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FILIFORM NILSOLITONS OF DIMENSION 8

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ABSTRACT. A Riemannian manifold (M, g) is said to be *Einstein* if its Ricci tensor satisfies $\operatorname{ric}(g) = cg$, for some $c \in \mathbf{R}$. In the homogeneous case, a problem that is still open is the so-called *Alekseevskii conjecture*. This conjecture says that any homogeneous Einstein space with negative scalar curvature (i.e., $c < 0$) is a *solvmanifold*: a simply connected solvable Lie group endowed with a left invariant Riemannian metric. The aim of this paper is to classify Einstein solvmanifolds of dimension 9 whose nilradicals are *filiform* (i.e., $(n-1)$-step nilpotent and n-dimensional).

1. Introduction. A Riemannian manifold (M, g) is said to be *Einstein* if its Ricci tensor satisfies $\operatorname{ric}(g) = cg$, for some $c \in \mathbf{R}$. Einstein metrics are often considered as the nicest, or most distinguished metrics on a given differentiable manifold (see for instance [**1**, Introduction]).

In the homogeneous case, a problem that is still open is the so-called *Alekseevskii conjecture* (see [**1**, 7.57]). This conjecture says that any homogeneous Einstein space with negative scalar curvature (i.e., $c < 0$) is a *solvmanifold*: a simply connected solvable Lie group endowed with a left invariant Riemannian metric. It is important to note that, nowadays, it is still unknown which solvable Lie groups admit a left invariant Einstein metric.

In [**6**], Lauret has proved that any Einstein solvmanifold S is *standard* (i.e., $[\mathfrak{a}, \mathfrak{a}] = 0$, where $\mathfrak{a} := [\mathfrak{s}, \mathfrak{s}]^{\perp}$, $\mathfrak{s}$ the Lie algebra of S). The study of standard Einstein solvmanifolds has been reduced to the rank-one case, that is, $\dim \mathfrak{a} = 1$, where strong structural and uniqueness results are well known (see [**3**]).

A nilpotent Lie algebra $\mathfrak{n}$ is called an *Einstein nilradical* if it is the nilradical (i.e., maximal nilpotent ideal) of the Lie algebra of an Einstein solvmanifold. It is proved in [**4**] that $\mathfrak{n}$ is an Einstein nilradical

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if and only if $\mathfrak{n}$ admits an inner product $\langle\cdot,\cdot\rangle$ such that

$$\text{Ric}_{\langle\cdot,\cdot\rangle} = cI + \phi, \quad c \in \mathbf{R},\ \phi \in \text{Der}(\mathfrak{n}), \tag{1}$$

where $\text{Ric}_{\langle\cdot,\cdot\rangle}$ denotes the Ricci operator of the nilpotent Lie group N endowed with the left invariant metric determined by $\langle\cdot,\cdot\rangle$. These metrics are called *nilsolitons*, as they are Ricci soliton metrics: solutions of the Ricci flow which evolves only by scaling and the action of diffeomorphisms. Nilsolitons are unique up to isometry and scaling, and $\mathfrak{s}$ is completely determined by the Lie algebra $\mathfrak{n} = [\mathfrak{s},\mathfrak{s}]$. So the study of Einstein solvmanifolds is actually a problem on nilpotent Lie algebras (see the survey [**7**] for further information).

In [**13**], it is proved that any nilpotent Lie algebra of dimension ≤ 6 is an Einstein nilradical. A first obstruction for a nilpotent Lie algebra to be an Einstein nilradical is that it has to admit an $\mathbf{N}$-*gradation* (i.e., $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_r$ such that $[\mathfrak{n}_i,\mathfrak{n}_j] \subset \mathfrak{n}_{i+j}$) (see [**3**]). This condition is necessary but it is not sufficient, the first examples of $\mathbf{N}$-graded Lie algebra which are not Einstein nilradicals were found in [**8**] and some of them are 7-dimensional.

A nilpotent Lie algebra $\mathfrak{n}$ is said to be *filiform* if $\dim\mathfrak{n} = n$ and $\mathfrak{n}$ is $(n-1)$-step nilpotent. The classification of filiform Einstein nilradicals for $\dim\mathfrak{n} \leq 7$ has been obtained in [**8**].

The aim of this paper is to classify nilpotent filiform Lie algebras of dimension 8 which are Einstein nilradicals. After some preliminaries in Section 2, we will begin our study in Section 3 by classifying nilpotent filiform $\mathbf{N}$-graded Lie algebras of dimension 8 up to isomorphism. We will follow the paper [**9**], which is in turn based on results given in [**2**]. Table 1 shows a complete classification up to isomorphism of $\mathbf{N}$-graded filiform Lie algebras of dimension 8. In Section 4, we use Table 1 and a characterization given in [**9**] to obtain the classification of filiform Lie algebras of dimension 8 which are Einstein nilradicals, which is the content of Table 2.

2. Preliminaries.

We consider the vector space

$$\begin{aligned} V &= \Lambda^2(\mathbf{R}^n)^* \otimes \mathbf{R}^n \\ &= \{\mu : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n : \mu \text{ bilinear and skew-symmetric}\}, \end{aligned}$$

then

$$\mathcal{N} = \{\mu \in V : \mu \text{ satisfies Jacobi and is nilpotent}\}$$

is an algebraic subset of V as the Jacobi identity and the nilpotency condition can both be written as zeroes of polynomial function. There is a natural action of $\mathrm{GL}_n := \mathrm{GL}_n(\mathbf{R})$ on V given by

$$g.\mu(X,Y) = g\mu(g^{-1}X, g^{-1}Y), \quad X,Y \in \mathbf{R}^n, \ g \in \mathrm{GL}_n, \ \mu \in V.$$

Note that each $\mu \in \mathcal{N}$ defines a nilpotent Lie algebra given by $(\mathbf{R}^n, \mu)$; thus, $\mathcal{N}$ parameterizes the set of all nilpotent Lie algebras of dimension n. Note also that μ and λ are isomorphic if and only if they lie in the same GL_n-orbit and so $\mathcal{N}/\mathrm{GL}_n$ parameterizes the set of isomorphism classes.

In the filiform case most of the standard invariants from Lie theory (descending and ascending central series, center, rank, etc.) coincide, so the next result will be a very useful tool to show that two filiform Lie algebras are not isomorphic.

Lemma 2.1. *Let $\mathfrak{n}$ be a Lie algebra, and let $C_j(\mathfrak{n})$ be its descending central series (i.e., $C_0(\mathfrak{n}) = \mathfrak{n}$ and $C_{i+1}(\mathfrak{n}) = [\mathfrak{n}, C_i(\mathfrak{n})]$, $i \geq 0$). Consider*

$$\mathrm{I}_{s,j}(\mathfrak{n}) = \{[x] \in \mathfrak{n}/C_j(\mathfrak{n}) : \dim \mathrm{Im}\,(\mathrm{ad}_x) = s\}, \quad s,j \in \mathbf{N}.$$

If $\varphi : \mathfrak{n} \to \mathfrak{n}'$ is an isomorphism of Lie algebras, then $\widetilde{\varphi}(\mathrm{I}_{s,j}(\mathfrak{n})) = \mathrm{I}_{s,j}(\mathfrak{n}')$, where $\widetilde{\varphi} : \mathfrak{n}/C_j(\mathfrak{n}) \to \mathfrak{n}'/C_j(\mathfrak{n}')$ is the morphism of Lie algebras given by $\widetilde{\varphi}[x] = [\varphi(x)]'$, $x \in \mathfrak{n}$.

Proof. Observe first that $\widetilde{\varphi}$ is an isomorphism of Lie algebras since $C_j(\mathfrak{n})$ and $C_j(\mathfrak{n}')$ are ideals of $\mathfrak{n}$ and $\mathfrak{n}'$, respectively, and φ is an isomorphism of Lie algebras.

Let $x \in \mathfrak{n}$, and we define $V = \mathrm{Im}\,(\mathrm{ad}_x)$ and $W = \mathrm{Im}\,(\mathrm{ad}_{\varphi(x)})$, so $W = \varphi(V)$ because

$$\begin{aligned} W &= \{[\varphi(x), z] : z \in \mathfrak{n}'\} = \{[\varphi(x), \varphi(y)] : y \in \mathfrak{n}\} \\ &= \{\varphi([x,y]) : y \in \mathfrak{n}\} \\ &= \varphi(\{[x,y] : y \in \mathfrak{n}\}) = \varphi(V). \end{aligned}$$

We define also $\overline{V} = \mathrm{Im}\,(\mathrm{ad}_{[x]})$ y $\overline{W} = \mathrm{Im}\,(\mathrm{ad}_{[\varphi(x)]'})$. Then $\overline{V} = \{[[x],[y]] : y \in \mathfrak{n}\} = \pi(V)$ y $\overline{W} = \{[[\varphi(x)]', [\varphi(y)]'] : y \in \mathfrak{n}\} = \pi'(W)$, where π and π' denote the canonical projections, so, as $\widetilde{\varphi}$ is an isomorphism $\overline{W} = \pi'(W) = \pi'(\varphi(V)) = \widetilde{\varphi}(\pi(V))$ and then

$$\begin{aligned}\mathrm{Im}\,(\mathrm{ad}\,_{[\varphi(x)]'}) &= \pi'(\mathrm{Im}\,\mathrm{ad}\,_{\varphi(x)}) = \pi'(\varphi(\mathrm{Im}\,\mathrm{ad}\,_x)) \\ &= \widetilde{\varphi}(\pi(\mathrm{Im}\,\mathrm{ad}\,_x)) \\ &= \widetilde{\varphi}(\mathrm{Im}\,\mathrm{ad}\,_{[x]}).\end{aligned}$$

Finally $\widetilde{\varphi}(\mathrm{I}\,_{s,j}(\mathfrak{n})) = \mathrm{I}\,_{s,j}(\mathfrak{n}')$. □

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A left invariant metric on G will always be identified with the inner product $\langle \cdot, \cdot \rangle$ determined on $\mathfrak{g}$. The pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ will be referred to as a *metric Lie algebra*.

A consequence of the Alekseevskii conjecture (see [**1**, 7.57]), is that a Lie group with a left invariant metric which is Einstein should be either solvable or compact. This conjecture is still open.

Any solvable Lie group admits at most one standard Einstein metric up to isometry and scaling. If S is standard Einstein, then for some distinguished element $H \in \mathfrak{a}$, the eigenvalues of $\mathrm{ad}\, H|_{\mathfrak{n}}$ are all positive integers without a common divisor, say $k_1 < \cdots < k_r$. These results were proved by Jens Heber in [**3**].

Definition 2.2. Let S be a standard Einstein solvmanifold, and let $d_1, \ldots, d_r$ be the corresponding multiplicities of $k_1 < \cdots < k_r$. Then the tuple

$$(k; d) = (k_1 < \cdots < k_r; d_1, \ldots, d_r)$$

is called the *eigenvalue type* of S.

In every dimension, only finitely many eigenvalue types occur (see [**3**]).

Definition 2.3. Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be a metric nilpotent Lie algebra. A metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle \cdot, \cdot \rangle')$ such that $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{n}$ is called a *metric solvable extension* of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ if $\mathfrak{n}$ is an ideal of $\mathfrak{s}$, $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{n}$ and $\langle \cdot, \cdot \rangle'|_{\mathfrak{n} \times \mathfrak{n}} = \langle \cdot, \cdot \rangle$.

Definition 2.4. A real semisimple derivation ϕ of a nilpotent Lie algebra $\mathfrak{n}$ is called *pre-Einstein* if $\mathrm{tr}\,(\phi \circ \psi) = \mathrm{tr}\,\psi$ for all $\psi \in \mathrm{Der}\,(\mathfrak{n})$.

A pre-Einstein derivation always exists, is unique up to conjugation and its eigenvalues are rational (see [**10**, Theorem 1]). If $\mathfrak{n}$ is an Einstein nilradical, then the derivation ϕ of (1), which is a multiple of $\operatorname{ad}(H)$, is a pre-Einstein derivation (up to scaling).

Lemma 2.5. *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be a metric nilpotent Lie algebra. Then there exists at most one pre-Einstein derivation ϕ of $\mathfrak{n}$ symmetric with respect to $\langle\cdot,\cdot\rangle$.*

Proof. Let $\mathfrak{p} = \operatorname{Der}(\mathfrak{n}) \cap \operatorname{sym}(\mathfrak{n}, \langle\cdot,\cdot\rangle)$. We define on $\mathfrak{p}$ the next inner product: $(A, B) = \operatorname{tr}(AB)$, $A, B \in \mathfrak{p}$.

As $f : \mathfrak{p} \to \mathbf{R}$, $f(A) = \operatorname{tr}(A)$ is a linear functional, then there exists an only $B \in \mathfrak{p}$ such that $\operatorname{tr}(A) = \operatorname{tr}(AB)$ for all $A \in \mathfrak{p}$, and if $\phi \in \mathfrak{p}$ then $\phi = B$. $\square$

It follows from Lemma 2.5 that a pre-Einstein derivation ϕ is symmetric with respect to an inner product $\langle\cdot,\cdot\rangle$. Then $\langle\cdot,\cdot\rangle$ is a nilsoliton if and only if $\langle\cdot,\cdot\rangle$ satisfies condition (1) for a multiple of ϕ.

Let $\mathfrak{n} = (\mathbf{R}^n, \mu)$ be a nilpotent Lie algebra of dimension n, and let $\phi \in \operatorname{Der}(\mathfrak{n})$ be a pre-Einstein derivation of $\mathfrak{n}$. Suppose that all the eigenvalues of ϕ are simple. Let $\{e_i\}$ be the basis of eigenvectors for ϕ and

$$\mu(e_i, e_j) = \sum_{k=1}^{n} c_{ij}^k e_k,$$

where μ is a Lie bracket of $\mathfrak{n}$ (note that for every pair (i, j), no more than one of the c_{ij}^k is nonzero). In the Euclidean space $\mathbf{R}^n$ with the inner product $(\cdot,\cdot)$ and orthonormal basis $\{f_1, \dots, f_n\}$, define the finite subset $\mathbf{F} = \{\alpha_{ij}^k = f_k - f_i - f_j : c_{ij}^k \neq 0\}$. Let L be the affine span of $\mathbf{F}$, the smallest affine subspace of $\mathbf{R}^n$ containing $\mathbf{F}$.

Theorem 2.6 [**9**, Theorem 1]. *Let $\mathfrak{n}$ be a nilpotent Lie algebra whose pre-Einstein derivation has all the eigenvalues simple. The algebra $\mathfrak{n}$ is an Einstein nilradical if and only if the orthogonal projection of the origin of $\mathbf{R}^n$ to L lies in the interior of the convex hull of $\mathbf{F}$.*

If $N{=}\#\mathbf{F}$ and we introduce a matrix $Y \in \mathbf{R}^{n\times N}$ whose vector-columns are the vectors α_{ij}^k in some fixed order. Define the vector

$[1]_N = (1, \dots, 1)^t \in \mathbf{R}^N$ and the matrix $U \in \mathbf{R}^{N\times N}$ by $U = Y^tY$. An equivalent way to write Theorem 2.6 is the following

Corollary 2.7 [**9**, Corollary 1]. *A nilpotent Lie algebra* $\mathfrak{n}$ *whose pre-Einstein derivation has all the eigenvalues simple is an Einstein nilradical if and only if there exists a vector* $v \in \mathbf{R}^N$ *all of whose coordinates are positive such that*

$$Uv = [1]_N. \tag{2}$$

3. Classification of filiform N-graded Lie algebras of dimension 8. In this section, we will study filiform Lie algebras of dimension 8 admitting an **N**-gradation.

The *rank* of a Lie algebra $\mathfrak{n}$, denoted by $\operatorname{rank}\mathfrak{n}$, is the dimension of the maximal abelian subalgebra of $\operatorname{Der}(\mathfrak{n})$ consisting of real semisimple elements. If $\mathfrak{n}$ is a n-dimensional nilpotent filiform **N**-graded Lie algebra, then its rank is at most two. It is known from [**2**, Section 3] that if $\operatorname{rank}\mathfrak{n} = 2$ then $\mathfrak{n}$ is isomorphic to

$$\begin{array}{ll} \mathfrak{m}_0(n): & \mu(e_1,e_i) = e_{i+1}, \quad i = 2,\dots,n-1, \\ \mathfrak{m}_1(n),\ n \text{ even}: & \mu(e_1,e_i) = e_{i+1}, \quad i = 2,\dots,n-1, \\ & \mu(e_i,e_{n-i+1}) = (-1)^i e_n, \quad i = 2,\dots,n-1, \end{array}$$

(see also [**12**]), and if $\operatorname{rank}\mathfrak{n} = 1$ then $\mathfrak{n}$ belongs to one and only one of the following classes:

$$\begin{array}{lll} A_r,\ 2 \le r \le n-3: & \mu(e_1,e_i) = e_{i+1}, & i = 2,\dots,n-1, \\ (3) & \mu(e_i,e_j) = c_{ij}e_{i+j+r-2}, & i,j \ge 2, i+j+r-2 \le n, \end{array}$$

and

$$\begin{array}{lll} B_r,\ 2 \le r \le n-4: & \mu(e_1,e_i) = e_{i+1}, & i = 2,\dots,n-2, \\ (4) & \mu(e_i,e_j) = c_{ij}e_{i+j+r-2}, & i,j \ge 2, i+j+r-2 \le n-1, \\ & \mu(e_i,e_{n+1-i}) = (-1)^{i+1}e_n, & \end{array}$$

with at least one $c_{ij} \neq 0$.

The rank cannot be zero because in that case it would not admit an $\mathbf{N}$-gradation.

For a given dimension, to classify Lie algebras in the classes A_r and B_r is involved. The numbers c_{ij}'s must satisfy the Jacobi condition, and even after finding those which do, it is difficult to obtain a complete list of these algebras up to isomorphism. The main difficulty comes from the fact that most common invariants from Lie theory usually coincide for all algebras in one of these families, especially in a curve.

Observe that two algebras which lie in different classes cannot be isomorphic, because if they were so they would have the same pre-Einstein derivation (recall that when the rank is 1 the semisimple derivation is unique up to conjugation and scaling) and that is absurd because the eigenvalues of this derivation for A_r are proportional to $(1, r, r+1, \ldots, n+r-2)$ and for B_r are proportional to $(1, r, r+1, \ldots, n+r-3, n+2r-3)$ (see [**9**]).

In this section we will study Lie algebras of the classes A_r and B_r for $n = 8$ and then will show in Table 1 a complete list up to isomorphism.

Lemma 3.1. *For $t \in \mathbf{R}$, let $(\mathbf{R}^8, \mu_t)$ be the Lie algebra defined by*

$$\begin{aligned} &\mu_t(e_1, e_j) = e_{j+1}, \quad j = 2, \ldots, 7, && \mu_t(e_2, e_3) = te_5, \\ &\mu_t(e_2, e_4) = te_6, && \mu_t(e_2, e_5) = (t-1)e_7, \\ &\mu_t(e_2, e_6) = (t-2)e_8, && \mu_t(e_3, e_4) = e_7, \\ &\mu_t(e_3, e_5) = e_8. \end{aligned}$$

Then $\mathfrak{n}_t$ is isomorphic to $\mathfrak{n}_s$ if and only if $t = s$.

Proof. If $\mathfrak{n}_t \simeq \mathfrak{n}_s$, then there exists a $\varphi : \mathfrak{n}_t \to \mathfrak{n}_s$ isomorphism of Lie algebras, so $\varphi\langle e_j, \ldots, e_8\rangle = \langle e_j, \ldots, e_8\rangle$, $j = 3, \ldots, 8$, since

$$\varphi(\mu_t(\mathfrak{n}_t, \mu_t(\mathfrak{n}_t, \ldots, \mu_t(\mathfrak{n}_t, \mathfrak{n}_t)\ldots))) = \mu_s(\mathfrak{n}_s, \mu_s(\mathfrak{n}_s, \ldots, \mu_s(\mathfrak{n}_s, \mathfrak{n}_s)\ldots)).$$

Therefore the matrix of φ respect to the basis $\{e_1, \ldots, e_8\}$ is given by

$$\varphi = \begin{bmatrix} a_1 & b_1 & & & & & & \\ a_2 & b_2 & & & & & & \\ a_3 & b_3 & c_3 & & & & & \\ a_4 & b_4 & c_4 & d_4 & & & & \\ a_5 & b_5 & c_5 & d_5 & l_5 & & & \\ a_6 & b_6 & c_6 & d_6 & l_6 & f_6 & & \\ a_7 & b_7 & c_7 & d_7 & l_7 & f_7 & g_7 & \\ a_8 & b_8 & c_8 & d_8 & l_8 & f_8 & g_8 & h_8 \end{bmatrix}.$$

It is easy to see that $b_1 = 0$ by using that $\operatorname{rank}(\operatorname{ad}_{\varphi(e_2)}) = \operatorname{rank}(\operatorname{ad}_{e_2})$.

Since φ is a morphism of Lie algebras, $\varphi(\mu_t(e_i, e_j)) = \mu_s(\varphi(e_i), \varphi(e_j))$ for all i, j and therefore:

$$\begin{aligned} &c_3 = a_1 b_2, \quad d_4 = a_1 c_3, \quad l_5 = a_1 d_4, \qquad f_6 = a_1 l_5,\\ &g_7 = a_1 f_6, \quad h_8 = a_1 g_7, \quad t l_5 = s b_2 c_3, \quad (t-1) g_7 = (s-1) b_2 l_5. \end{aligned}$$

Then we get that $t a_1^2 = s b_2$ and $(t-1) a_1^2 = (s-1) b_2$ and so $t a_1^2 - a_1^2 = s b_2 - b_2$. Finally, $a_1^2 = b_2$ and therefore $t = s$. □

Lemma 3.2. *Let $\mathfrak{n}$ be a Lie algebra of dimension 8 of class A_2. Then $\mathfrak{n}$ is isomorphic to one and only one of the following algebras*:

$$\begin{array}{lll} \mathfrak{m}_2(8) & \mu(e_1, e_i) = e_{i+1}, & i = 2, \ldots, 7,\\ & \mu(e_2, e_i) = e_{i+2}, & i = 3, \ldots, 6.\\ \mathfrak{g}_\alpha(8), \alpha \in \mathbf{R} & \mu(e_1, e_i) = e_{i+1}, & i = 2, \ldots, 7,\\ & \mu(e_2, e_3) = (2+\alpha) e_5,\ \mu(e_2, e_4) = (2+\alpha) e_6, & \\ & \mu(e_2, e_5) = (1+\alpha) e_7,\ \mu(e_3, e_4) = e_7, & \\ & \mu(e_3, e_5) = e_8, \qquad \mu(e_2, e_6) = \alpha e_8. & \end{array}$$

Proof. From the Jacobi identity and (3), it follows that $c_{23} = c_{24}$, $c_{25} = c_{24} - c_{34}$, $c_{26} = c_{25} - c_{35}$ y $c_{34} = c_{35}$. Therefore, the Lie algebras of class A_2 are:

$$\begin{array}{lll} \mu_{a,b}(e_1, e_j) = e_{j+1}, & j = 2, \ldots, 7, & \mu_{a,b}(e_2, e_3) = a e_5,\\ \mu_{a,b}(e_2, e_4) = a e_6, & & \mu_{a,b}(e_2, e_5) = (a-b) e_7,\\ \mu_{a,b}(e_2, e_6) = (a-2b) e_8, & & \mu_{a,b}(e_3, e_4) = b e_7,\\ \mu_{a,b}(e_3, e_5) = b e_8, & & \end{array}$$

with $a, b \in \mathbf{R}$, some non zero.

If $b = 0$, we get $\mu_{a,0} \simeq \mu_{1,0}$ since $\mu_{1,0} = g_a . \mu_{a,0}$ by taking $g_a \in \mathrm{GL}_8$, $g_a = \operatorname{diag}(1, a, a, a, a, a, a, a)$, where $\operatorname{diag}(a, b, c, d, e, f, g, h)$ denotes the diagonal matrix with entries $a, b, c, \ldots, h$. If $b \neq 0$ then $\mu_{a,b} \simeq \mu_{a/b,1}$ since $\mu_{a/b,1} = g_b . \mu_{a,b}$ for $g_b \in \mathrm{GL}_8$, $g_b = \operatorname{diag}(1, b, \ldots, b)$. Let $t = a/b$. Then $\mu_{t,1} = \mu_t$, where μ_t is the bracket of Lemma 3.1 and so we know that $\mu_t \simeq \mu_s$ if and only if $t = s$.

Let us now see that there does not exist $t \in \mathbf{R}$ such that $\mu_{t,1}$ is isomorphic to $\mu_{1,0}$.

Let $x = c_1e_1 + c_2e_2 + c_3e_3$. The matrices of ad_x relative to $\mu_{t,1}$ and $\mu_{1,0}$ in terms of the basis $\{e_1, \dots, e_8\}$ are respectively given by

$$\mathrm{ad}_x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_2 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_3 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -tc_3 & tc_2 & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & tc_2 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_3 & (t-1)c_2 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_3 & (t-2)c_2 & c_1 & 0 \end{bmatrix},$$

$$\mathrm{ad}_x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_2 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_3 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_3 & c_2 & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & c_1 & 0 \end{bmatrix}.$$

Then, by Lemma 2.1, $\mu_{t,1}$ is not isomorphic to $\mu_{1,0}$ because the first matrix never has rank 2 while the second matrix does ($c_1 = 0, c_2 = 0, c_3 \neq 0$). □

We will omit the proof of the following two lemmas because they are similar to those for Lemmas 3.1 and 3.2.

Lemma 3.3. *For $t \in \mathbf{R}$, let $(\mathbf{R}^8, \mu_t)$ be the Lie algebra defined by:*

$$\begin{aligned} &\mu_t(e_1, e_j) = e_{j+1}, \quad j = 2, \dots, 7, && \mu_t(e_2, e_3) = (t+1)e_6, \\ &\mu_t(e_2, e_4) = (t+1)e_7, && \mu_t(e_2, e_5) = te_8, \\ &\mu_t(e_3, e_4) = e_8. \end{aligned}$$

Then $\mathfrak{n}_t$ is isomorphic to $\mathfrak{n}_s$ if and only if $t = s$.

Lemma 3.4. *If* $\mathfrak{n}$ *is a Lie algebra of class* A_3 *and dimension* 8*, then* $\mathfrak{n}$ *is isomorphic to one and only one of the next algebras:*

$$\begin{array}{lllll} \mathfrak{a}_t(8), t \in \mathbf{R} & \mu(e_1,e_i) = e_{i+1}, & i = 2,\dots,7, & \mu(e_2,e_3) = (t+1)e_6, \\ & \mu(e_2,e_4) = (t+1)e_7, & & \mu(e_2,e_5) = te_8, \\ & \mu(e_3,e_4) = e_8. & & \\ \mathfrak{c}_{1,0}(8) & \mu(e_1,e_i) = e_{i+1}, & i = 2,\dots,7, & \mu(e_2,e_3) = e_6, \\ & \mu(e_2,e_4) = e_7, & & \mu(e_2,e_5) = e_8. \end{array}$$

Lemma 3.5. *Up to isomorphism there is only one Lie algebra of class* A_4 *and dimension* 8*, defined by*:

$$\begin{array}{ll} \mathfrak{d}_1(8) & \mu(e_1,e_i) = e_{i+1}, \quad i = 2,\dots,7, \\ & \mu(e_2,e_3) = e_7, \quad \mu(e_2,e_4) = e_8. \end{array}$$

Proof. From the Jacobi identity and (3) it follows that $c_{23} = c_{24}$, and therefore:

$$\mu_a(e_1,e_j) = e_{j+1}, \quad j = 2,\dots,7,$$
$$\mu_a(e_2,e_3) = ae_7, \quad \mu_a(e_2,e_4) = ae_8.$$

with $a \neq 0$. Since $a \neq 0$, $\mu_a \simeq \mu_1$ for $g_a = \operatorname{diag}(1,(1/a),\dots,(1/a))$ because $\mu_a = g_a.\mu_1$. □

Lemma 3.6. *If* $\mathfrak{n}$ *is a Lie algebra of class* A_5 *and dimension* 8*, then* $\mathfrak{n}$ *is isomorphic to*

$$\mathfrak{h}_1(8) \quad \mu(e_1,e_i) = e_{i+1}, \qquad i = 2,\dots,7, \quad \mu(e_2,e_3) = e_8.$$

Proof. From (3) we get that the algebras of class A_5 and rank 1 are:

$$\mu_a(e_1,e_j) = e_{j+1}, \quad j = 2,\dots,7, \qquad \mu_a(e_2,e_3) = ae_8, \quad a \neq 0.$$

Since $a \neq 0$, then $\mu_a \simeq \mu_1$ by taking $g_a = \operatorname{diag}(1,(1/a),\dots,(1/a))$. □

Lemma 3.7. *Let $\mathfrak{n}$ be a Lie algebra of class B_2 and dimension 8. Then $\mathfrak{n}$ is isomorphic to*

$$\mathfrak{b}(8)\quad \mu(e_1,e_i)=e_{i+1},\quad i=2,\dots,6,\quad \mu(e_2,e_3)=-\tfrac{1}{2}e_5,$$
$$\mu(e_2,e_4)=-\tfrac{1}{2}e_6,\quad \mu(e_2,e_5)=-\tfrac{3}{2}e_7,$$
$$\mu(e_3,e_4)=e_7,\quad \mu(e_i,e_{9-i})=(-1)^{i+1}e_8,$$
$$i=2,3,4.$$

Proof. From (4) and the Jacobi identity follows:

$$\mu_a(e_1,e_j)=e_{j+1},\quad j=2,\dots,6,\quad \mu_a(e_2,e_3)=ae_5,$$
$$\mu_a(e_2,e_4)=ae_6,\quad \mu_a(e_2,e_5)=3ae_7,$$
$$\mu_a(e_i,e_{9-i})=(-1)^{i+1}e_8,\quad i=2,3,4,\quad \mu_a(e_3,e_4)=-2ae_7,$$

with $a\neq 0$.

Like $a\neq 0$, then $\mu_a\simeq\mu_{-1/2}$ since by taking $g_a=\operatorname{diag}(1,a,\dots,a,a^2)$ we get that $g_a.\mu_a=\mu_1$. □

Lemma 3.8. *Let $\mathfrak{n}$ be a Lie algebra of class B_3 of dimension 8. Then $\mathfrak{n}$ is isomorphic to*

$$\mathfrak{k}_1(8)\quad \mu(e_1,e_i)=e_{i+1},\quad i=2,\dots,6,\quad \mu(e_2,e_3)=e_6,$$
$$\mu(e_2,e_4)=e_7,\quad \mu(e_i,e_{9-i})=(-1)^{i+1}e_8,$$
$$i=2,3,4.$$

Proof. From (4) and the Jacobi identity we obtain:

$$\mu_a(e_1,e_j)=e_{j+1},\quad j=2,\dots,6,\quad \mu_a(e_2,e_3)=ae_6$$
$$\mu_a(e_2,e_4)=ae_7,\quad \mu_a(e_i,e_{9-i})=(-1)^{i+1}e_8,$$
$$i=2,3,4,$$

with $a\neq 0$.

Then $g_a.\mu_a=\mu_1$ for $g_a=\operatorname{diag}(1,a,a,\dots,a,a^2)$. □

Lemma 3.9. *Up to isomorphism there is only one Lie algebra of class B_4 of dimension 8, defined by:*

$$\mathfrak{s}_1(8)\quad \mu(e_1,e_j)=e_{j+1},\quad j=2,\dots,6,\quad \mu(e_2,e_3)=e_7,$$
$$\mu(e_i,e_{9-i})=(-1)^{i+1}e_8,\quad i=2,3,4.$$

Proof. The algebras of (4) for $r = 4$ are:

$$\begin{array}{lll} \mu_a(e_1, e_j) = e_{j+1}, & j = 2, \dots, 6, & \mu_a(e_2, e_3) = ae_7, \\ \mu_a(e_i, e_{9-i}) = (-1)^{i+1}e_8, & i = 2, 3, 4, & \end{array}$$

with $a \neq 0$.

Let $g_a = \operatorname{diag}(1, a, \dots, a, a^2)$; then $\mu_a \simeq \mu_1$, since $g_a.\mu_a = \mu_1$. □

The results obtained in this section can be summarized in the following theorem.

Theorem 3.10. *Let $\mathfrak{n}$ be an $\mathbf{N}$-graded, filiform Lie algebra of dimension 8. Then $\mathfrak{n}$ is isomorphic to one and only one of the Lie algebras in Table 1.*

4. Classification of filiform Einstein nilradicals of dimension 8. In this section we will determine which filiform Lie algebras of dimension 8 are Einstein nilradicals. To do so we will apply Corollary 2.7.

Recall that if $\operatorname{rank} \mathfrak{n} = 0$ then $\mathfrak{n}$ does not admit an $\mathbf{N}$-gradation so $\mathfrak{n}$ is not Einstein nilradical. If $\operatorname{rank} \mathfrak{n} = 2$, then $\mathfrak{n}$ is Einstein nilradical (see [**5**, Theorem 4.2] for $\mathfrak{m}_0(8)$ and [**11**, Theorem 35] for $\mathfrak{m}_1(8)$). Therefore, to obtain a complete classification of filiform Einstein nilradicals of dimension 8 we must study only the algebras of Table 1 which have rank 1.

Every algebra $\mathfrak{n}$ from A_r or B_r has only one semisimple derivation, which is automatically a pre-Einstein derivation (up to conjugation and scaling). As all eigenvalues of such a derivation are simple (they are proportional to $(1, r, r+1, \dots, n+r-2)$ for A_r and to $(1, r, r+1, \dots, n+r-3, n+2r-3)$ for B_r), the question of whether or not $\mathfrak{n}$ is an Einstein nilradical is answered by Corollary 2.7.

We will only analyze in detail three illustrative cases, as the proof is completely analogous for any other Lie algebra appearing in Table 1. The first case will be a Lie algebra which is not an Einstein nilradical, the second one an algebra which turns out to be an Einstein nilradical,

TABLE 1. **N**-graded filiform Lie algebras of dimension 8 up to isomorphism.

$\mathfrak{m}_0(8)$	$\mu(e_1, e_i) = e_{i+1},\ i = 2, \ldots, 7.$
$\mathfrak{m}_1(8)$	$\mu(e_1, e_i) = e_{i+1},\ i = 2, \ldots, 7,$ $\mu(e_i, e_{9-i}) = (-1)^i e_8,\ i = 2, \ldots, 7.$
$\mathfrak{m}_2(8)$	$\mu(e_1, e_i) = e_{i+1},\ i = 2, \ldots, 7,$ $\mu(e_2, e_i) = e_{i+2},\ i = 3, \ldots, 6.$
$\mathfrak{g}_\alpha(8)$, $\alpha \in \mathbf{R}$	$\mu(e_1, e_i) = e_{i+1},\ i = 2, \ldots, 7,$ $\mu(e_2, e_3) = (2+\alpha)e_5,$ $\mu(e_2, e_4) = (2+\alpha)e_6,$ $\mu(e_2, e_5) = (1+\alpha)e_7,$ $\mu(e_2, e_6) = \alpha e_8,$ $\mu(e_3, e_4) = e_7,$ $\mu(e_3, e_5) = e_8.$
$\mathfrak{a}_t(8)$, $t \in \mathbf{R}$	$\mu(e_1, e_i) = e_{i+1},\ i = 2, \ldots, 7,$ $\mu(e_2, e_3) = (t+1)e_6,\ \mu(e_2, e_4) = (t+1)e_7,$ $\mu(e_2, e_5) = te_8,\ \mu(e_3, e_4) = e_8.$
$\mathfrak{c}_{1,0}(8)$	$\mu(e_1, e_i) = e_{i+1},\ i = 2, \ldots, 7,$ $\mu(e_2, e_3) = e_6,\ \mu(e_2, e_4) = e_7,$ $\mu(e_2, e_5) = e_8.$
$\mathfrak{d}_1(8)$	$\mu(e_1, e_i) = e_{i+1},\ i = 2, \ldots, 7,$ $\mu(e_2, e_3) = e_7,\ \mu(e_2, e_4) = e_8.$
$\mathfrak{h}_1(8)$	$\mu(e_1, e_i) = e_{i+1},\ i = 2, \ldots, 7,$ $\mu(e_2, e_3) = e_8.$
$\mathfrak{b}(8)$	$\mu(e_1, e_i) = e_{i+1},\ i = 2, \ldots, 6,$ $\mu(e_2, e_3) = -\frac{1}{2}e_5,\ \mu(e_2, e_4) = -\frac{1}{2}e_6,$ $\mu(e_2, e_5) = -\frac{3}{2}e_7,\ \mu(e_3, e_4) = e_7,$ $\mu(e_i, e_{9-i}) = (-1)^{i+1}e_8,\ i = 2, 3, 4.$
$\mathfrak{k}_1(8)$	$\mu(e_1, e_i) = e_{i+1},\ i = 2, \ldots, 6,$ $\mu(e_2, e_3) = e_6,\ \mu(e_2, e_4) = e_7,$ $\mu(e_i, e_{9-i}) = (-1)^{i+1}e_8.$
$\mathfrak{s}_1(8)$	$\mu(e_1, e_i) = e_{i+1},\ i = 2, \ldots, 6,$ $\mu(e_2, e_3) = e_7,$ $\mu(e_i, e_{9-i}) = (-1)^{i+1}e_8,\ i = 2, 3, 4.$

and in the third case we will consider a curve of Lie algebras that depend on a real parameter. This curve has only one Lie algebra which is not an Einstein nilradical.

We consider in $\mathbf{R}^n$ the canonical inner product $(\cdot,\cdot)$ and $\{f_1,\dots,f_n\}$ the canonical basis.

Case 1. In this case, we consider the Lie algebra $\mathfrak{c}_{1,0}(8)$ of dimension 8. $\mathfrak{c}_{1,0}(8)$ is nilpotent and has a pre-Einstein derivation with simple eigenvalues, $\phi(e_1)=e_1$, $\phi(e_i)=(i+1)e_i$, $i\geq 2$, with eigenvalues $\{1,3,4,5,6,7,8,9\}$.

The finite set $\mathbf{F}=\{\alpha_{ij}^k : c_{ij}^k\neq 0\}$ is given by:

$$\{(-1,-1,1,0,0,0,0,0),(-1,0,-1,1,0,0,0,0),(-1,0,0,-1,1,0,0,0),$$
$$(-1,0,0,0,-1,1,0,0),(-1,0,0,0,0,-1,1,0),(-1,0,0,0,0,0,-1,1),$$
$$(0,-1,-1,0,0,1,0,0),(0,-1,0,-1,0,0,1,0),(0,-1,0,0,-1,0,0,1)\},$$

and with respect to this enumeration the matrix U defined in Section 2 is

$$U=\begin{bmatrix} 3&0&1&1&1&1&0&1&1\\ 0&3&0&1&1&1&1&-1&0\\ 1&0&3&0&1&1&0&1&-1\\ 1&1&0&3&0&1&1&0&1\\ 1&1&1&0&3&0&-1&1&0\\ 1&1&1&1&0&3&0&-1&1\\ 0&1&0&1&-1&0&3&1&1\\ 1&-1&1&0&1&-1&1&3&1\\ 1&0&-1&1&0&1&1&1&3 \end{bmatrix}.$$

Then, by using Maple, we obtain that the solutions to the system (2) are

$$v=\Big(-\frac{9}{281},\frac{100}{281}-t_1,\frac{166}{281}-t_1-t_2,\frac{28}{281},\frac{8}{281}+t_1,$$
$$-\frac{55}{281}+t_1+t_2,t_1,t_2,\frac{161}{281}-t_1-t_2\Big),\quad t_1,t_2\in\mathbf{R}.$$

Then, the system does not have a positive solution because the first coordinate is always negative, and therefore by Corollary 2.7 $\mathfrak{c}_{1,0}(8)$ is not Einstein nilradical.

Case 2. We consider the Lie algebra of rank 1, of class A_4 and dimension 8, $\mathfrak{d}_1(8)$, it is nilpotent and has pre-Einstein derivation given by $\phi(e_1) = e_1$, $\phi(e_i) = (i+2)e_i$, $2 \leq i \leq 7$, with eigenvalues $\{1, 4, 5, 6, 7, 8, 9, 10\}$).

The finite set $\mathbf{F} = \{\alpha_{ij}^k : c_{ij}^k \neq 0\}$ is:

$$\{(-1,-1,1,0,0,0,0,0),\ (-1,0,-1,1,0,0,0,0),\ (-1,0,0,-1,1,0,0,0),\ (-1,0,0,0,-1,1,0,0),\ (-1,0,0,0,0,-1,1,0),\ (-1,0,0,0,0,0,-1,1),\ (0,-1,-1,0,0,0,1,0),\ (0,-1,0,-1,0,0,0,1)\},$$

and with respect to this enumeration, the matrix U is

$$U = \begin{bmatrix} 3 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 3 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 3 & 1 \\ 1 & -1 & 1 & 0 & 0 & 1 & 1 & 3 \end{bmatrix}.$$

Solutions to (2) are

$$v = \left(\frac{3}{62}, -\frac{7}{186} + t_1, \frac{29}{186}, \frac{20}{93}, \frac{13}{93}, \frac{32}{93} - t_1, \frac{77}{186} - t_1, t_1\right), \quad t_1 \in \mathbf{R}.$$

Then by Corollary 2.7, $\mathfrak{d}_1(8)$ is Einstein nilradical because the system has a positive solution considering $7/186 < t_1 < 64/186$.

Case 3. We consider the Lie algebras of rank 1, class A_2 and dimension 8, $\mathfrak{g}_\alpha(8)$, $\alpha \in \mathbf{R}$, which has pre-Einstein derivation with all the eigenvalues simple, $(\phi(e_1) = e_1, \phi(e_i) = ie_i, i \geq 2$, with eigenvalues $\{1, 2, 3, 4, 5, 6, 7, 8\}$).

If $\alpha \neq -2, -1, 0$ then the finite set $\mathbf{F} = \{\alpha_{ij}^k : c_{ij}^k \neq 0\}$ is given by:

$$\{(-1,-1,1,0,0,0,0,0), (-1,0,-1,1,0,0,0,0), (-1,0,0,-1,1,0,0,0), (-1,0,0,0,-1,1,0,0), (-1,0,0,0,0,-1,1,0), (-1,0,0,0,0,0,-1,1), (0,0,-1,-1,0,0,1,0), (0,0,-1,0,-1,0,0,1), (0,-1,-1,0,1,0,0,0), (0,-1,0,-1,0,1,0,0), (0,-1,0,0,-1,0,1,0), (0,-1,0,0,0,-1,0,1)\}.$$

Solutions of (2) are

$$v=\Big(-\frac{3}{17}+t_2+t_1,-\frac{9}{17}+t_2+t_4+t_1+t_5+t_3,-\frac{4}{17}+t_2+t_4+t_5,\\ \frac{12}{17}-t_2-t_4-t_1-t_3,\frac{11}{17}-t_2-t_4-t_1-t_5,\frac{7}{17}-t_2-t_5,t_1,t_2,\\ \frac{14}{17}-t_2-t_4-t_1-t_5-t_3,t_3,t_4,t_5\Big),\qquad t_1,t_2,t_3,t_4,t_5\in\mathbf{R}.$$

Therefore, if $\alpha\neq -2,-1,0,\mathfrak{g}_\alpha(8)$ is Einstein nilradical by Corollary 2.7 because the system has a positive solution, for example, taking $t_1=2/17$, $t_2=2/17$, $t_3=1/17$, $t_4=5/17$, $t_5=1/17$.

If $\alpha=0$ then the finite set $\mathbf{F}=\{\alpha_{ij}^k : c_{ij}^k\neq 0\}$ is

$$\{(-1,-1,1,0,0,0,0,0),\ (-1,0,-1,1,0,0,0,0),\ (-1,0,0,-1,1,0,0,0),\\ (-1,0,0,0,-1,1,0,0),\ (-1,0,0,0,0,-1,1,0),\ (-1,0,0,0,0,0,-1,1),\\ (0,0,-1,-1,0,0,1,0),\ (0,0,-1,0,-1,0,0,1),\ (0,-1,-1,0,1,0,0,0),\\ (0,-1,0,-1,0,1,0,0),\ (0,-1,0,0,-1,0,1,0)\}.$$

We note that the matrix U for the case $\alpha=0$ is obtained by erasing the last row and the last column of the corresponding matrix in the case $\alpha\neq -2,-1,0$, as the only vector which is missing in the new set $\mathbf{F}$ is α_{26}^8.

Solutions to $Uv=[1]$ are

$$v=\Big(\frac{11}{17}-t_2-t_3-t_4,\frac{5}{17}-t_2,\frac{10}{17}-t_2-t_3-t_1,-\frac{2}{17}+t_2,t_2+t_3-\frac{3}{17},\\ -\frac{7}{17}+t_2+t_3+t_1+t_4,t_1,\frac{14}{17}-t_2-t_3-t_1-t_4,t_2,t_3,t_4\Big),$$

$$t_1,t_2,t_3,t_4\in\mathbf{R}.$$

This implies that $\mathfrak{g}_0(8)$ is Einstein nilradical because by taking $t_1=2/17$, $t_2=4/17$, $t_3=1/17$, $t_4=1/17$. we obtain a positive solution.

If $\alpha = -1$, $\mathbf{F} = \{\alpha_{ij}^k : c_{ij}^k \neq 0\}$ is given by

$$\{(-1,-1,1,0,0,0,0,0),\ (-1,0,-1,1,0,0,0,0),\ (-1,0,0,-1,1,0,0,0),\\ (-1,0,0,0,-1,1,0,0),\ (-1,0,0,0,0,-1,1,0),\ (-1,0,0,0,0,0,-1,1),\\ (0,0,-1,-1,0,0,1,0),\ (0,0,-1,0,-1,0,0,1),\ (0,-1,-1,0,1,0,0,0),\\ (0,-1,0,-1,0,1,0,0),\ (0,-1,0,0,0,-1,0,1)\}.$$

Solutions of (2) are

$$v = \left(\frac{8}{17} - t_1 - t_4, \frac{2}{17} - t_1 + t_3, -\frac{4}{17} + t_4 + t_2, \frac{1}{17} + t_1 - t_3 + t_4, t_1,\right.\\ \left.\frac{7}{17} - t_4 - t_2, \frac{11}{17} - t_1 - t_4 - t_2, t_2, \frac{3}{17} + t_1 - t_3, t_3, t_4\right),$$

$$t_1, t_2, t_3, t_4 \in \mathbf{R}.$$

Therefore, $\mathfrak{g}_{-1}(8)$ is Einstein nilradical taking, for example, $t_1 = 1/17$, $t_2 = 4/17$, $t_3 = 1/17$, $t_4 = 1/17$.

If $\alpha = -2$, the finite set $\mathbf{F}$ is

$$\{(-1,-1,1,0,0,0,0,0),\ (-1,0,-1,1,0,0,0,0),\ (-1,0,0,-1,1,0,0,0),\\ (-1,0,0,0,-1,1,0,0),\ (-1,0,0,0,0,-1,1,0),\ (-1,0,0,0,0,0,-1,1),\\ (0,0,-1,-1,0,0,1,0),\ (0,0,-1,0,-1,0,0,1),\ (0,-1,0,0,-1,0,1,0),\\ (0,-1,0,0,0,-1,0,1)\}.$$

Solutions of system (2) are

$$v = \left(-\frac{3}{17} + t_2 + t_1, \frac{5}{17}, \frac{10}{17} - t_1, -\frac{2}{17} + t_3, -\frac{3}{17}, \frac{7}{17} - t_2 - t_3, t_1, t_2,\right.\\ \left.\frac{14}{17} - t_2 - t_1 - t_3, t_3\right), \qquad t_1, t_2, t_3 \in \mathbf{R}.$$

So $\mathfrak{g}_{-2}(8)$ is not Einstein nilradical because the system does not have a positive solution since the fifth coordinate of v is always negative.

Theorem 4.1. *Let $\mathfrak{n}$ be a filiform Lie algebra of dimension 8. Then $\mathfrak{n}$ is an Einstein nilradical if and only if $\mathfrak{n}$ is not any of the following algebras*: $\mathfrak{m}_2(8)$, $\mathfrak{g}_{-2}(8)$, $\mathfrak{a}_{-1}(8)$, $\mathfrak{c}_{1,0}(8)$ (*see Table* 2).

TABLE 2. Classification of filiform Einstein nilradicals of dimension 8.

Algebra	Einstein nilradical	No Nilradical Einstein	eigenvalue type
$\mathfrak{m}_0(8)$	✓		$1 < 26 < 27 < 28 < 29 < 30 < 31 < 32$
$\mathfrak{m}_1(8)$	✓		$10 < 123 < 133 < 143 < 153 < 163 < 173 < 296$
$\mathfrak{m}_2(8)$		✓	
$\mathfrak{g}_\alpha(8)$, $\alpha \neq -2$	✓		$1 < 2 < 3 < 4 < 5 < 6 < 7 < 8$
$\mathfrak{g}_{-2}(8)$		✓	
$\mathfrak{a}_t(8)$, $t \neq -1$	✓		$1 < 3 < 4 < 5 < 6 < 7 < 8 < 9$
$\mathfrak{a}_{-1}(8)$		✓	
$\mathfrak{c}_{1,0}(8)$		✓	
$\mathfrak{d}_1(8)$	✓		$1 < 4 < 5 < 6 < 7 < 8 < 9 < 10$
$\mathfrak{h}_1(8)$	✓		$1 < 5 < 6 < 7 < 8 < 9 < 10 < 11$
$\mathfrak{b}(8)$	✓		$1 < 2 < 3 < 4 < 5 < 6 < 7 < 9$
$\mathfrak{k}_1(8)$	✓		$1 < 3 < 4 < 5 < 6 < 7 < 8 < 11$
$\mathfrak{s}_1(8)$	✓		$1 < 4 < 5 < 6 < 7 < 8 < 9 < 13$

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