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BOUNDARY DECOMPOSITION OF THE BERGMAN KERNEL

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ABSTRACT. We study the Bergman kernel on a domain having smooth boundary with several connected components, and relate it to the Bergman kernel of simpler domains having only some of these boundary components. Results both in one and several complex variables are obtained.

1. Introduction. The Bergman kernel has, in the past 50 years, become an important tool in the complex analysis of both one and several complex variables (see [**2, 5, 9**], for example). Its reproducing properties, its biholomorphic invariance, and its relationship to the Bergman metric are all of fundamental importance.

It is important to obtain concrete information about the Bergman kernel. That said, we must confess that it is generally quite difficult to obtain specific, calculable information about this kernel. On the disc, the ball and the polydisc, the kernel may be computed with an explicit formula (see [**5**]). Analogous work was performed on the bounded symmetric domains of Cartan in [**4**]. But, for more general domains, a formula is certainly not feasible; one might hope instead for an asymptotic expansion (see, for instance, [**2**] or [**9**]).

This paper explores a slightly different avenue for getting one's hands on the Bergman kernel of a domain. The general approach is perhaps best illustrated with an example. Let

$$\Omega = \{\zeta \in \mathbf{C} : 1 < |\zeta| < 2\}.$$

This is the annulus, and any explicit representation of its Bergman kernel will involve elliptic functions (see [**1**]). One might hope, however, to relate the Bergman kernel K_Ω of Ω to the Bergman kernels K_{Ω_1} and K_{Ω_2} of

$$\Omega_1 = \{\zeta \in \mathbf{C} : |\zeta| < 2\}$$

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and

$$\Omega_2 = \{\zeta \in \mathbf{C} : 1 < |\zeta|\}.$$

The first of these has an explicitly known Bergman kernel (see [**5**]), and the second domain is the inversion of a disc, so its kernel is known explicitly as well.

One could pose a similar question for domains of higher connectivity. The question also makes sense, with a suitable formulation, in several complex variables. Our purpose here is to come up with precise formulations of results such as these and to prove them. In one complex variables, we can make decisive use of classical results relating the Bergman kernel to the Green's function (see [**6**]). In several complex variables there are analogous results of Garabedian (see [**3**]) that will serve in good stead.

In Section 2 we introduce appropriate definitions and notation. In Section 3 we prove a basic, representative result in the plane. Section 4 proves a more general result in the plane. Section 5 treats the multi-dimensional result. Section 6 sums up the work.

2. Definitions and notation. If $\Omega \subseteq \mathbf{C}^n$ is a bounded domain, then we let $K_\Omega(z, \zeta)$ denote its Bergman kernel. This is the reproducing kernel for

$$A^2(\Omega) \equiv \{f \in L^2(\Omega) : f \text{ is holomorphic on } \Omega\}.$$

It is known, for planar domains, that $K_\Omega(z, \zeta)$ is related to the Green's function $G_\Omega(z, \zeta)$ for Ω by this formula:

$$K_\Omega(z, \zeta) = 4 \cdot \overline{\frac{\partial^2}{\partial\zeta\partial\overline{z}} G_\Omega(\zeta, z)}.$$

Of course it is essential for our analysis to realize that the Green's function is known quite explicitly on any given domain. If

$$\Gamma(\zeta, z) = \frac{1}{2\pi} \log |\zeta - z|$$

is the fundamental solution for the Laplacian (on all of $\mathbf{C}$), then we construct the Green's function as follows:

Given a bounded domain $\Omega \subseteq \mathbf{C}$ with smooth boundary, the *Green's function* is posited to be a function $G_\Omega(\zeta, z)$ that satisfies

$$G_\Omega(\zeta, z) = \Gamma(\zeta, z) - F_z^\Omega(\zeta),$$

where $F_z^\Omega(\zeta) = F^\Omega(\zeta, z)$ is a particular harmonic function in the ζ variable. It is mandated that F^Ω be chosen (and is in fact uniquely determined by the condition) so that $G(\,\cdot\,, z)$ vanishes on the boundary of Ω. One constructs the function $F^\Omega(\,\cdot\,, z)$, for each fixed z, by solving a suitable Dirichlet problem. Again, the reference [**5**, p. 40] has all the particulars. It is worth noting that the Green's function is a symmetric function of its arguments.

In our proof, we shall be able to exploit known properties of the Poisson kernel (see especially [**7**]) and of the solution to the Dirichlet problem (see [**8**]) to get the estimates that we need.

We shall first formulate and solve our problem for domains in the plane. Afterward we shall treat matters in higher-dimensional complex space.

3. A representative result. We first prove our main result for the domain

$$\Omega = \{\zeta \in \mathbf{C} : 1 < |\zeta| < 2\}.$$

This argument will exhibit all the key ideas–at least in one complex variable. The later exposition will be clearer because we took the time to treat this case carefully.

Let

$$\Omega_1 = \{\zeta \in \mathbf{C} : |\zeta| < 2\}$$

and

$$\Omega_2 = \{\zeta \in \mathbf{C} : 1 < |\zeta|\}.$$

For convenience in what follows, we let S_1 be the boundary curve of Ω_1 and S_2 the boundary curve of Ω_2. Of course it then follows that $\partial\Omega = S_1 \cup S_2$.

We claim that

$$K_\Omega(z,\zeta) = \frac{1}{2}\left[K_{\Omega_1}(z,\zeta) + K_{\Omega_2}(z,\zeta)\right] + \mathcal{E}(z,\zeta),$$

where $\mathcal{E}$ is an error term that is smooth on $\overline{\Omega \times \Omega}$. In particular, $\mathcal{E}$ is bounded with all derivatives bounded on that domain.

For the proof, we write

$$\begin{aligned}
&\frac{1}{8}\left[\overline{K_{\Omega_1}(z,\zeta) + K_{\Omega_2}(z,\zeta)}\right] \\
&\qquad = \frac{1}{2}\frac{\partial^2}{\partial\zeta\partial\overline{z}}\left[\big(\Gamma(\zeta,z) - F^{\Omega_1}(\zeta,z)\big) + \big(\Gamma(\zeta,z) - F^{\Omega_2}(\zeta,z)\big)\right] \\
&\qquad = \frac{\partial^2}{\partial\zeta\partial\overline{z}}\left(\Gamma(\zeta,z) - \frac{1}{2}\left[F^{\Omega_1}(\zeta,z) + F^{\Omega_2}(\zeta,z)\right]\right).
\end{aligned}$$

Now we claim that

$$F^{\Omega_1}(\zeta,z) + F^{\Omega_2}(\zeta,z) = 2F^{\Omega}(\zeta,z) + \mathcal{E}(z,\zeta)$$

for a suitable error term $\mathcal{E}$. We must analyze

$$G(\zeta,z) \equiv \left[F^{\Omega_1}(\zeta,z) + F^{\Omega_2}(\zeta,z)\right] - 2F^{\Omega}(\zeta,z).$$

We think of G as the solution of a Dirichlet problem on Ω, and we must analyze the boundary data. What we see is this:

• For z near S_1, F^Ω and F^{Ω_1} agree on S_1 (in the variable ζ) and equal 0. And F^{Ω_2} is smooth and bounded by $C \cdot |\log(1/2)|$, just by the form of the Green's function. All three functions are plainly smooth and bounded on S_2 (for z still near S_1) by similar reasoning. In conclusion, G is smooth and bounded on $\overline{\Omega}$ for z near S_1.

• For z near S_2, F^Ω and F^{Ω_2} agree on S_2 (in the variable ζ) and equal 0. And F^{Ω_1} is smooth and bounded by $C \cdot |\log(1/2)|$, just by the form of the Green's function. All three functions are plainly smooth and bounded on S_1 (for z still near S_2) by similar reasoning. In conclusion, G is smooth and bounded on $\overline{\Omega}$ for z near S_2.

• For z away from both S_1 and S_2–in the interior of Ω–it is clear that all the terms are bounded and smooth on $\partial\Omega$. So the solution G of the Dirichlet problem will also be smooth as desired.

As a result of these considerations, G is smooth on $\overline{\Omega}$.

That completes our argument and gives, altogether, the error term $\mathcal{E}$. Thus

$$F^{\Omega_1} + F^{\Omega_2} - 2F^{\Omega} = \mathcal{E}.$$

It follows that

$$\begin{aligned} \frac{1}{2}[K_{\Omega_1}(z,\zeta) + K_{\Omega_2}(z,\zeta)] &= 4\overline{\frac{\partial^2}{\partial\zeta\partial\overline{z}}\Big(\Gamma(\zeta,z) - F^{\Omega}(\zeta,z)\Big)} + \mathcal{E}' \\ &= K_{\Omega}(z,\zeta) + \mathcal{E}. \end{aligned}$$

4. The more general result in the plane. Now consider a smoothly bounded domain $\Omega \subseteq \mathbf{C}$ with k connected components in its boundary, $k \geq 2$. We denote the boundary components by $S_1, \ldots, S_k$; for specificity, we let S_1 be the component of the boundary that bounds the unbounded component of the complement of Ω. Let Ω_1 be the *bounded* region in the plane bounded by the single Jordan curve S_1. Let $\Omega_2, \ldots, \Omega_k$ be the unbounded regions bounded by $S_2, S_3, \ldots, S_k$, respectively.

Then we may analyze, just as in the last section, the expression

$$K_{\Omega} - \frac{1}{k}\left[K_{\Omega_1} + K_{\Omega_2} + \cdots + K_{\Omega_k}\right]$$

to obtain a smooth error term

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_k.$$

That completes our analysis of a smooth, finitely connected domain in the plane.

5. Domains in higher-dimensional complex space. The elegant paper [**3**] contains the necessary information about the relationship of the Bergman kernel and a certain Green's function in several complex variables so that we may carry out our program in that more general context.

Fix a smoothly bounded domain Ω in $\mathbf{C}^k$. Let $t = (t_1, \ldots, t_k)$ be a fixed point in Ω. Following Garabedian's notation, we set

$$r = \sqrt{\sum_{j=1}^{k} |z_j - t_j|^2}.$$

Let σ_k be constants chosen so that

$$\lim_{\varepsilon \to 0} \sigma_k \int_{\Gamma_\varepsilon} B \cdot \sum_{j=1}^{k} \frac{\partial r^{-2k+2}}{\partial z_j} \alpha_j \, d\sigma + B(t) = 0,$$

where Γ_ε is the sphere of radius ε about t, B is some continuous function, and $(\alpha_1, \ldots, \alpha_k)$ is a collection of complex-valued direction cosines.

Now set $\theta(z, t)$ to be that function

$$(*) \qquad \theta = \sigma_k r^{-2k+2} + \text{ regular terms}$$

on Ω so that

$$\sum_{j=1}^{k} \frac{\partial \theta}{\partial \overline{z}_j} \cdot \overline{\alpha}_j = 0$$

on $\partial\Omega$,

$$\frac{\partial}{\partial \overline{z}_j} \triangle \theta = 0$$

on Ω (for $j = 1, \ldots, k$) and such that

$$\int_{\Omega} \theta \overline{f} \, dV = 0,$$

for all functions f analytic in Ω. It follows from standard elliptic theory that such a θ exists.

In fact, according to [**3**], this function θ that we have constructed is a Green's function for the boundary value problem

$$\frac{\partial}{\partial \overline{z}_j} \triangle \beta = 0 \quad \text{on } \Omega, \ j = 1, \ldots, k$$
$$\sum_{j=1}^{k} \frac{\partial \beta}{\partial \overline{z}_j} \cdot \overline{\alpha}_j = 0 \quad \text{on } \partial\Omega.$$

Garabedian goes on to prove that the Bergman kernel for Ω is related to the Green's function θ in this way:

$$K_\Omega(z,t) = \triangle_z \theta(z,t).$$

This is just the information that we need to apply the machinery that has been developed here.

In order to flesh out the argument in the context of several complex variables, our primary task is to argue that our new Green's function has a form similar to the classical Green's function from one complex variable. But in fact this is immediate from equation $(*)$. It follows from this that the argument in Section 3 using the maximum principle will go through as before, and we may establish a version of the result in Sections 3 and 4 in the context of several complex variables. The theorem is this:

Theorem 1. *Let Ω be a smoothly bounded domain in $\mathbf{C}^n$ with boundary having connected components $S_1, S_2, \ldots, S_k$. For specificity, say that S_1 is the boundary component that bounds the unbounded portion of the complement of $\overline{\Omega}$. Let K_Ω be the Bergman kernel for Ω, let K_1 be the Bergman kernel for the bounded domain having S_1 as its single boundary element, and let K_j, for $j \geq 2$, be the Bergman kernel for the unbounded domain having S_j as its single boundary component. Then*

$$K_\Omega = K_1 + K_2 + \cdots + K_k + \mathcal{E},$$

where $\mathcal{E}$ is an error term that is bounded with bounded derivatives.

The reader can see that this new theorem is completely analogous to the results of Sections 3 and 4 in the one variable setting. But it must be confessed that this theorem is something of a *canard*. For, when $j \geq 2$, any function holomorphic on the unbounded domain with boundary S_j will (by the Hartogs extension phenomenon) extend analytically to all of $\mathbf{C}^n$. And of course there are no L^2 holomorphic functions on all of $\mathbf{C}^n$. So it follows that $K_j \equiv 0$. So the theorem really says that

$$K_\Omega = K_1 + \mathcal{E}.$$

This is an interesting fact, but not nearly as important or provocative as the one-variable result. The one other point worth noting is that the statement of the result is now a bit different from that in one complex variable, just because we are dealing with a different Green's function for a different boundary value problem. Basically what we are seeing is that $K_2, \ldots, K_k$ do not count at all, and K_1 is the principal and only term.

6. Concluding remarks. It is always a matter of interest to find means to get control of the Bergman kernel of any domain. This paper offers a simple device–more meaningful in the one-variable context than in the several-variable context–for doing so. In practice, asymptotic expansions seem to be the most powerful device for getting hard analytic information about a Bergman kernel. The decomposition presented here could be the first step in such an expansion.

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