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DELOCALIZED BETTI NUMBERS AND MORSE TYPE INEQUALITIES

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ABSTRACT. In this paper we state and prove delocalized Morse type inequalities for Morse functions as well as for closed differential 1-forms. These inequalities involve delocalized Betti numbers. As an immediate consequence, we prove the vanishing of delocalized Betti numbers of manifolds fibering over the circle under a vanishing condition on the delocalizing conjugacy class.

1. Introduction. Given a manifold M and a real Morse function f on M the following Morse inequalities establish relations between the topology of M and the number of critical points of order j denoted by C_j (cf. [**7**])

$$C_k - C_{k-1} + \cdots \pm C_0 \geq \beta^k - \beta^{k-1} + \cdots \pm \beta^0.$$

Here $\beta^j = \dim H^j(M, \mathbf{R})$ is the j-th Betti number of M. These relations have been the subject of many significant generalizations. Novikov and Shubin have proved in [**9**] that these inequalities hold if the Betti numbers are replaced by the L^2-Betti numbers. The L^2-Betti numbers (or von Neumann Betti numbers) were introduced by Atiyah in his investigation on equivariant index theorem (see [**1**]). The Morse theory for closed 1-forms has been introduced by Novikov and he has proved in [**8**] that the Morse inequalities can be generalized to closed 1-forms if one replaces the Betti numbers by the so-called Novikov numbers. In [**3**, Theorem 1] it is shown that the Novikov-Shubin inequalities hold as well for closed 1-forms. In this paper we are interested in the delocalized Betti numbers which were introduced by Lott in [**5**]. These delocalized Betti numbers are not yet well studied and enjoy properties which are not satisfied by the ordinary or L^2-Betti numbers, e.g., the delocalized Betti numbers of any manifold with

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free abelian fundamental group and of hyperbolic manifolds vanish. In this paper we show that some appropriate combinations of delocalized Betti numbers satisfy the Novikov-Shubin and the Farber inequalities (see Theorems 3 and 5). We prove the delocalized Novikov-Shubin inequalities following Roe's account [**11**] of the Witten approach to Morse theory [**12**]. As a consequence of this method we reprove the vanishing of the delocalized Euler character of M. To prove the delocalized Morse inequalities for closed 1-forms we assume that closed 1-form vanishes on the delocalizing conjugacy class. Then we use the Morse inequalities for Morse functions and a method invented by .Farber. As a consequence of the Morse inequalities for closed 1-forms, it turns out that the delocalized Betti numbers of a fibration $M \xrightarrow{p} S^1$ vanish provided that the conjugacy class belongs to the kernel of $p_* : \pi_1(M) \to \pi_1(S^1)$. This vanishing theorem for L^2-Betti numbers was conjectured by Gromov and proved by Lück in [**6**].

In Section 2 we state and prove the Morse inequalities in a very general analytic framework. In Section 3 we use these inequalities to prove the Morse inequalities for Morse functions and delocalized Betti numbers. In these two sections we follow closely the methods used by Roe in [**11**, Chapter 14]. We use the main theorem of Section 3 to prove the Morse inequalities for closed 1-forms and delocalized Betti numbers in Section 4.

2. General analytic Morse inequalities. Let (M, g) be a closed, oriented Riemannian manifold, and denote by Δ^k the Laplacian operator acting on differential k-forms on M. Let $(\widetilde{M}, \widetilde{g})$ be the Riemannian universal covering of M with $\widetilde{g} = \pi^* g$, where π is the covering map. We denote by G the fundamental group of M and by $\widetilde{\Delta}^k = d^*d + dd^*$ the Laplacian operator acting on L^2-elements of $\Omega^k(\widetilde{M})$. For $0 \leq k \leq n$, let T_k be a real valued non-negative continuous trace on the space of all smoothing G-invariant operators on $L^2(\widetilde{M}, \Lambda^k T\widetilde{M})$. The continuity is understood with respect to the uniform convergence of Schwartz kernels on compact subsets of $\widetilde{M} \times \widetilde{M}$. We usually omit the subfix k and denote these traces by the same symbol T. Let $\widetilde{P}^k$ denote the orthogonal projection on $\ker \widetilde{\Delta}^k$ which is a smoothing operator on $L^2(\widetilde{M}, \Lambda^k T\widetilde{M})$, cf., [**1**]. We define the k-th T-Betti number by the following relation

$$\beta^k_{\mathrm{T}} := \mathrm{T}(\widetilde{P}^k). \tag{2.1}$$

For ϕ a rapidly decreasing non-negative smooth function on $\mathbf{R}^{\geq 0}$ with $\phi(0) = 1$, the operator $\phi(\widetilde{\Delta}^k)$ is a smoothing operator and so we can define $\mu^k_{\mathrm{T}} = \mathrm{T}(\phi(\widetilde{\Delta}^k))$. Notice that β^k_T and μ^k_T are both non-negative real numbers.

Theorem 1 (analytic Morse inequalities). *For $0 \leq k \leq n = \dim M$ we have the following inequalities*

$$\mu^k_{\mathrm{T}} - \mu^{k-1}_{\mathrm{T}} + \cdots \pm \mu^0_{\mathrm{T}} \geq \beta^k_{\mathrm{T}} - \beta^{k-1}_{\mathrm{T}} + \cdots \pm \beta^0_{\mathrm{T}},$$

and the equality holds for $k = n$.

Proof. Let $\{\phi_m\}_m$ be a sequence of non-negative rapidly decreasing smooth functions on $\mathbf{R}^{\geq 0}$ which converges to zero outside of $0 \in \mathbf{R}$ and $\phi_m(0) = 1$. The operators $\phi_m(\widetilde{\Delta}^k)$ are non-negative smoothing operators with smooth Schwartz kernel K_m. The sequence of kernels K_m converges uniformly on compact subsets of $\widetilde{M} \times \widetilde{M}$ to the kernel $\widetilde{K}$ of the projection $\widetilde{P}^k$. In fact the spectral theorem for self-adjoint operator $\widetilde{\Delta}^k$ (see, e.g., [**10**, Theorem VIII. 5]) implies that $\phi_m(\widetilde{\Delta}^k)\omega \to \widetilde{P}^k\omega$ for any L^2-differential k-form ω, i.e., $\phi_m(\widetilde{\Delta}^k) \to \widetilde{P}^k$ weakly in $\mathcal{L}(\mathcal{D}, \mathcal{D}')$. Since the strong and the weak topology of $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ coincide on bounded subsets, we conclude the convergence $\phi_m(\widetilde{\Delta}^k) \to \widetilde{P}^k$ in the topological space $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ with strong topology. The Schwartz kernel theorem asserts that this topological space is isomorphic to $\mathcal{D}'(\widetilde{M} \times \widetilde{M}, \Lambda^k T^*\widetilde{M} \times \Lambda^k T^*\widetilde{M})$. With respect to this isomorphism, the convergence $\phi_m(\widetilde{\Delta}^k) \to \widetilde{P}^k$ means the convergence of the kernels $K_m \to \widetilde{K}$ which implies the assertion. Therefore by continuity of T we obtain

$$\beta^k_T = \lim_{m\to\infty} \mathrm{T}(\phi_m(\widetilde{\Delta}^k)). \tag{2.2}$$

The function $\phi - \phi_m$ is non-negative, rapidly decreasing and vanishing at 0, so it takes the form $x\psi^2_m(x)$ where ψ_m is non-negative and rapidly decreasing. We have

$$\begin{aligned}\mathrm{T}(dd^*\psi^2_m(\widetilde{\Delta}^k)) &= \mathrm{T}(\psi_m(\widetilde{\Delta}^k)d\, d^*\psi_m(\widetilde{\Delta}^k))\\ &= \mathrm{T}(d^*\psi^2_m(\widetilde{\Delta}^k)\, d)\\ &= \mathrm{T}(d^*d\psi^2_m(\widetilde{\Delta}^{k-1})).\end{aligned}$$

In the last step we have used the commutation relation $\widetilde{\Delta}^k\, d = d\, \widetilde{\Delta}^{k-1}$. Therefore

$$(\mu_{\mathrm{T}}^k - T(\phi_m(\widetilde{\Delta}^k)) - (\mu_{\mathrm{T}}^{k-1} - T(\phi_m(\widetilde{\Delta}^{k-1})) + \cdots \pm (\mu_{\mathrm{T}}^0 - T(\phi_m(\widetilde{\Delta}^0)) = T(d^* d\, \psi_m(\widetilde{\Delta}^k)^2)).$$

Since $d^* d\, \psi_m(\widetilde{\Delta}^k)^2$ is a non-negative smoothing operator, the right side of the above equality is non-negative for $k < n$ and is zero for $k = n$. By tending m toward infinity and using relation (2.2) we get the desired inequalities. □

3. Delocalized Novikov-Shubin inequalities. We recall the definition of *delocalized Betti numbers* as they are introduced by Lott in [**5**]. We keep the notation of the previous section. Let $\widetilde{P}$ be a smoothing G-invariant operator acting on $\Lambda^k T^* \widetilde{M}$ with Schwartz kernel $\widetilde{K}$ which is rapidly decreasing far from the diagonal of $\widetilde{M} \times \widetilde{M}$. For each $\widetilde{x} \in \widetilde{M}$ and $h \in G$ one can identify both $\Lambda^k T^*_{\widetilde{x}} \widetilde{M}$ and $\Lambda^k T^*_{h.\widetilde{x}} \widetilde{M}$ with $\Lambda^k T^*_x M$ where $x = \pi(\widetilde{x})$. Henceforth $\widetilde{K}(\widetilde{x}, h.\widetilde{x})$ can be considered as an element of $\mathrm{End}\,(\Lambda^k T^*_x M)$. Consequently, given a conjugacy class $\langle g \rangle$ of G the sum $\sum_{h \in \langle g \rangle} \widetilde{K}^k(\widetilde{x}, h.\widetilde{x})$ is finite and, as a function of $\widetilde{x} \in \widetilde{M}$, is invariant with respect to the action of G. Therefore this function pushes down and defines a smooth section $K_{\langle g \rangle}$ of the bundle $\mathrm{End}\,(\Lambda^k(T^* M))$ over M. The following relation defines the delocalized trace $\mathrm{Tr}_{\langle g \rangle}$

$$\mathrm{Tr}_{\langle g \rangle}(\widetilde{P}) := \int_M \mathrm{tr}\, K_{\langle g \rangle}\, d\mu_g. \tag{3.1}$$

As in the previous section, let $\widetilde{\Delta}^k$ denote the Laplacian operator acting on L^2-sections of $\Lambda^k T\widetilde{M}$. The orthogonal projection $\widetilde{P}^k$ on $\ker(\widetilde{\Delta}^k)$ is a smoothing G-invariant operator on $L^2(\widetilde{M}, \Lambda^k T\widetilde{M})$ with kernel $\widetilde{K}^k$. The k-th delocalized Betti number $\beta^k_{\langle g \rangle}$ is defined as follows

$$\beta^k_{\langle g \rangle} = \mathrm{Tr}_{\langle g \rangle}(\widetilde{P}^k). \tag{3.2}$$

Equivalently if ϕ_m is a sequence of real functions as in Theorem 1, e.g., $\phi_m(x) = e^{-x/m}$, then by (2.2)

$$\beta^k_T = \lim_{m \to \infty} \mathrm{Tr}_{\langle g \rangle}(\phi_m(\widetilde{\Delta}^k)). \tag{3.3}$$

Remark 1. Notice that the trace $\mathrm{Tr}_{\langle g\rangle}$ is not always finite since it is not known weather the kernel $\widetilde{K}_{\langle g\rangle}$ is rapidly decreasing far from the diagonal of $\widetilde{M}\times\widetilde{M}$. Nevertheless if $\langle g\rangle$ is a finite set, then the above delocalized Betti numbers are well defined. This is why we restrict ourself, from now on, to the class $\mathcal{C}(G)$ of finite conjugacy classes.

Since the delocalized traces $\mathrm{Tr}_{\langle g\rangle}$ are not in general non-negative, we cannot apply Theorem 1 to these traces (except for $k = n$). Instead we consider the following linear functional which are introduced in [**4**] and are proved to be non-negative traces

$$\mathrm{T}_{\langle g\rangle} := \mathrm{Tr}_{\langle e\rangle} + \frac{1}{|\langle g\rangle|}\mathrm{Tr}_{\langle g\rangle} \tag{3.4}$$

Lemma 2. *The linear functional* $\mathrm{T}_{\langle g\rangle}$ *is a non-negative trace on the space of the* G*-invariant smoothing operator.*

Proof. For each $g\in G$ the linear functional $\mathrm{Tr}_{\langle g\rangle}$ is a trace (cf., [**5**, Lemma 2]) so $\mathrm{T}_{\langle g\rangle}$ is also a trace. Below we show that it is non-negative, i.e., $\mathrm{T}_{\langle g\rangle}(\widetilde{P})\geq 0$ if $\widetilde{P}=\widetilde{Q}^*\widetilde{Q}$ where $\widetilde{Q}$ is a G invariant smoothing operator on $H := L^2(\widetilde{M},\Lambda^kT^*\widetilde{M})$. Let $\{\theta_k\}_{k\in\mathbf{N}}$ be an orthonormal basis for H, and for $h\in G$ put $(h.\theta)(\widetilde{x}) := \theta(h.\widetilde{x})$. We identify M with a distinguished fundamental domain of G in $\widetilde{M}$. With this identification we have

$$\begin{aligned}
\mathrm{Tr}_{\langle e\rangle}(\widetilde{P}) &= \sum_k \int_M \langle \widetilde{Q}\theta_k\,(\widetilde{x}), \widetilde{Q}\theta_k\,(\widetilde{x})\rangle\, d\mu_{\widetilde{g}} \\
&= \left(\sum_k \int_M \langle \widetilde{Q}\theta_k\,(\widetilde{x}), \widetilde{Q}\theta_k\,(\widetilde{x})\rangle\, d\mu_{\widetilde{g}}\right)^{1/2} \\
&\quad\times \left(\sum_k \int_M \langle \widetilde{Q}\theta_k\,(\widetilde{x}), \widetilde{Q}\theta_k\,(\widetilde{x})\rangle\, d\mu_{\widetilde{g}}\right)^{1/2} \\
&= \left(\sum_k \int_M \langle \widetilde{Q}\theta_k\,(\widetilde{x}), \widetilde{Q}\theta_k\,(\widetilde{x})\rangle\, d\mu_{\widetilde{g}}\right)^{1/2} \\
&\quad\times \left(\sum_k \int_M \langle \widetilde{Q}\theta_k\,(h.\widetilde{x}), \widetilde{Q}\theta_k\,(h.\widetilde{x})\rangle\, d\mu_{\widetilde{g}}\right)^{1/2}
\end{aligned}$$

$$\geq \sum_k \left(\int_M \langle \widetilde{Q}\theta_k\,(\widetilde{x}), \widetilde{Q}\theta_k\,(\widetilde{x})\rangle \, d\mu_{\widetilde{g}} \right)^{1/2} \\ \times \left(\int_M \langle \widetilde{Q}\theta_k\,(h.\widetilde{x}), \widetilde{Q}\theta_k\,(h.\widetilde{x})\rangle \, d\mu_{\widetilde{g}} \right)^{1/2}$$

Here the third equality follows from the G-invariance of the Riemannian volume element, while the last inequality is the Cauchy-Schwarz inequality for L^2-sequences. The following pairing is a symmetric bilinear function on H

$$\langle \omega, \eta \rangle := \int_M \langle \omega(\widetilde{x}), \eta(\widetilde{x})\rangle \, d\mu_{\widetilde{g}}.$$

By applying the Cauchy-Schwarz inequality arising from this bilinear function to the last expression in the above, we obtain

$$\mathrm{Tr}_{\langle e\rangle}(\widetilde{P}) \geq \left| \sum_k \int_M \langle \widetilde{Q}\theta_k\,(\widetilde{x}), \widetilde{Q}\theta_k\,(\widetilde{h}.x)\rangle \, d\mu_{\widetilde{g}} \right|.$$

This inequality with the following relation

$$\mathrm{Tr}_{\langle g\rangle}(\widetilde{P}) = \sum_{h\in\langle g\rangle} \sum_k \int_M \langle \widetilde{Q}\theta_k\,(\widetilde{x}), \widetilde{Q}\theta_k\,(\widetilde{h}.x)\rangle \, d\mu_{\widetilde{g}}$$

implies the following relation which proves the assertion of the lemma

$$|\langle g\rangle| \mathrm{Tr}_{\langle e\rangle}(\widetilde{P}) \geq |\mathrm{Tr}_{\langle g\rangle}(\widetilde{P})|. \qquad \square$$

Using the trace $\mathrm{T}_{\langle g\rangle}$ we define the following combination of delocalized Betti numbers

$$\gamma^k_{\langle g\rangle} := \mathrm{T}_{\langle g\rangle}(\widetilde{P}^k) = \beta^k_{\langle e\rangle} + \frac{1}{|\langle g\rangle|}\beta^k_{\langle g\rangle}.$$

Now let f be a Morse function on M, and denote by $\widetilde{f}$ its lifting to $\widetilde{M}$. For $s > 0$ put $d_s := e^{-s\widetilde{f}} d e^{s\widetilde{f}}$ and $d^*_s := e^{s\widetilde{f}} d^* e^{-s\widetilde{f}}$ and define the

Witten deformed Laplacian, acting on L^2-elements of $\Omega^k(\widetilde{M})$, by the following relation

$$\widetilde{\Delta}_s^k : d_s d_s^* + d_s^* d_s.$$

This deformed Laplacian is a perturbation of the Laplacian $\widetilde{\Delta}^k$ by differential operators of order zero. So it is an elliptic second order differential operator and its L^2-kernel consists of smooth differential k-forms and the projection on this kernel is a smoothing operator, cf., [**1**]. Just as above we can define the deformed k-th delocalized Betti number $\beta^k_{\langle g\rangle}(s)$. In fact these deformed delocalized Betti numbers $\beta^k_{\langle g\rangle}(s)$ are independent of s. To see that consider the deformed de-Rham complex of L^2-differential forms

$$\cdots \longrightarrow \Omega^k(T^*\widetilde{M}) \xrightarrow{d_s^k} \Omega^{k+1}(T^*\widetilde{M}) \longrightarrow \cdots .$$

This complex is equivariant with respect to the action of the group G, so the k-th cohomology vector space of this complex $H_s^k(\widetilde{M}, \mathbf{R})$ is a G-vector space. This real G-vector space is isomorphic to the kernel of $\widetilde{\Delta}_s^k$. The isomorphism associates to an element in $\ker \widetilde{\Delta}_s^k$ its class in $H_s^k(\widetilde{M}, \mathbf{R})$ and is clearly G-equivariant. Moreover the above deformed complex is isomorphic to the ordinary de-Rham complex (corresponding to $s = 0$) through conjugation with $e^{s\widetilde{f}}$ which is G-equivariant as well. We conclude that there is a G-equivariant isomorphism between $\ker \widetilde{\Delta}_s^k$ and $\ker \widetilde{\Delta}^k$. Since the action of G on L^2-differential form is symmetric, this implies that the orthogonal projections on $\ker \widetilde{\Delta}_s^k$ and on $\ker \widetilde{\Delta}^k$ are G-similar. Therefore by taking the $\mathrm{Tr}_{\langle g\rangle}$ we conclude that the deformed Betti numbers are actually independent of s. In particular

$$\gamma^k_{\langle g\rangle}(s) = \gamma^k_{\langle g\rangle}; \quad 0 \le k \le n. \tag{3.5}$$

These relations and Theorem 1 applied to the deformed Laplacian imply the following inequalities (equality holds for $k = n$)

$$\mu^k_{\langle g\rangle}(s) - \mu^{k-1}_{\langle g\rangle}(s) + \cdots \pm \mu^0_{\langle g\rangle}(s) \ge \gamma^k_{\langle g\rangle} - \gamma^{k-1}_{\langle g\rangle} + \cdots \pm \gamma^0_{\langle g\rangle}, \tag{3.6}$$

where $\mu^k_{\langle g\rangle}(s) := \mathrm{T}_{\langle g\rangle}\phi(\widetilde{\Delta}_s^k)$. We shall study the behavior of $\mu^k_{\langle g\rangle}(s)$ when s goes to infinity in order to prove the following theorem.

Theorem 3 (delocalized Novikov-Shubin inequalities). *Let f be a Morse function on M, and denote by C_k the number of critical points of Morse index k. For $0 \leq k \leq n = \dim M$, we have the following inequalities*

$$C_k - C_{k-1} + \cdots \pm C_0 \geq \gamma^k_{\langle g \rangle} - \gamma^{k-1}_{\langle g \rangle} + \cdots \pm \gamma^0_{\langle g \rangle}, \tag{3.7}$$

and the equality holds for $k = n$.

Proof. At first we recall from [**11**] that the deformed Laplacian has the following form

$$\widetilde{\Delta}^k_s = \widetilde{\Delta}^k + sL_0 + s^2 |d\widetilde{f}|^2 \tag{3.8}$$

where L_0 is a zeroth order operator and $|d\widetilde{f}|$ is the endomorphism given by the multiplication by $|d\widetilde{f}|$. There is a positive constant C such that $|d\widetilde{f}(\widetilde{x})| \geq C$ when $\widetilde{x}$ is outside the union $\widetilde{U}_r$ of the r-neighborhoods of critical points of $\widetilde{f}$. Here r is sufficiently small so that the $4r$-neighborhoods of critical points are disjoint. For the proof of the following lemma we refer to [**11**, Lemma 12.10] where the proof is given for compact manifolds but remain true for non-compact manifolds as well.

Lemma 4. *Let ϕ be a rapidly decreasing function as above such that the Fourier transform of the function ψ defined by $\psi(t) := \phi(t^2)$ is supported in $(-r, r)$. Let $\widetilde{K}$ denote the kernel of the smoothing operator $\phi(\widetilde{\Delta}^k_s)$. Then $\widetilde{K}(.,.)$ tends uniformly to zero on $\widetilde{M} \setminus \widetilde{U}_{2r} \times \widetilde{M} \setminus \widetilde{U}_{2r}$ when s goes to infinity.*

On the other hand, the Schwartz kernel of $\phi(\widetilde{\Delta}^k_s)$ is supported in the distance r of the diagonal of $\widetilde{M} \times \widetilde{M}$. So if ω is a differential k-form which is supported within the distance $2r$ of a critical point of $\widetilde{f}$, then $\phi(\widetilde{\Delta}^k_s)\omega$ is supported within the distance $3r$ of the same critical point. To see this, let $\widetilde{D}_s := d_s + d^*_s$; then $\widetilde{\nabla}_s = (\widetilde{D}_s)^2$ and by the condition on the support of $\widehat{\psi}$ we have

$$\phi(\widetilde{\Delta}^k_s)\, \omega(\widetilde{x}) = \psi(\widetilde{D}_s)\, \omega(\widetilde{x}) = \int_{-r}^{r} \widehat{\psi}(t) e^{-it\widetilde{D}_s} \omega\,(\widetilde{x})\, dt. \tag{3.9}$$

This relation and the unit propagation speed property for the deformed Dirac operator $\widetilde{D}_s$ imply that $\phi(\widetilde{\Delta}_s^k)\omega$ is supported within distance $3r$ of the support of ω. Since the action of G on $\widetilde{M}$ is uniformly properly discontinuous, for r sufficiently small and for a non-trivial element $h \in G$, the element $(\widetilde{x}, h.\widetilde{x})$ is not in the distance $2r$ of the diagonal of $\widetilde{M} \times \widetilde{M}$. Consequently the previous lemma and the above discussion show that for a non-trivial conjugacy class $\langle g \rangle$ the delocalized traces $\mathrm{Tr}_{\langle g \rangle}$ have no contribution to the value of $\lim \mathrm{T}_{\langle g \rangle}\phi(\widetilde{\Delta}_s^k)$ when s goes to infinity, so

$$\lim_{s\to\infty} \mu^k_{\langle g \rangle}(s) = \lim_{s\to\infty} \mathrm{Tr}_{\langle e \rangle}\phi(\widetilde{\Delta}_s^k). \tag{3.10}$$

Now we shall to prove the following equality which prove the desired inequalities of the theorem

$$\lim_{s\to\infty} \mathrm{Tr}_{\langle e \rangle}\phi(\widetilde{\Delta}_s^k) = C_k. \tag{3.11}$$

For this purpose Let $\widetilde{\beta} = p^*\beta$ be a G-invariant smooth function on $\widetilde{M}$ which is supported in $\widetilde{U}_{3r}$ and is equal to 1 on $\widetilde{U}_{2r}$. Lemma 4 shows that

$$\lim_{s\to\infty} \mathrm{Tr}_{\langle e \rangle}\phi(\widetilde{\Delta}_s^k) = \lim_{s\to\infty} \mathrm{Tr}_{\langle e \rangle}(\widetilde{\beta}\phi(\widetilde{\Delta}_s^k)), \tag{3.12}$$

where $\widetilde{\beta}$ is the pointwise multiplication by $\widetilde{\beta}$. So, the next step is to study the asymptotic behavior of $\mathrm{Tr}_{\langle e \rangle}(\widetilde{\beta}\phi(\widetilde{\Delta}_s^k))$ when s goes to infinity. Since the kernel of $\phi(\widetilde{\Delta}_s^k)$ is supported in the distance r of the diagonal, the differential forms which are supported outside $\widetilde{U}_{4r}$ have no contribution in the value of the expression in the right hand side of (3.12). So to evaluate the value of this expression we can consider only those differential forms which are supported in $\widetilde{U}_{4r}$. In fact Lemma 4 and the condition on the support and values of β show that we may consider those differential forms which are supported in $\widetilde{U}_{3r}$. As for $\widetilde{\Delta}_s^k$, the kernel of $\phi(\Delta_s^k)$ is supported in the distance r of the diagonal of $M \times M$. So for r sufficiently small one can lift the smoothing operator $\phi(\Delta_s^k)$ to $\widetilde{M}$. We have the following equality

$$\phi(\widetilde{\Delta}_s^k)\,\omega = p^*\phi(\Delta_s^k)\,\omega; \qquad \mathrm{supp}\,(\omega) \subset \widetilde{U}_{3r}. \tag{3.13}$$

To prove this equality, by relations (3.9), it suffices to show that

$$e^{-it\widetilde{D}_s}\,\omega = p^*(e^{-itD_s})\,\omega; \quad -r \le t \le r.$$

It is clear that we may assume that the support of ω is included in a small ball of radius $3r$. For $|t| \le r$ both sides of the above equality define smooth differential forms which are solutions of the wave equation $\partial_t + i\widetilde{D}_s = 0$. Moreover they have the same initial condition ω for $t = 0$. Therefore by uniqueness of wave operator we get the desired equality. Consequently

$$\mathrm{Tr}_{\langle e\rangle}(\widetilde{\beta}\phi(\widetilde{\Delta}_s^k)) = \mathrm{Tr}_{\langle e\rangle} p^*(\beta\phi(\Delta_s^k)) = \mathrm{Tr}\,(\beta\phi(\Delta_s^k)). \tag{3.14}$$

Now the argument leading to relation (3.12) can be applied to Δ_s^k to deduce

$$\lim_{s\to\infty} \mathrm{Tr}\,(\beta\phi(\Delta_s^k)) = \lim_{s\to\infty} \mathrm{Tr}\,\phi(\Delta_s^k) = C_k.$$

The last equality is the main step of the analytical proof of the Morse inequalities for the Morse function f on M, cf., [**11**, page 192]. This equality and the above discussion prove the relation (3.11). This complete the proof of the theorem. □

Inequality (3.7) can be written in the following form

$$C_k - C_{k-1} + \cdots \pm C_0 \ge \beta_{\langle e\rangle}^k - \beta_{\langle e\rangle}^{k-1} + \cdots \pm \beta_{\langle e\rangle}^0 + B_{\langle g\rangle}^k \tag{3.15}$$

where

$$B_{\langle g\rangle}^k := \frac{1}{|\langle g\rangle|}(\beta_{\langle g\rangle}^k - \beta_{\langle g\rangle}^{k-1} + \cdots \pm \beta_{\langle g\rangle}^0).$$

Inequalities (3.15) without the term $B_{\langle g\rangle}^k$ at the right hand side are the Morse inequalities for L^2-Betti numbers established by Novikov and Shubin [**9**]. The proof of Theorem 3 can be applied to the localized trace $\mathrm{Tr}_{\langle e\rangle}$ and provides an analytical proof for the Novikov-Shubin inequalities.

Remark 2. It is clear from its proof that when $k = n$ the equality of Theorem 1 holds even if the trace T is not real. If we apply this equality to $\mathrm{Tr}_{\langle g\rangle}$ and follow the proof of Theorem 3 we get the vanishing of the delocalized Euler character

$$\chi_{\langle g\rangle}(M) := \beta_{\langle g\rangle}^n - \beta_{\langle g\rangle}^{n-1} \cdots \pm \beta_{\langle g\rangle}^0 = 0.$$

Of course this result can be proved by the local properties of heat equation [**2**, Chapter 2].

4. Delocalized Morse inequalities for closed 1-forms. In this section we use the main theorem of the previous section to prove a delocalized version of the Morse inequalities for closed 1-forms. Following [**3**] we then use these inequalities to prove the vanishing of the delocalized Betti numbers for spaces which fiber over the circle. Vanishing of the L^2-Betti numbers of such spaces was conjectured by Gromov and proved by Lück [**6**].

Let ω be a closed differential 1-form on M. In a small open subset U one has $\omega = df_U$, where f_U is a smooth function on U uniquely determined up to an additive constant. A point $p \in U$ is a non-degenerate critical point of ω with index j if it is a non-degenerate critical point of f_U with index j. As in the previous section we denote by C_j the number of these points. Since ω is closed the map

$$\gamma \xrightarrow{\xi} \int_\gamma \omega \tag{4.1}$$

defines a homeomorphism between the fundamental groups G and the additive group $(\mathbf{R}, +)$.

As in the previous section let C_j denote the number of critical points of index j. The following theorem reduces to Theorem 3 if ω is an exact form.

Theorem 5 (delocalized Morse inequalities for closed 1-forms). *Let ω be a closed Morse 1-form on M and $\langle g\rangle$ a finite conjugacy class in $G = \pi_1(M)$. If ω vanishes on $\langle g\rangle$, i.e., $\xi(g) = 0$, then for $0 \le k \le n = \dim M$ the following inequalities hold*

$$C_k - C_{k-1} + \cdots \pm C_0 \ge \gamma^k_{\langle g\rangle} - \gamma^{k-1}_{\langle g\rangle} + \cdots \pm \gamma^0_{\langle g\rangle}, \tag{4.2}$$

and the equality takes place for $k = n$.

Proof. For $m \in \mathbf{Z}^{\ge 0}$, define the normal subgroup G_m of G by $G_m := \xi^{-1}(m\mathbf{Z})$. Let M_m denote the corresponding cyclic m-sheeted normal covering space of M with $\pi_1(M_m) = G_m$. Denote by ω_m the lifting of ω to M_m. We have clearly

$$C_j(\omega_m) = m.C_j(\omega).$$

The relevance of these covering spaces is that the number of critical points of ω_m can be approximated by the number of critical points of an exact form on M_m. More precisely there is a constant C, independent of m, and an exact Morse 1-form $\widetilde{\omega}_m$ on M_m such that

$$C_j(\omega_m) \leq C_j(\widetilde{\omega}_m) \leq C_j(\omega_m) + C; \quad \text{for } 0 \leq j \leq n. \tag{4.3}$$

For the proof of this fact we refer to [**3**, Section 2]. The Morse theory for the exact 1-form $\widetilde{\omega}_m = d\widetilde{f}_m$ is exactly the Morse theory for the Morse function $\widetilde{f}_m$. Notice that the linear map ξ on $\langle g \rangle$ is constant, so $\xi(g) = 0$ implies the inclusion $\langle g \rangle \subset G_m$ for $m \in \mathbf{Z}$; in other words, $\langle g \rangle \subset \pi_1(M_m)$. Therefore in relation (3.1) we can (instead of M) integrate over the m-sheeted covering M_m to get a non-negative trace $\mathrm{T}_{\langle g \rangle, m}$ on the space of G-invariant smoothing operators on $\widetilde{M}$. Let P be a smoothing G-invariant operator on $\widetilde{M}$ with kernel K. Fix m elements of G which represent the classes in the quotient G/G_m. The symbol k stands for each one of these elements. After identifying M_m and M with fundamental domains in $\widetilde{M}$, we have

$$\begin{aligned}
\mathrm{T}_{\langle g \rangle, m}(P) &= \int_{M_m} \sum_{h \in \langle g \rangle} K(x, h.x) \\
&= \sum_k \int_{k.M} \sum_{h \in \langle g \rangle} K(x, h.x) \\
&= \sum_k \int_M \sum_{h \in \langle g \rangle} K(k.x, kh.x) \\
&= m. \sum_k \int_M \sum_{h \in \langle g \rangle} K(x, h.x).
\end{aligned}$$

Therefore $\mathrm{T}_{\langle g \rangle, m}(P) = m.\mathrm{T}_{\langle g \rangle}(P)$ which implies that the delocalized Betti numbers defined by $\mathrm{T}_{\langle g \rangle, m}$ are m-times the delocalized Betti numbers defined by $\mathrm{T}_{\langle g \rangle}$. The arguments leading to Theorem 3 can be applied to this situation as well and the resulting Morse inequalities take the following form (with equality for $k = n$)

$$C_k(\widetilde{\omega}_m) - C_{k-1}(\widetilde{\omega}_m) + \cdots \pm C_0(\widetilde{\omega}_m) \geq m.\gamma^k_{\langle g \rangle} - m.\gamma^{k-1}_{\langle g \rangle} + \cdots \pm m.\gamma^0_{\langle g \rangle}.$$

Using these inequalities and relations (4.3), we get

$$\begin{aligned}\sum_{j=0}^{k}(-1)^{k-j}C_j(\omega) &= \frac{1}{m}\sum_{j=0}^{k}(-1)^{k-j}C_j(\omega_m)\\ &\geq \frac{1}{m}\sum_{j=0}^{k}(-1)^{k-j}C_j(\widetilde{\omega}_m) - \frac{k+1}{m}\,C\\ &\geq \sum_{j=0}^{k}(-1)^{k-j}\gamma^j_{\langle g\rangle} - \frac{k+1}{m}\,C.\end{aligned}$$

Now, taking the limit when m goes to infinity, we obtain the claim of the theorem. □

Corollary 6 (Vanishing theorem). *Let $M \xrightarrow{p} S^1$ be a fibration, and let $\langle g\rangle$ be a finite conjugacy class in $\pi_1(M)$. If $p_*(g) = 0 \in \pi_1(S^1)$, then the delocalized Betti numbers $\beta^p_{\langle g\rangle}$ vanish.*

Proof. The pull-back 1-form $\omega = p^*(d\theta)$ on M has no critical point. Applying the inequalities of Theorem 5 we obtain the following vanishing result for $0 \leq j \leq n$

$$\gamma^j_{\langle g\rangle}(M) = \beta^j_{\langle e\rangle} + \frac{1}{|\langle g\rangle|}\beta^j_{\langle g\rangle} = 0.$$

Since $\beta^j_{\langle e\rangle} = 0$ by the previously mentioned result of Lück, we conclude that $\beta^j_{\langle g\rangle} = 0$. □

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