A DIRICHLET ANALOGUE OF THE FREE MONOGENIC INVERSE SEMIGROUP VIA MÖBIUS INVERSION

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ABSTRACT. The MÖbius inversion formula of the free monogenic inverse semigroup is represented by the MÖbius function for Cauchy product. In this short note we describe a Dirichlet analogue of this inverse semigroup.

1. Introduction. The Cauchy product and the Dirichlet product are familiar convolutions of arithmetical functions. The corresponding MÖbius functions \( \mu_C \) and \( \mu_D \) (as convolution inverses of the zeta function) are the following ones:

\[
\mu_C(n) = \begin{cases} 
1 & \text{if } n = 0 \\
-1 & \text{if } n = 1 \\
0 & \text{if } n > 1 
\end{cases}
\]

\[
\mu_D(n) = \begin{cases} 
1 & \text{if } n = 1 \\
(-1)^k & \text{if } n = p_1 \cdots p_k \text{ where } p_i \text{ are distinct primes} \\
0 & \text{if } p^2 \mid n \text{ for some prime } p 
\end{cases}
\]

The reduced standard division category \( C_F(S) \) of an inverse monoid \( S \) relative to an idempotent transversal \( F \) of the \( D \)-classes of \( S \) with \( 1 \in F \), \( \text{Ob} C_F(S) = F \); \( \text{Hom} (e,f) = \{(s,e) \in S \times F | s^{-1}e \leq e, es^{-1} = f \} \); \( e \xrightarrow{(s,e)} f \xrightarrow{(t,f)} g = e \xrightarrow{(t,f)(s,e)^{-1}} (t,e) \) is a MÖbius category if and only if \( S \) is combinatorial and \( (E(S), \subseteq) \) is locally finite (see [10]). If an inverse semigroup \( S \) is without identity it may be converted into an inverse monoid by adjoining an identity. We call the MÖbius inversion formula of such MÖbius category \( C_F(S) \), the MÖbius inversion formula of the inverse monoid (semigroup) \( S \). In [11] there are given MÖbius inversion formulas of some inverse semigroups \( S \):

\[
F,G : \mathbb{N} \longrightarrow \mathbb{C}, \quad F(n) = \sum_{i=0}^{n} G(i) \Leftrightarrow G(n) = \sum_{i=0}^{n} \mu_C(n-i)F(i)
\]
is the M"obius inversion formula of the bicyclic semigroup $B$. The bicyclic semigroup is a combinatorial, $E$–unitary and bisimple inverse monoid. From [10, 11] the classical M"obius inversion formula

(1.4) $F, G: \mathbb{N}^* \rightarrow \mathbb{C}, F(n) = \sum_{d|n} G(d) \Leftrightarrow G(n) = \sum_{d|n} \mu_d \left( \frac{n}{d} \right) F(d)$

is the M"obius inversion formula of the inverse monoid $B_M = \mathbb{N}^* \times \mathbb{N}^*$ with the multiplication

(1.5) $(a, b) \cdot (c, d) = \left( \frac{ma}{b}, \frac{md}{c} \right),$

where $m$ is the least common multiple of $b$ and $c$. The multiplicative (or Dirichlet) analogue $B_M$ of the bicyclic semigroup $B$ is combinatorial, $E$–unitary and bisimple like $B$. In [11] it was shown that the M"obius inversion formula of the free monogenic inverse semigroup $I$ (using Scheiblich's [9] representation with identity adjoined) is the following one:

$$F, G: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{C}$$

$$F(a, b) = \sum_{u=0}^{a} \sum_{v=0}^{b} G(u, v) \iff$$

(1.6) $$G(a, b) = \sum_{u=0}^{a} \sum_{v=0}^{b} \mu_C(a - u) \mu_C(b - v) F(u, v)$$

The free monogenic inverse semigroup is $E$-unitary, combinatorial and completely semisimple.

Starting with the M"obius category of the integer affine maps, the purpose of this paper is to give a Dirichlet analogue $I_D$ (via the M"obius inversion formula (1.6)) different from the standard Dirichlet analogue, namely the multiplicative analogue of the free monogenic inverse semigroup. This “new” Dirichlet analogue $I_D$ is combinatorial, $E$–unitary and completely semisimple inverse semigroup like the free monogenic inverse semigroup. It arises as the Leech monoid of a triangular bimorphic division category.

2. The M"obius category of integer affine maps. A M"obius category is a decomposition-finite category $C$ (i.e., a small category in
which any morphism \( \alpha \) has only finitely many nontrivial factorizations 
\( \alpha = \beta \gamma \) such that an incidence function \( \xi : \text{Mor} \ C \to \mathbf{C} \) has a

corollary inverse if and only if \( \xi(\alpha) \neq 0 \) for each identity morphism 
\( \alpha \). The convolution \( \xi * \eta \) of two incidence functions \( \xi \) and \( \eta \) is defined by

\[
(\xi * \eta)(\alpha) = \sum_{\beta \gamma = \alpha} \xi(\beta)\eta(\gamma).
\]

A triangular category is a special M"obius category. A category \( C \) is
called triangular if the set \( \text{Ob}C \) of objects of \( C \) is the set of non-negative
integers \( \mathbb{N} \) (or the set of positive integers \( \mathbb{N}^* \)) and the cardinalities
\( A(k, n) = |\text{Hom}(\langle k, n \rangle)| \) of sets of morphisms constitute a triangular
family of numbers:

\[
A(n, n) = 1 \text{ for all } n; \quad A(k, n) = 0 \text{ if } k > n.
\]

For further details on M"obius categories and triangular categories, we refer the reader to \([1, 6]\).

Now, let \( C_{\alpha} \) be the category of integer affine maps defined by:

- \( \text{Ob}C_{\alpha} = \mathbb{N}^* \);
- \( \{\text{Hom}(\langle k, n \rangle) = \{f, k \mid f : \mathbb{R} \to \mathbb{R}, f(x) = ax + b \text{ with } a \in \mathbb{N}^*, b \in \mathbb{N} \text{ and } f(k) = n\} \};
- The composition \( (g, n)(f, k) \) of two morphisms \( (f, k) : k \to n \) and
  \( (g, n) : n \to m \) is given by: \( (g, n)(f, k) = (g \circ f, k) \), where \( g \circ f \) is the
usual composition of maps.

Straightforward verification shows that:

**Proposition 1.** \( C_{\alpha} \) is a triangular category in which every morphism
is a bimorphism (with the corresponding family of triangular numbers:
\( A(k, n) = [n/k] \), where \([n/k]\) is the integer part of \( n/k \)).

**Proposition 2.** The convolution \( \xi * \eta \) of two incidence functions,
\( \xi, \eta : \text{Mor} C_{\alpha} \to \mathbf{C} \) is given by

\[
(\xi * \eta)(ax + b, k) = \sum_{u|a} \sum_{v=0}^{[ub/a]} \xi\left(\frac{a}{u}x + b - \frac{a}{u}v, uk + v\right)\eta(ux + v, k).
\]
Proof. Let $(ax + b, k) : k \to n$ be a morphism of $C_a$ and $m \in \mathbb{N}^*$ such that $k \leq m \leq n$. Given a morphism $(ux + v, k) : k \to m$, there exists (a necessarily unique) morphism $(sx + t, m) : m \to n$ such that $(ax + b, k) = (sx + t, m)(ux + v, k)$ if and only if $u \mid a$ and $v \leq (ub)/a$. Then $s = a/u$ and $t = b - (a/u)v$. Thus the convolution $\xi \ast \eta$ of two incidence functions $\xi$ and $\eta$ of $C_a$ is given by (2.3). □

An (abstract) division category $D$ is a small category with pushouts and a quasi initial object $I$ (i.e., an object $I$ with at least one morphism into each object of $D$) in which every morphism is an epimorphism (see [2, page 268]). We have:

**Proposition 3.** The category $C_a$ is a division category.

Proof. 1 is a quasi initial object and the following square is a pushout

\[
\begin{array}{ccc}
k & \xrightarrow{(f_2-a_2x+b_2,k)} & n \\
(f_1-a_1x+b_1,k) & \downarrow & (g_2-c_2x+d_2,n_2) \\
n_1 & \xleftarrow{(g_1-c_1x+d_1,n_1)} & n
\end{array}
\]

(2.4)

where $c_1 = m/a_1$, $c_2 = m/a_2$, $m = l.c.m\{a_1, a_2\}$, $n = \max\{c_1n_1, c_2n_2\}$ and $d_1 = n - c_1n_1$, $d_2 = n - c_2n_2$. □

In what follows we shall evaluate the Möbius function of the triangular-division category $C_a$.

If $C$ is a Möbius category, the convolution inverse $\xi^{-1}$ of an incidence function $\xi$ of $C$ with $\xi(\alpha) \neq 0$ for each identity morphism $\alpha$, is given recursively by:

(2.5)

\[
\xi^{-1}(\alpha) = \begin{cases} 1/\xi(\alpha) & \text{if } \alpha \text{ is an identity} \\ -\xi^{-1}(1_{\text{Cdom } \alpha}) \sum_{\beta_1=\alpha \gamma \neq \alpha} \xi(\beta)\xi^{-1}(\gamma) & \text{otherwise} \end{cases}
\]

(the convolution identity is $\delta$, where $\delta(\alpha) = 1$ if $\alpha$ is an identity morphism and $\delta(\alpha) = 0$ otherwise).
The Möbius function $\mu$ is the convolution inverse of the incidence function $\zeta$ defined by: $\zeta(\alpha) = 1$ for each morphism $\alpha$ of the Möbius category $C$.

**Proposition 4.** The Möbius function of $C_\alpha$ is given by

$$\mu(ax + b, k) = \begin{cases} 
\mu_D(a) & \text{if } b = 0 \\
-\mu_D(a) & \text{if } b = a \\
0 & \text{otherwise}
\end{cases}$$

**Proof.** In $C_\alpha$, for $\xi = \zeta$, formula (2.5) becomes

$$\mu(ax + b, k) = \begin{cases} 
1 & \text{if } a = 1, b = 0 \\
- \sum_{u|b,a\neq a}^{[ub/a]} \sum_{v=0}^{b-1} \mu(ux + v, k) - \sum_{v=0}^{b-1} \mu(ax + v, k) & \text{otherwise.}
\end{cases}$$

Now, using (2.7) we prove (2.6) by induction on the positive integer $a$. If $a = 1$, it is easy to see that (2.6) holds for $b = 0, 1, 2$. Assume $b > 2$ and $\mu(x + b_1, k) = 0$ for $b_1 = 2, \ldots , b - 1$. Then:

$$\mu(x + b, k) = - \sum_{b_1=0}^{b-1} \mu(x + b_1, k) = -\mu(x, k) - \mu(x + 1, k) = 0.$$  

If $a \geq 2$, it is useful to consider the following cases: $b = 0, 0 < b < a$, $b = a$ and $b > a$. In each case it is easy to check that (2.6) holds.  

3. **The Leech inverse monoid of $C_\alpha$.** The category $C_\alpha$ is a Möbius category and a division category. We call such categories $MD$ categories (Möbius-division categories, see [12]). A $MD$ category is a small category $M$ with the following properties:

(i) $M$ has pushouts;

(ii) $M$ has a quasi initial object $I$;

(iii) Every morphism of $M$ is an epimorphism;

(iv) The identity morphisms are indecomposable (i.e., $1 = \beta \gamma \implies \beta = \gamma = 1$)
(v) $M$ is decomposition-finite.

If $M$ is an MD-category with quasi initial object $I$, then

$$L(M, I) = \{(\alpha, \beta) \in \text{Mor} \times \text{Mor} \mid \text{Dom} \alpha = \text{Dom} \beta = I, \text{Codom} \alpha = \text{Codom} \beta\}$$

with the multiplication

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\gamma_1 \alpha_1, \gamma_2 \beta_2),$$

where $[\beta_1, \alpha_2, \gamma_1, \gamma_2]$ is a pushout,

is the Leech inverse monoid of $M$.

**Theorem 1** [12]. Every MD-category $M$ with a quasi initial object $I$ is isomorphic to the reduced standard division category $C_{F}(L(M, I))$ for any idempotent transversal $F$ of the D-classes of $L(M, I)$ with $(1, 1) \in F$.

For us, the significance of this theorem follows from the fact that the forthcoming Leech inverse monoid $I_D \cong L(C_\alpha, 1)$ arises from an MD category, namely, $C_\alpha$. Thus the reduced standard division category $C_{F}(I_D)$ is isomorphic to $C_\alpha$. So, the Möbius inversion formula of the inverse monoid $I_D$ is the Möbius inversion formula of $C_\alpha$.

If in $C_\alpha$ the inner square

$$
\begin{array}{ccc}
1 & \xrightarrow{g_2} & f_2(1) = g_2(1) \\
\downarrow{f_1} & & \downarrow{q} \\
\bullet & \xrightarrow{p} & f_1(1) = g_1(1)
\end{array}
$$

is a pushout, then the set of integer affine maps pairs $\{(f, g) \mid f(1) = g(1)\}$ with the multiplication

$$(f_1, g_1)(f_2, g_2) = (p \circ f_1, q \circ g_2)$$

is the Leech inverse monoid $L(C_\alpha, 1)$. We can now easily check (taking into account (2.4), that the Leech inverse monoid $L(C_\alpha, 1)$ is isomorphic to $I'$ given by:

$$I' = \{(a, b, a', b') \in \mathbb{N}^4 \mid a \neq 0, a' \neq 0, a + b = a' + b'\},$$
\[(a, b, a', b')(c, d, c', d')
= \begin{cases} 
\left(\frac{ma}{a'}, \frac{mb}{b' - \frac{a'}{a}}, m \left(\frac{a'}{a} + \frac{b'}{c} - \frac{d}{e} \right)\right) & \text{if } \frac{b'}{a'} > \frac{d}{e} \\
\left(\frac{ma}{a'}, \frac{b}{a'} + \frac{d}{e}, \frac{m}{c'}, \frac{md}{c'} \right) & \text{if } \frac{b'}{a'} \leq \frac{d}{e} 
\end{cases}
\]

where \(m = \text{l.c.m.}(a', c)\). By \((u, r, v, s) \rightarrow (u, v, r, s)\), as in \([9]\), and by an adjustment of (3.6), we obtain the following isomorphic copy of \(L(C_\alpha, 1)\):

**Proposition 5.** The Leech inverse monoid \(L(C_\alpha, 1)\) of the MD-category of integer affine maps is isomorphic to the inverse monoid \(I_D\) given by

\[(3.7)\quad I_D = \{(a, b, a', b') \in \mathbb{N}^* \times \mathbb{N}^* \mid a + a' = b + b'\}\]

\[(3.8)\quad (a, b, a', b')(c, d, c', d')
= \left(\frac{ma}{b}, \frac{md}{c}, m' + \frac{md'}{b} - \frac{mb'}{b}, m' + \frac{md'}{c} - \frac{mc'}{c}\right)
\]

where \(m = \text{l.c.m.}(b, c)\) and \(m' = \max(\frac{mb'}{b}, \frac{mc'}{c})\).

Now, the Möbius inversion formula of \(I_D\) is the Möbius inversion formula of \(C_\alpha\). For two incidence functions \(\xi, \eta : \text{Mor} C_\alpha \rightarrow \mathbb{C}\) we have \(\xi = \zeta * \eta\) if and only if \(\eta = \mu * \xi\). That is,

\[(3.9)\quad \xi(ax + b, k) = \sum_{u|a} \sum_{v=0}^{[u(b)/a]} \eta(ux + v, k)\]

if and only if

\[(3.10)\quad \eta(ax + b, k) = \sum_{u|a} \sum_{v=0}^{[u(b)/a]} \mu \left(\frac{a}{u}x + b - \frac{a}{u}v, uk + v\right) \xi(ux + v, k).\]

By (2.6) it follows that (3.9) holds if and only if

\[(3.11)\quad \eta(ax + b, k) = \sum_{u|a; a|ub} \mu_D \left(\frac{a}{u}x + \frac{ub}{a}, k\right) - \sum_{u|a; a|u(bb' = 0)} \mu_D \left(\frac{a}{u}x + \frac{ub}{a} - 1, k\right).\]
So, we obtain the following M"obius inversion formula. Given functions \( F, G : \mathbb{N}^* \times \mathbb{N} \to \mathbb{C} \),

\[
F(a, b) = \sum_{u \mid a} \sum_{v=0}^{\lfloor ub/a \rfloor} G(u, v)
\]

for all \((a, b) \in \mathbb{N}^* \times \mathbb{N}\), if and only if

\[
G(a, b) = \begin{cases} 
\sum_{u \mid a} \mu_D \left( \frac{a}{u} \right) F(u, 0) & \text{if } b = 0 \\
\sum_{u \mid a \mid ab} \mu_D \left( \frac{a}{u} \right) \left[ F \left( u, \frac{ub}{a} \right) - F \left( u, \frac{ub}{a} - 1 \right) \right] & \text{if } b \neq 0.
\end{cases}
\]

By an adjustment of (3.13) we have:

**Theorem 2.** (The M"obius inversion formula for \( I_D \).) Given functions \( F, G : \mathbb{N}^* \times \mathbb{N} \to \mathbb{C} \),

\[
F(a, b) = \sum_{u \mid a} \sum_{v=0}^{\lfloor ub/a \rfloor} G(u, v)
\]

for all \((a, b) \in \mathbb{N}^* \times \mathbb{N}\), if and only if

\[
G(a, b) = \sum_{u \mid a \mid ab} \sum_{v=0}^{ub/a} \mu_D \left( \frac{a}{u} \right) \mu_C \left( \frac{ub}{a} - v \right) F(u, v).
\]

We say that the inverse semigroup \( I_D \) is a Dirichlet analogue of the free monogenic inverse semigroup \( I \) via M"obius inversion. The M"obius functions and the M"obius inversion formulas of these semigroups have a pronounced similarity as we can see in the table at the end.

By [10, Theorem 3.3, Corollary 3.4 and Theorem 4.1], as for \( I \), it follows

**Corollary 1.** The Dirichlet analogue \( I_D \) of \( I \) is a combinatorial completely semisimple \( E \)-unitary inverse monoid.
The free monogenic inverse semigroup $I$ (Scheiblich’s [9] representation with identity adjoined)

The Dirichlet analogue $I_D$

The multiplicative analogue $I_M$

$I = \{(a, b, a', b') \in \mathbb{N}^4 \mid a + a' = b + b'\},$

$(a, b, a', b') \in \{(c, d, c', d') \in \mathbb{N}^4 \mid a + a' = b + b', c + c' = d + d'\},$

$m^1 = \max(b, c), m^1' = \max(b', c'),$

where $m = \max(b,c), m' = \max(b',c').$

The Möbius function of $I$:

$$\mu : \mathbb{N} \times \mathbb{N} \to \mathbb{C},$$

$$\mu(a,b) = \begin{cases} \mu_{\max}(a) \mu_{\max}(b), & \text{if } a = 0, \\ -\mu_{\max}(a) \mu_{\max}(b), & \text{if } b = 1, \\ 0, & \text{otherwise} \end{cases}$$

The Möbius inversion formula for $I$:

$$F, G : \mathbb{N} \times \mathbb{N} \to \mathbb{C},$$

$$F(a,b) = \sum_{u=0}^{a-1} \sum_{v=0}^{b-1} G(u,v) \iff G(a,b) = \sum_{u=0}^{a-1} \sum_{v=0}^{b-1} \mu_{\max}(a) \mu_{\max}(b) F(u,v).$$

The Möbius function of $I_D$:

$$\mu : \mathbb{N} \times \mathbb{N} \to \mathbb{C},$$

$$\mu(a,b) = \begin{cases} \mu_{\max}(a) \mu_{\max}(b), & \text{if } b = 0, \\ -\mu_{\max}(a) \mu_{\max}(b), & \text{if } b = 1, \\ 0, & \text{otherwise} \end{cases}$$

The Möbius inversion formula for $I_D$:

$$F, G : \mathbb{N} \times \mathbb{N} \to \mathbb{C},$$

$$F(a,b) = \sum_{u=0}^{a-1} \sum_{v=0}^{b-1} G(u,v) \iff G(a,b) = \sum_{u=0}^{a-1} \sum_{v=0}^{b-1} \mu_{\max}(a) \mu_{\max}(b) F(u,v).$$

The Möbius function of $I_M$:

$$\mu : \mathbb{N} \times \mathbb{N} \to \mathbb{C},$$

$$\mu(a,b) = \begin{cases} \mu_{\max}(a) \mu_{\max}(b), & \text{if } b = 0, \\ -\mu_{\max}(a) \mu_{\max}(b), & \text{if } b = 1, \\ 0, & \text{otherwise} \end{cases}$$

The Möbius inversion formula for $I_M$:

$$F, G : \mathbb{N} \times \mathbb{N} \to \mathbb{C},$$

$$F(a,b) = \sum_{u=0}^{a-1} \sum_{v=0}^{b-1} G(u,v) \iff G(a,b) = \sum_{u=0}^{a-1} \sum_{v=0}^{b-1} \mu_{\max}(a) \mu_{\max}(b) F(u,v).$$

Remark 1. The multiplicative analogue of $I$, denoted $I_M$:

$$I_M = \{(a, b, a', b') \in \mathbb{N}^4 \mid aa' = bb'\}$$

with multiplication:

$$(a, b, a', b')(c, d, c', d') = \left(\frac{ma}{b}, \frac{md}{c}, \frac{m'a'}{b'}, \frac{m'd'}{c'}\right),$$

where $m = \text{l.c.m.}(b, c)$ and $m' = \text{l.c.m.}(b', c')$, is also a Leech inverse monoid arises from an $MD$ category, namely, the truly standard division category $D(\mathbb{N}^*)$ of the multiplicative monoid of all positive integers, $\mathbb{N}^*$. Since this is triangular and binorphic (cancellative), the multiplicative analogue is also a combinatorial, completely semisimple.
and $E$-unitary inverse semigroup. We can obtain, in a way similar to
the above, the M"obius function and the M"obius inversion formula of
$I_M$. We included them in the table on the previous page where the
central place is allocated for the Dirichlet analogue $I_D$.

Acknowledgments. The authors would like to thank the referee
for very valuable suggestions.

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