$

$$x(0) = w, \quad x'(0) = z, \tag{1.7}$$

where $h : I \to X$ is an integrable function, is studied in [**42**]. Similarly, the existence of solutions of semi-linear, second-order abstract Cauchy problems has been discussed in [**41**]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)w + S(t)z + \int_0^t S(t-s)h(s)\,ds, \quad t \in I, \tag{1.8}$$

is called a mild solution of Problem (1.6)–(1.7) and that, when $w \in E$, the function $x(\cdot)$ is of class C^1 on I and

$$x' = AS(t)w + C(t)z + \int_0^t C(t-s)h(s)\,ds, \quad t \in I. \tag{1.9}$$

In this work, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and satisfying the following axioms:

(A) If $x : (-\infty, \sigma + b) \to X$, for $b > 0$, is continuous on $[\sigma, \sigma + b)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b)$, the following conditions hold:

(i) x_t is in $\mathcal{B}$,

(ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$,

(iii) $\|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$

where $H > 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded, and H, K, M are independent of $x(\cdot)$.

(A-1) For the function $x(\cdot)$ in (A), the map $t \mapsto x_t$ is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + b)$.

(B) The space $\mathcal{B}$ is complete.

Example 1.1. The phase space $C_r \times L^p(\rho, X)$. Let $r \ge 0$, $1 \le p < \infty$, and let $\rho : (-\infty, -r] \to \mathbf{R}$ be a nonnegative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [**26**]. In other words, this means that ρ is locally integrable and that there exists a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \le \gamma(\xi)\rho(\theta)$, for all $\xi \le 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(\rho, X)$ consists of the classes of functions $\varphi : (-\infty, 0] \to X$

such that φ is continuous on $[-r,0]$, Lebesgue-measurable, and $\rho\|\varphi\|^p$ is Lebesgue integrable on $(-\infty,-r)$. The seminorm in $C_r \times L^p(\rho, X)$ is defined by

$$\|\varphi\|_{\mathcal{B}} = \sup\{\|\varphi(\theta)\| : -r \le \theta \le 0\} + \left(\int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|^p d\theta \right)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(\rho; X)$ satisfies axioms (A), (A-1) and (B). Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^{0} \rho(\theta)\, d\theta\right)^{1/2}$, for $t \ge 0$. (See [**26**, Theorem 1.3.8] for details).

Additional terminology and notations used in this work are standard. In particular, if $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, we indicate by $\mathcal{L}(Z,Y)$ the Banach space of bounded linear operators from Z into Y endowed with the operator norm, and we abbreviate this notation to $\mathcal{L}(Z)$ whenever $Z = Y$. Moreover, $C(I,Z)$ denotes the space of continuous functions from I into Z endowed with the norm of uniform convergence and $B_r(x,Z)$ denotes the closed ball with center at x and radius r in Z.

This paper has three sections. In the next section we discuss the existence of mild solutions of the abstract second-order neutral system (1.1)–(1.3) and, in Section 3 we consider some applications of our results.

2. Existence results. In order to study the problem (1.1)–(1.3), we introduce the following assumption.

(H1) There exists a Banach space $(Y, \|\cdot\|_Y)$ continuously included in X such that $AS(t) \in \mathcal{L}(Y,X)$, for all $t \in I$, and the function $AS(\cdot) : I \to \mathcal{L}(Y,X)$ is strongly continuous. Let $N_Y, \widetilde{N}_1$ be positive constants such that $\|y\| \le N_Y \|y\|_Y$, for all $y \in Y$, and $\|AS(t)\|_{\mathcal{L}(Y,X)} \le \widetilde{N}_1$, for all $t \in I$.

Remark 2.1. It is clear that if (H1) holds, then Y is continuously included in E. In fact, if we take $y \in Y$, then

$$C(t)y - y = A \int_0^t S(s)y\, ds = \int_0^t AS(s)y\, ds$$

which implies that the function $C(\cdot)y$ is continuously differentiable, and therefore $Y \subseteq E$. Moreover, for $a \geq 1$, it follows from the definition of the norm in E that

$$\|y\|_E = \|y\| + \sup_{0\leq t\leq 1} \|AS(t)y\| \leq (N_Y + \widetilde{N}_1)\|y\|_Y$$

which shows that the inclusion $\iota : Y \hookrightarrow E$ is continuous. Using the properties of cosine functions, we can show easily that this property also holds for $0 < a < 1$.

We also observe that the spaces $Y = [D(A)]$ and $Y = E$ satisfy assumption (H1).

To establish the concept of a mild solution, we assume that f and g satisfy appropriate conditions. In what follows, for a function $x(\cdot)$ defined on $(-\infty, a]$ we will consider $x'(0)$ as the right derivative of $x(\cdot)$ at zero.

(H2) The function $f : I\times\mathcal{B}\times X \to X$ satisfies the following conditions:

(i) The function $f(t,\cdot,\cdot) : \mathcal{B} \times X \to X$ is continuous almost everywhere $t \in I$ and $f(\cdot,\psi,x) : I \to X$ is strongly measurable for every $(\psi,x) \in \mathcal{B} \times X$.

(ii) There exist an integrable function $m_f : I \to [0,\infty)$ and a continuous nondecreasing function $W_f : [0,\infty) \to (0,\infty)$ such that

$$\|f(t,\psi,x)\| \leq m_f(t)W_f\left(\|\psi\|_{\mathcal{B}} + \|x\|\right), \quad (t,\psi,x) \in I \times \mathcal{B} \times X.$$

(H3) The function $g : I\times\mathcal{B} \to Y$, and for every function $x : (-\infty, a] \to X$ such that $x_0 = \varphi$, x is continuously differentiable on I and $x'(0) = z$, then the function $I \to Y$, $t \mapsto g(t, x_t)$, is continuously differentiable and $\frac{d}{dt}g(t,x_t)|_{t=0} = \eta$, independent of $x(\cdot)$.

In connection with these conditions it is worth pointing out the following comments.

Remark 2.2. Let $x : (-\infty, a] \to X$ be a function involved in (H3). If assumptions (H1) and (H3) hold, then the function $t \in I \mapsto g(t,x_t) \in Y$ is continuous and Bochner's criterion [**7**, Theorem 1.1.4, Proposition

1.3.4] implies that the function $s \in [0,t] \mapsto AS(t-s)g(s,x_s) \in X$ is integrable. Moreover, the function $t \in I \mapsto \int_0^t AS(t-s)g(s,x_s)\,ds \in Y$ is continuously differentiable, and we have

$$\begin{aligned}\frac{d}{dt}\int_0^t AS(t-s)g(s,x_s)\,ds \\ &= \frac{d}{dt}\int_0^t AS(s)g(t-s,x_{t-s})\,ds \\ &= AS(t)g(0,\varphi) + \int_0^t AS(t-s)\frac{d}{ds}g(s,x_s)\,ds.\end{aligned}$$

Similarly, if (H2) holds, then the function $s \in [0,t] \mapsto f(s,x_s,x'(s)) \in X$ is integrable and

$$\begin{aligned}\|f(s,x_s,x'(s))\| &\le m_f(s)W_f(\|x_s\|_{\mathcal{B}} + \|x'(s)\|) \\ &\le m_f(s)W_f(K(s)\sup_{0\le\xi\le s}\|x(\xi)\| \\ &\quad + M(s)\|\varphi\|_{\mathcal{B}} + \|x'(s)\|).\end{aligned}$$

Remark 2.3. Assumption (H3) is easy to satisfy. For instance, without specifying the phase space and neglecting for the moment technical details, we mention that assumption (H3) is satisfied by functions g of type $g(t,\psi) = \int_{-\infty}^0 Q(-s)\psi(s)\,ds$, where $Q(\cdot) \in \mathcal{L}(X,Y)$ is an appropriate operator valued map.

Motivated by equation (1.8), we consider the following concept of mild solutions of Problem (1.1)–(1.3). Here we assume that conditions (H1), (H2) and (H3) hold.

Definition 2.1. A function $x : (-\infty, b] \to X$, with $0 < b \le a$, is a mild solution of Problem (1.1)–(1.3) on $[0,b]$ if $x(\cdot)$ is a function of class C^1 on $[0,b]$ that satisfies conditions (1.2) and (1.3), and verifies the integral equation

$$\begin{aligned}x(t) = C(t)(\varphi(0) - g(0,\varphi)) + S(t)(z-\eta) + g(t,x_t) \\ + \int_0^t AS(t-s)g(s,x_s)\,ds \\ + \int_0^t S(t-s)f(s,x_s,x'(s))\,ds,\ 0 \le t \le b.\end{aligned}$$

We now establish our first existence result. In what follows, we will denote by y the function $y : (-\infty, a] \to X$ such that $y_0 = \varphi$ and $y(t) = C(t)\varphi(0) + S(t)z$, for $0 \le t \le a$. We set $\mathcal{F}(a)$ for the space consisting of functions $u : (-\infty, a] \to X$ such that $u_0 = 0$, u is continuously differentiable on $[0, a]$ and $u'(0) = 0$. We consider the space $\mathcal{F}(a)$ endowed with the norm $\|u\|_1 = \sup_{0\le t\le a} \|u'(t)\|$. We denote $P : \mathcal{F}(a) \to C(I, Y)$ the function given by $P(u)(t) = \frac{d}{dt} g(t, u_t + y_t)$. Moreover, since $K(\cdot)$ is a continuous function, by substituting $K(t)$ by $\max_{0\le s\le t} K(s)$ we may assume that $K(\cdot)$ is nondecreasing.

Theorem 2.1. *Suppose that assumptions* (H1), (H2) *and* (H3) *are fulfilled,* $\varphi(0) \in E$ *and that the following conditions hold*:

(a) *For each* $r > 0$, *the set* $\{f(t, u_t + y_t, u'(t) + y'(t)) : t \in I,\ u \in \mathcal{F}(a),\ \|u\|_1 \le r\}$ *is relatively compact in* X.

(b) *There exists a continuous function* $L_P : I \times [0, \infty) \to [0, \infty)$ *such that, for each* $t \ge 0$, *the function* $L_P(t, \cdot)$ *is nondecreasing, and for all* $u, v \in \mathcal{F}(a)$, *with* $\|u\|_1,\ \|v\|_1 \le r$, *we have*

$$\sup_{0\le t\le b} \|P(u)(t) - P(v)(t)\|_Y \le L_P(b, r)\|u - v\|_1.$$

Then the following properties are verified

(i) *If* $\liminf_{\xi\to\infty}(W_f(\xi))/\xi < \infty$ *and* $L_P(t, r) \to 0$ *as* $t \to 0$, *for each* $r > 0$, *then there exists* $b > 0$ *and a mild solution of Problem* (1.1)–(1.3) *on* $[0, b]$.

(ii) *If* $\lim_{r\to\infty} L_P(b, r) = L_P(b)$, *for some* $0 < b \le a$, *and if*

$$(1 + b\widetilde{N}_1)L_P(b) + N(bK(b) + 1)\liminf_{\xi\to\infty} \frac{W_f(\xi)}{\xi} \int_0^b m_f(s)\, ds < 1, \tag{2.1}$$

then there exists a mild solution of Problem (1.1)–(1.3) *on* $[0, b]$.

Proof. We define the operator Γ on the space $\mathcal{F}(a)$ by

$$\begin{aligned} \Gamma(u)(t) = &-C(t)g(0, \varphi) - S(t)\eta + g(t, u_t + y_t) \\ &+ \int_0^t AS(t - s)g(s, u_s + y_s)\, ds \\ &+ \int_0^t S(t - s)f(s, u_s + y_s, u'(s) + y'(s))\, ds, \quad t \in I. \end{aligned} \tag{2.2}$$

It is clear from Remark 2.2 that $\Gamma u(t)$ is well defined and that the function $\Gamma u(\cdot)$ is continuously differentiable, with

$$\begin{aligned} \frac{d}{dt}\Gamma(u)(t) &= -C(t)\eta + P(u)(t) \\ &\quad + \int_0^t AS(t-s)P(u)(s)\,ds \\ &\quad + \int_0^t C(t-s)f(s,u_s+y_s,u'(s)+y'(s))\,ds \end{aligned} \tag{2.3}$$

so that $\frac{d}{dt}\Gamma(u)|_{t=0} = 0$. This shows that Γ maps $\mathcal{F}(a)$ into $\mathcal{F}(a)$. Moreover, condition (H2), the continuity of P and a standard application of Lebesgue's dominated convergence theorem allow us to conclude that $\Gamma : \mathcal{F}(a) \to \mathcal{F}(a)$ is continuous.

We next show that there exists an $r > 0$ such that $\Gamma(B_r(0,\mathcal{F}(b))) \subseteq B_r(0,\mathcal{F}(b))$, for some $0 < b \le a$. To prove this assertion, we estimate $\|\frac{d}{dt}\Gamma u(t)\|$. In the expressions that follow we denote by c a generic constant, independent of u. It follows from (2.3) that

$$\begin{aligned} \left\|\frac{d}{dt}\Gamma(u)(t)\right\| &\le c + \|P(u)(t)\| \\ &\quad + \left\|\int_0^t AS(t-s)[Pu(s)-P(0)(s)]\,ds\right\| \\ &\quad + \left\|\int_0^t AS(t-s)P(0)(s)\,ds\right\| \\ &\quad + N\int_0^t m_f(s)W_f(\|u_s+y_s\|_{\mathcal{B}} + \|u'(s)+y'(s)\|)\,ds \\ &\le c + L_P(t,r)\|u\|_1 + \widetilde{N}_1 t L_P(t,r)\|u\|_1 \\ &\quad + N\int_0^t m_f(s)W_f\left(K(s)\max_{0\le\xi\le s}\|u(\xi)\| + \|u'(s)\| + c\right)ds \\ &\le c + (1+b\widetilde{N}_1)L_P(b,r)r \\ &\quad + NW_f\left((bK(b)+1)r+c\right)\int_0^b m_f(s)\,ds, \end{aligned} \tag{2.4}$$

for all $0 \le t \le b$ and $\|u\|_1 \le r$.

In case (i), we can select $r > 0$ large enough, and $b > 0$ sufficiently small so that

$$\frac{c}{r} + (1 + b\widetilde{N}_1)L_P(b, r) + N\frac{W_f((bK(b)+1)r + c)}{r}\int_0^b m_f(s)\,ds < 1. \tag{2.5}$$

Similarly, in case (ii), we can choose $r > 0$ sufficiently large such that (2.5) holds. Therefore, in both cases, it follows from (2.4) and (2.5) that $\|\frac{d}{dt}\Gamma(u)(t)\| \le r$, which establishes our assertion.

On the other hand, we decompose $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_i, for $i = 1, 2$, are defined by

$$\begin{aligned}\Gamma_1(u)(t) &= -C(t)g(0,\varphi) - S(t)\eta + g(t, u_t + y_t)\\ &\quad + \int_0^t AS(t-s)g(s, u_s + y_s)\,ds,\\ \Gamma_2(u)(t) &= \int_0^t S(t-s)f(s, u_s + y_s, u'(s) + y'(s))\,ds.\end{aligned}$$

In addition, for $r > 0$ and $b > 0$ selected as was previously established, we consider Γ defined on $B_r(0, \mathcal{F}(b))$. For $u, v \in \mathcal{F}(b)$, with $\|u\|_1$, $\|v\|_1 \le r$, we find that

$$\begin{aligned}\left\|\frac{d}{dt}(\Gamma_1 u(t) - \Gamma_1 v(t))\right\| &\le \|P(u)(t) - P(u)(t)\|\\ &\quad + \left\|\int_0^t AS(t-s)[P(u)(s) - P(v)(s)]\,ds\right\|\\ &\le (1 + b\widetilde{N}_1)L_P(b, r)\|u - v\|_1,\end{aligned}$$

and condition (2.5) implies that Γ_1 is a contraction mapping. Moreover, applying the Ascoli-Arzelà theorem, we obtain that the set $\{\frac{d}{dt}\Gamma_2(u) : u \in B_r(0, \mathcal{F}(b))\}$ is relatively compact in $C([0, b])$. Since $\Gamma_2(u)(0) = 0$, for all $u \in B_r(0, \mathcal{F}(b))$, we have that the set $\{\Gamma_2(u) : u \in B_r(0, \mathcal{F}(b))\}$ is relatively compact in $\mathcal{F}(b)$. Consequently, Γ is a condensing map which allows us to establish the existence of a fixed point u for Γ (see [**35**, Theorem IV.3.2]). Clearly, $x = u + y$ is a mild solution of Problem (1.1)–(1.3) on the interval $[0, b]$, which completes the proof. □

Remark 2.4. The values $x(t)$ for $0 \leq t \leq b$ of the mild solution constructed in Theorem 2.1 belong to the space E. In fact, it follows from the properties of cosine functions that $y(t) \in E$. Since $x(t) = u(t) + y(t)$, it only remains to prove that $u(t) \in E$. In view of the fact that $u(t) = \Gamma(u)(t)$ we shall establish that $\Gamma(u)(t) \in E$. Since the values $g(t, u_t + y_t) \in E$ for all $t \in I$, and applying again the properties of cosine functions, it is clear that $-C(t)g(0, \varphi) - S(t)\eta + g(t, u_t + y_t) \in E$. Moreover, directly using the properties of the group $\mathcal{H}$ mentioned in the Introduction, we obtain that

$$\int_0^t S(t-s)f(s, u_s + y_s, u'(s) + y'(s))\, ds \in E.$$

Similarly, since the function $t \mapsto g(t, u_t + y_t) \in Y$ is continuously differentiable, it follows from Pazy ([**38**, Corollary 4.2.5]) that

$$\int_0^t \mathcal{H}(t-s) \begin{bmatrix} \widetilde{g}(s) \\ 0 \end{bmatrix} ds = \int_0^t \begin{bmatrix} C(t-s)\widetilde{g}(s) \\ AS(t-s)\widetilde{g}(s) \end{bmatrix} ds \in D(\mathcal{A}) = D(A) \times E$$

where $\widetilde{g}(s) = g(s, u_s + y_s)$. Hence, $\int_0^t AS(t-s)\widetilde{g}(s)\, ds \in E$. Combining these properties with (2.2) the assertion follows.

A particular situation is obtained when the function g is continuously differentiable. To establish this result we introduce the following assumption.

(H4) The function $g : I \times \mathcal{B} \to Y$ is of class C^1, and there are a positive constant L_g^1 and a continuous function $L_g^2 : I \times [0, \infty) \to [0, \infty)$ such that

$$\|D_1 g(t, \psi_1) - D_1 g(t, \psi_2)\|_Y \leq L_g^1 \|\psi_1 - \psi_2\|_{\mathcal{B}},$$
$$\|D_2 g(t, \psi_1) - D_2 g(t, \psi_2)\|_{\mathcal{L}(\mathcal{B}, Y)} \leq L_g^2(t, r) \|\psi_1 - \psi_2\|_{\mathcal{B}},$$

for every $t \in I$, $r > 0$, and $\psi_i \in B_r(0, \mathcal{B})$.

Lemma 2.1. *Assume g satisfies condition* (H4), *$\varphi(0) \in E$, the function $s \mapsto y_s$ is differentiable at zero with $\frac{dy_t}{dt}|_{t=0} = \psi \in \mathcal{B}$ and $\psi(0) = z$. Then, for every $u \in \mathcal{F}(a)$ the function $t \mapsto g(t, u_t + y_t)$ is continuously differentiable,*

$$\eta = \frac{d}{dt} g(t, u_t + y_t)|_{t=0} = D_1 g(0, \varphi) + D_2 g(0, \varphi)(\psi) \tag{2.6}$$

is independent of u*, and the map* P *given by* $P(u)(t) = \frac{d}{dt}g(t, u_t + y_t)$ *satisfies the condition*

$$\|P(u)(t) - P(v)(t)\|_Y \leq L_P(b, r)\|u - v\|_1,$$

where $L_P : I \times [0, \infty)$ *is continuous and* $L_P(t, \cdot)$ *is nondecreasing for each* $t \geq 0$*. Moreover, if* $D_2 g(0, \varphi) = 0$*, then* $L_P(b, r) \to 0$*,* $b \to 0$*, for each* $r > 0$*.*

Proof. We define $w : (-\infty, a] \to X$ by $w_0 = \psi$ and $w(t) = y'(t)$, for $0 \leq t \leq a$. From the axioms of phase space we have that $w_t \in \mathcal{B}$, for $t \geq 0$, and that the function $I \to \mathcal{B}$, $t \mapsto w_t$, is continuous. We shall show that $t \to y_t$ is differentiable and $\frac{d}{dt}y_t = w_t$. To establish our assertion, for $h > 0$ we define $\xi : \mathbf{R} \to X$ by $\xi(s) = y(s + h)$. Turning to use of the axioms of phase space we obtain

$$\begin{aligned}\left\|\frac{y_{t+h} - y_t}{h} - w_t\right\|_{\mathcal{B}} &= \left\|\frac{\xi_t - y_t}{h} - w_t\right\|_{\mathcal{B}} \\ &\leq K(t) \max_{0 \leq s \leq t}\left\|\frac{y(s+h) - y(s)}{h} - w(s)\right\| \\ &\quad + M(t)\left\|\frac{y_h - \varphi}{h} - \psi\right\|_{\mathcal{B}}.\end{aligned}$$

Since the right hand side of the above inequality converges to zero as $h \to 0^+$, we infer that the function y_t has a continuous right derivative and, therefore y_t is continuously differentiable. Similarly, for each $u \in \mathcal{F}(a)$ the function u_t is continuously differentiable and $\frac{d}{dt}u_t = u'_t$.

On the other hand, it follows from the chain rule that the function $I \to Y$, $t \mapsto g(t, u_t + y_t)$ is continuously differentiable and

$$\frac{d}{dt}g(t, u_t + y_t) = D_1 g(t, u_t + y_t) + D_2 g(t, u_t + y_t)(u'_t + w_t).$$

Hence we obtain (2.6). Moreover, we can estimate $\|P(u)(t) - P(v)(t)\|_Y$. Previously, we point out that

$$\|u_t\|_{\mathcal{B}} \leq K(t) \max_{0 \leq s \leq t}\|u(s)\| \leq tK(t) \max_{0 \leq s \leq t}\|u'(s)\|,$$

for all $u \in \mathcal{F}(a)$. Therefore, $\|u_t + y_t\|_{\mathcal{B}} \leq c(r) = bK(b)r + \max_{0\leq t\leq b}\|y_t\|_{\mathcal{B}}$, for all $u \in \mathcal{F}(a)$ with $\|u\|_1 \leq r$ and $0 \leq t \leq b$. Similarly, $\|u'_t + w_t\|_{\mathcal{B}} \leq d(r) = K(b)r + \max_{0\leq t\leq b}\|w_t\|_{\mathcal{B}}$, for all $u \in \mathcal{F}(a)$ with $\|u\|_1 \leq r$, and $0 \leq t \leq b$. Now, we can estimate

$$\begin{aligned}\|D_2g(t, v_t + y_t)\| &\leq \|D_2g(t, v_t + y_t) - D_2g(t, \varphi)\|_{\mathcal{L}(\mathcal{B},Y)} \\ &\quad + \|D_2g(t, \varphi)\|_{\mathcal{L}(\mathcal{B},Y)} \\ &\leq L_g^2(t, c(r))(\|v_t\|_{\mathcal{B}} + \|y_t - \varphi\|_{\mathcal{B}}) \\ &\quad + \|D_2g(t, \varphi)\|_{\mathcal{L}(\mathcal{B},Y)}.\end{aligned}$$

Consequently, it follows from (H4) that

$$\begin{aligned}&\|P(u)(t) - P(v)(t)\|_Y \\ &\quad\leq \|D_1g(t, u_t + y_t) - D_1g(t, v_t + y_t)\|_Y \\ &\qquad + \|D_2g(t, u_t + y_t)(u'_t + w_t) - D_2g(t, v_t + y_t)(v'_t + w_t)\|_Y \\ &\quad\leq L_g^1\|u_t - v_t\|_{\mathcal{B}} + \|[D_2g(t, u_t + y_t) \\ &\qquad - D_2g(t, v_t + y_t)](u'_t + w_t)\|_Y \\ &\qquad + \|D_2g(t, v_t + y_t)(u'_t - v'_t)\|_Y \\ &\quad\leq tL_g^1K(t)\max_{0\leq s\leq t}\|u'(s) - v'(s)\| \\ &\qquad + L_g^2(t, c(r))\|u_t - v_t\|_{\mathcal{B}}\|u'_t + w_t\|_{\mathcal{B}} \\ &\qquad + L_g^2(t, c(r))(\|v_t\|_{\mathcal{B}} + \|y_t - \varphi\|_{\mathcal{B}})K(t)\max_{0\leq s\leq t}\|u'(s) - v'(s)\| \\ &\qquad + \|D_2g(t, \varphi)\|_{\mathcal{L}(\mathcal{B},Y)}K(t)\max_{0\leq s\leq t}\|u'(s) - v'(s)\| \\ &\quad\leq K(t)[tL_g^1 + \|D_2g(t, \varphi)\|_{\mathcal{L}(\mathcal{B},Y)}]\max_{0\leq s\leq t}\|u'(s) - v'(s)\| \\ &\qquad + K(t)L_g^2(t, c(r))[td(r) + rtK(t) \\ &\qquad + \|y_t - \varphi\|_{\mathcal{B}}]\max_{0\leq s\leq t}\|u'(s) - v'(s)\|.\end{aligned}$$

Hence, by defining

$$\begin{aligned}L_P(b, r) = K(b)&\left(bL_g^1 + \max_{0\leq t\leq b}\|D_2g(t, \varphi)\|_{\mathcal{L}(\mathcal{B},Y)}\right) \\ &+ \left(\max_{0\leq t\leq b} L_g^2(t, c(r))[bd(r) + rbK(b) + \|y_t - \varphi\|_{\mathcal{B}}]\right),\end{aligned}$$

we complete the proof that P fulfills the stated conditions. □

We can establish the following consequence.

Corollary 2.1. *Assume that assumptions* (H1), (H2) *and* (H4), *with* $D_2(0,\varphi)=0$, *are fulfilled. Suppose, further, that* $\varphi(0)\in E$, *the function* $s\mapsto y_s$ *is differentiable at zero, with* $\frac{dy_t}{dt}|_{t=0}=\psi\in\mathcal{B}$ *and* $\psi(0)=z$, *and that condition* (a) *of Theorem* 2.1 *holds. Then there exists a mild solution of Problem* (1.1)–(1.3) *on* $[0,b]$, *for some* $b>0$.

Proof. It follows from Lemma 2.1 that assumption (H3) is fulfilled, with η given by (2.6), and that P has the properties considered in case (i) of Theorem 2.1. □

The conditions considered in this result are not necessary to obtain existence of mild solutions.

Example 2.2. Let $\mathcal{B}=C_0\times L^p(\rho,X)$ be the phase space defined in Example 1.1, with $p=1$ and $\rho=1$. Let $Q:X\to E$ be a bounded linear operator. Let $g:I\times\mathcal{B}\to E$ be given by $g(t,\psi)=Q\int_{-t}^0\psi(\theta)\,d\theta$. It is easy to see that g fulfills the conditions considered in Theorem 2.1, for every $\varphi\in\mathcal{B}$. However, we can select φ so that the function $t\mapsto y_t$ is not differentiable.

We complete this section with an existence result when f satisfies a Lipschitz condition.

Theorem 2.2. *Assume that assumptions* (H1) *and* (H3) *are fulfilled,* $\varphi(0)\in E$ *and the following conditions hold*:

(a) *The function* $f(\cdot,\psi,x)$ *is integrable on* I, *for each* $\psi\in\mathcal{B}$ *and* $x\in X$.

(b) *There exist a constant* $L_f>0$ *and a continuous function* $L_P: I\times[0,\infty)\to[0,\infty)$, *with* $L_P(t,\cdot)$ *nondecreasing, for each* $t\geq 0$, *such that*

$$\|f(t,\psi_1,x_1)-f(t,\psi_2,x_2)\|\leq L_f(\|\psi_1-\psi_2\|_{\mathcal{B}}+\|x_1-x_2\|),$$
$$\sup_{0\leq t\leq b}\|P(u)(t)-P(v)(t)\|_Y\leq L_P(b,r)\|u-v\|_1,$$

for all $\psi_i \in \mathcal{B}$, $x_i \in X$, *for* $i = 1, 2$, *and* $u, v \in \mathcal{F}(a)$, *with* $\|u\|_1,\ \|v\|_1 \le r$.

Then the following properties are verified:

(i) *If* $L_P(t, r) \to 0$, *as* $t \to 0$, *for each* $r > 0$, *then there exists a* $b > 0$ *and a unique mild solution of Problem* (1.1)–(1.3) *on* $[0, b]$.

(ii) *If* $\lim_{r\to\infty} L_P(b, r) = L_P(b)$, *for some* $0 < b \le a$, *and*

$$(1 + b\widetilde{N}_1)L_P(b) + bNL_f(bK(b) + 1) < 1, \tag{2.7}$$

then there exists a unique mild solution of Problem (1.1)–(1.3) *on* $[0, b]$.

Proof. We proceed as in the proof of Theorem 2.1. We define Γ by (2.2). Initially we only mention that $\Gamma : \mathcal{F}(a) \to \mathcal{F}(a)$ is a continuous map. We claim that there exist $b, r > 0$ such that $\Gamma(B_r(0, \mathcal{F}(b))) \subseteq B_r(0, \mathcal{F}(b))$. To prove this assertion we estimate $\|\frac{d}{dt}\Gamma u(t)\|$. Since

$$\begin{aligned}
&\left\|\frac{d}{dt}\Gamma(u)(t)\right\| \\
&\quad\le \left\|\frac{d}{dt}[\Gamma(u)(t) - \Gamma(0)(t)]\right\| + \|\Gamma(0)(t)\| \\
&\quad\le \|P(u)(t) - P(0)(t)\| \\
&\qquad + \left\|\int_0^t AS(t-s)[P(u)(s) - P(0)(s)]\,ds\right\| \\
&\qquad + \left\|\int_0^t C(t-s)[f(s, u_s + y_s, u'(s) + y'(s)) - f(s, y_s, y'(s))]\,ds\right\| \\
&\qquad + \|\Gamma(0)(t)\| \\
&\quad\le L_P(b, r)r + \ b\widetilde{N}_1 L_P(b, r)r \\
&\qquad + bNL_f(bK_b + 1)r + \|\Gamma(0)(t)\|
\end{aligned} \tag{2.8}$$

for all $0 \le t \le b$ and $\|u\|_1 \le r$. It is clear that in case (i) we can choose $r > 0$ large enough and $b > 0$ sufficiently small such that

$$(1 + b\widetilde{N}_1)L_P(b, r) + bNL_f(bK_b + 1) + \frac{\|\Gamma(0)(t)\|}{r} < 1. \tag{2.9}$$

Similarly, in case (ii), it follows from (2.7) that we can choose $r > 0$ sufficiently large such that (2.9) holds. Thus, in both cases, our assertion is consequence of the estimate (2.8). Moreover, the same type of estimate shows that Γ is a contraction mapping on $B_r(0, \mathcal{F}(b))$, which completes the proof. □

3. Applications. The literature for neutral differential equations with $x(t) \in \mathbf{R}^n$ is extensive (see [**8, 9, 18, 21, 28, 29, 31, 43, 44**] for a list of applications). In this case, our results are easily applicable. In fact, the operator A is a matrix of order $n \times n$ which generates the uniformly continuous cosine function $C(t) = \cosh(tA^{1/2})$. We take $Y = X = \mathbf{R}^n$ and condition (H1) is satisfied automatically. Moreover, we can estimate $\|S(t)\| \leq \|A\|^{-1/2} \sinh(t\|A\|^{1/2})$. Our next result is an immediate consequence of Theorem 2.1.

Proposition 3.1. *Suppose that assumptions* (H2), *with* $m_f \in L^\infty(I)$, (H3) *and condition* (b) *of Theorem* 2.1 *are fulfilled. Then the following properties hold*:

(i) *If* $\liminf_{\xi \to \infty} (W_f(\xi))/\xi < \infty$ *and* $L_P(t, r) \to 0$ *as* $t \to 0$, *for each* $r > 0$, *then there exists* $b > 0$ *and a mild solution of Problem* (1.1)–(1.3) *on* $[0, b]$.

(ii) *If* $\lim_{r \to \infty} L_P(b, r) = L_P(b)$, *for some* $0 < b \leq a$, *and*

$$(1 + b\|A\|^{1/2} \sinh(b\|A\|^{1/2}))L_P(b) + \sinh(b\|A\|^{1/2})(bK(b) + 1) \liminf_{\xi \to \infty} \frac{W_f(\xi)}{\xi} \int_0^b m_f(s)\, ds < 1,$$

then there exists a mild solution of Problem (1.1)–(1.3) *on* $[0, b]$.

On the other hand, the class of equations under consideration also has important applications. Abstract second order neutral functional differential equations arise, for instance, in the theory of heat conduction in materials with fading memory developed by Gurtin and Pipkin in [**19**]. For other applications of systems modeled by neutral hyperbolic partial differential equations with delay we refer to [**9, 28**], and for applications to control theory we mention [**10, 37**].

We next consider an example of a partial neutral differential equation with infinite delay. Initially we introduce the required technical framework. Let $X = L^2([0, \pi])$, and let $A : D(A) \subseteq X \to X$ be the linear map defined by $Af = f''$, where $D(A) = \{f \in X : f'' \in X, f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbf{R}}$ on X. Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbf{N}$, with corresponding eigenvectors $z_n(\xi) = (2/\pi)^{1/2}\sin(n\xi)$, and the following three properties hold.

(a) The set $\{z_n : n \in \mathbf{N}\}$ is an orthonormal basis of X and $A\varphi = -\sum_{n=1}^{\infty} n^2\langle\varphi, z_n\rangle z_n$, for $\varphi \in D(A)$.

(b) For $\varphi \in X$, we have $C(t)\varphi = \sum_{n=1}^{\infty}\cos(nt)\langle\varphi, z_n\rangle z_n$. It follows from this expression that $S(t)\varphi = \sum_{n=1}^{\infty}(\sin(nt)/n)\langle\varphi, z_n\rangle z_n$, which implies that the operator $S(t)$ is compact for all $t \in \mathbf{R}$ and that $\|C(t)\| = \|S(t)\| = 1$, for all $t \in \mathbf{R}$.

(c) If Φ is the group of translations on X defined by $\Phi(t)x(\xi) = \widetilde{x}(\xi + t)$, where $\widetilde{x}(\cdot)$ is the extension of $x(\cdot)$ with period 2π, then $C(t) = (\Phi(t) + \Phi(-t))/2$ and $A = B^2$, where B is the infinitesimal generator of Φ and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$ (see [**17**] for details). In particular, we observe that the inclusion $\iota : E \to X$ is compact.

To model our system, we shall use the phase space $\mathcal{B} = C_r \times L^2(\rho, X)$, with $r = 0$, defined in the Example 1.1.

We consider the partial neutral differential equation

$$\begin{aligned}(3.1)\quad \frac{\partial^2}{\partial t^2}\left[u(t,\tau) - \int_{-\infty}^{t}\int_0^{\pi} b(t-s,\eta,\tau)u(s,\eta)\,d\eta\,ds\right] &= \frac{\partial^2}{\partial \tau^2}u(t,\tau)\\ &+ \int_{-\infty}^{t} a_1(t-s)u(s,\tau)\,ds + \int_0^{\pi} a_2\left(\tau,\eta,\frac{\partial}{\partial t}u(t,\eta)\right)d\eta\end{aligned}$$

for $t \in I = [0,a]$, $\tau \in J = [0,\pi]$, with boundary conditions

$$(3.2)\qquad u(t,0) = u(t,\pi) = 0, \quad t \in I,$$

and initial conditions
(3.3)

$$u(s,\tau) = \varphi(s,\tau), \qquad \frac{\partial}{\partial t}u(0,\tau) = z(\tau), \quad s \in (-\infty, 0],\ \tau \in [0,\pi],$$

where we assume that $\varphi \in \mathcal{B}$, with the identification $\varphi(s)(\tau) = \varphi(s,\tau)$, $\varphi(0,\cdot) \in H^1([0,\pi])$ and $z \in X$.

We define the functions $f : \mathcal{B} \times X \to X$ and $g : \mathcal{B} \to X$ by

$$g(\psi)(\tau) = \int_{-\infty}^{0} \int_{0}^{\pi} b(-s, \eta, \tau)\psi(s, \eta)\, d\eta\, ds,$$

$$f(\psi, \xi)(\tau) = \int_{-\infty}^{0} a_1(s)\psi(s, \tau)\, ds + \int_{0}^{\pi} a_2(\tau, \eta, \xi(\eta))\, d\eta.$$

We assume that the functions $a_1(\cdot), a_2(\cdot)$ and $b(\cdot)$ satisfy the following five conditions:

(i) The functions $b(\theta, \eta, \zeta)$, $(\partial/\partial\zeta)b(\theta, \eta, \zeta)$, $(\partial^2/\partial\zeta^2)\, b(\theta, \eta, \zeta)$ are continuous on $(-\infty, 0] \times I \times J$; for every $(\theta, \eta) \in (-\infty, 0] \times I$, $b(\theta, \eta, \pi) = b(\theta, \eta, 0) = 0$ and

$$L_g = \max\left\{ \left[\int_0^{\pi} \int_{-\infty}^{0} \int_0^{\pi} \frac{1}{\rho(s)} \left(\frac{\partial^i b(s, \eta, \tau)}{\partial \tau^i} \right)^2 d\eta\, ds\, d\tau \right]^{1/2} : i = 0, 1, 2 \right\} < \infty.$$

(ii) The function $(\partial/\partial\theta)b(\theta, \eta, \zeta)$ exists and

$$\int_{-\infty}^{0} \int_0^{\pi} \frac{1}{\rho(s)} \left(\frac{\partial}{\partial s} b(s, \eta, \tau) \right)^2 d\eta\, ds < \infty.$$

(iii) The functions $(\partial^2/\partial\zeta\partial\theta)b(\theta, \eta, \zeta)$ and $(\partial/\partial\zeta)b(0, \eta, \zeta)$ exist, and we have

$$\int_0^{\pi} \int_{-\infty}^{0} \int_0^{\pi} \frac{1}{\rho(s)} \left(\frac{\partial^2}{\partial\tau\partial s} b(s, \eta, \tau) \right)^2 d\eta\, ds\, d\tau < \infty,$$

$$\int_0^{\pi} \int_0^{\pi} \left(\frac{\partial}{\partial\tau} b(0, \eta, \tau) \right)^2 d\eta\, d\tau < \infty.$$

(iv) The function $a_1(\cdot)$ is continuous and $L_f = (\int_{-\infty}^{0} (a_1^2(\theta)/\rho(\theta))\, d\theta)^{1/2} < \infty$.

(v) The function $a_2 : J \times J \times \mathbf{R} \to \mathbf{R}$ satisfies the Carathéodory conditions, $a_2(\cdot, \cdot, 0) \in L^2(J \times J)$, and there exists a positive function $\gamma \in L^2(J \times J)$ such that

$$|a_2(\tau, \eta, \alpha) - a_2(\tau, \eta, \beta)| \leq \gamma(\tau, \eta)|\alpha - \beta|.$$

As a consequence of conditions (iv) and (v), the function $f(t,\psi,x)$ is independent of t and uniformly Lipschitz continuous at the variable (ψ,x). Moreover, g is a bounded linear operator and, using condition (i), we can show that $g(\cdot)$ is $D(A)$-valued and that $Ag:\mathcal{B}\to X$ is a bounded linear operator with $\|g(\cdot)\|_{\mathcal{L}([D(A)],X)}\le L_g$. Since $\varphi(0)\in E$, we take $Y=E$. It follows from the introduction that if $\iota:Y\hookrightarrow X$ is the inclusion, then $\|\iota(x)\|\le\|x\|_E$ as well as the function $t\mapsto AS(t)$ is uniformly continuous into $\mathcal{L}(Y,X)$ and $\|AS(t)\|_{\mathcal{L}(Y,X)}\le 1$.

On the other hand, the function g also satisfies (H3). In fact, for every function $x:(-\infty,a]\to X$ given by $x(t)(\tau)=w(t,\tau)$ such that x is continuous on $[0,a]$, we have that the derivative

$$\frac{d}{dt}g(x_t)=\int_0^\infty\int_0^\pi\frac{\partial b(s,\eta,\tau)}{\partial s}w(t-s,\eta)\,d\eta\,ds+\int_0^\pi b(0,\eta,\tau)w(0,\eta)\,d\eta$$

exists. In particular,

$$\begin{aligned}\frac{d}{dt}g(x_t)|_{t=0}&=\int_0^\infty\int_0^\pi\frac{\partial b(s,\eta,\tau)}{\partial s}\varphi(-s,\eta)\,d\eta\,ds\\&\quad+\int_0^\pi b(0,\eta,\tau)\varphi(0,\eta)\,d\eta\\&=\Lambda,\end{aligned}$$

exists and it is independent of x. In addition, using the notations introduced in Section 2, the map P given by

$$P(w(\cdot,\tau))(t)=\frac{d}{dt}g(w_t(\cdot,\tau)+y_t)=\frac{d}{dt}g(w_t(\cdot,\tau))+\frac{d}{dt}g(y_t)$$

for $w\in\mathcal{F}(a)$, is an affine mapping. Employing condition (iii) we see that $P(w)(t)\in E$, the function $P(w)(\cdot)\in C(I,E)$, and $P:\mathcal{F}(a)\to C(I,E)$ verifies a uniform Lipschitz condition

$$\sup_{0\le t\le b}\|P(v)(t)-P(w)(t)\|_E\le L_P(b)\|v-w\|_1,$$

for $v,w\in\mathcal{F}(a)$, where $L_P(b)\to 0$ as $b\to 0$.

As consequence of these remarks, the neutral system (3.1)–(3.3) can be written in the abstract form (1.1)–(1.33) and the following result is obtained directly from Theorem 2.2.

Proposition 3.2. *If conditions* (i)–(v) *hold, then there exists a unique mild solution of system* (3.1)–(3.3) *on an interval* $[0, b]$, *for some* $b > 0$.

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