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ON ℓ_ψ SPACES AND INFINITE ψ-DIRECT SUMS OF BANACH SPACE

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ABSTRACT. In this paper, the sequence Banach spaces ℓ_ψ are defined for a class of convex functions ψ and properties of these spaces and their dual spaces are proved. It can be seen that some well-known sequence Banach spaces are spaces of this type. The ψ-direct sum of a sequence $(X_n)_{n\in\mathbf{N}}$ of Banach spaces is also defined. The modulus of convexity of this space is estimated in terms of the modulus of convexity of the spaces ℓ_ψ and X_n, $n = 1, 2, \dots$. Based on this estimate, conditions are proved under which uniform convexity, uniform smoothness and uniform non-squareness are inherited by ψ-direct sums.

1. Introduction. A norm $\|\cdot\|$ on $\mathbf{C}^2$ is called absolute if $\|(z_1, z_2)\| = \|(|z_1|, |z_2|)\|$ for $(z_1, z_2) \in \mathbf{C}^2$ and is called normalized if $\|(1, 0)\| = \|(0, 1)\| = 1$. For a continuous and convex function ψ on $[0, 1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1 - s, s\} \le \psi(s) \le 1$ for every $0 \le s \le 1$, Bonsall and Duncan [**3**] defined the norm

$$\|(z_1, z_2)\|_\psi = \begin{cases} (|z_1| + |z_2|)\psi(\frac{|z_2|}{|z_1|+|z_2|}) & \text{if } (z_1, z_2) \neq (0, 0) \\ 0 & \text{if } (z_1, z_2) = (0, 0) \end{cases}$$

on $\mathbf{C}^2$, and they proved that a norm $\|\cdot\|$ on $\mathbf{C}^2$ is absolute and normalized if and only if there is a function ψ with the above properties such that $\|\cdot\| = \|\cdot\|_\psi$. ℓ_p norms are typical examples of such norms, but there are plenty non ℓ_p-type norms on $\mathbf{C}^2$ which are absolute and normalized.

For every ψ as above, Takahashi, Kato and Saito [**19**] introduced the ψ-direct sum $X \bigoplus_\psi Y$ of Banach spaces X and Y equipped with

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the norm $\|(x,y)\| = \|(||x||,||y||)\|_\psi$ for $x \in X$, $y \in Y$, and proved that this space is strictly convex if and only if X, Y and ψ are strictly convex. Saito and Kato [**16**] proved that the space $X \bigoplus_\psi Y$ is uniformly convex if and only if X, Y are uniformly convex and ψ is strictly convex and Kato, Saito and Tamura [**9**] that it is uniformly non-square if $\|\cdot\|_\psi \neq \|\cdot\|_1$ and $\|\cdot\|_\psi \neq \|\cdot\|_\infty$.

Saito, Kato and Takahashi [**18**] extended the result of Bonslall-Duncan in the absolute and normalized norms on $\mathbf{C}^n$ for $n = 2, 3, \dots,$ using a family of continuous and convex functions defined on a certain convex subset of $[0,1]^{n-1}$. They also proved that such a norm is strictly convex if and only if ψ is strictly convex. Mitani, Saito and Suzuki [**14**] proved an equivalent condition in terms of ψ in order that $(\mathbf{C}^n, \|\cdot\|_\psi)$ be smooth. Kato, Saito and Tamura [**8**] defined the ψ-direct sum of a finite family of Banach spaces $X_1, \dots, X_n$ and proved that this space is strictly (respectively uniformly, locally uniformly) convex if and only if the spaces $X_1, \dots, X_n$ are strictly (respectively uniformly, locally uniformly) convex and the function ψ is strictly convex. As Dowling [**6**] remarked the construction of ψ-direct sum is a special case of a general construction of Day [**4**]. Following this, Dowling in [**6**] gave a simple proof that the ψ direct sum of $X_1, \dots, X_n$ is strictly convex (respectively uniformly convex, locally uniformly convex, uniformly convex in every direction) if and only if ψ is strictly convex and $X_1, \dots, X_n$ is strictly convex (respectively uniformly convex, locally uniformly convex, uniformly convex in every direction). But this proof does not work for smoothness, uniform smoothness and uniform non-squareness. Mitani, Oshiro and Saito [**12**] proved that the ψ-direct sum of a finite family of Banach spaces $X_1, \dots, X_n$ is smooth (respectively uniformly smooth) if and only if the space $(\mathbf{C}^n, \|\cdot\|_\psi)$ is smooth and X_i is smooth (respectively uniformly smooth) for $i = 1, \dots, n$.

In Section 3 of this paper, following the construction of Kato, Saito and Tamura [**8**], the space ℓ_ψ is introduced for a convex function ψ, defined on a certain convex subset of $[0,1]^{\mathbf{N}}$, with some appropriate conditions, and properties of this space and its dual are proved. Also, the ψ-direct sum of a sequence of Banach spaces $(X_n)_{n\in\mathbf{N}}$ is introduced and its dual space is characterized. In Section 4 an estimate of the modulus of convexity of the ψ-direct sum of a finite family of Banach spaces is proved (Theorem 4.2). From this result the well-known theorem about uniform convexity of ψ-direct sums ([**6, 8**]) is improved, estimating the modulus of convexity of ψ-direct sum, and a new result about uniform non-squareness of ψ-direct sum follows. Also the modulus of convexity of the ψ-direct sum of a sequence of Banach spaces $(X_n)_{n\in\mathbf{N}}$ is estimated (Theorem 4.5). From this theorem, conditions under which uniform convexity, uniform smoothness and uniform non-squareness are inherited by ψ-direct sum of a sequence of Banach spaces $(X_n)_{n\in\mathbf{N}}$ follow (Corollary 4.7 and Corollary 4.8).

2. Preliminaries. Let $n \geq 2$. A norm $\|\cdot\|$ on $\mathbf{C}^n$ is called absolute if $\|(z_1,\dots,z_n)\| = \|(|z_1|,\dots,|z_n|)\|$ for every $(z_1,\dots,z_n) \in \mathbf{C}^n$ and is called normalized if $\|(1,0,\dots,0)\| = \|(0,1,0,\dots,0)\| = \cdots = \|(0,\dots,0,1)\| = 1$. The family of all absolute and normalized norms on $\mathbf{C}^n$ is denoted by AN_n.

Let $\Delta_n = \{(s_1,\dots,s_{n-1}) \in [0,1]^{n-1} : \sum_{i=1}^{n-1} s_i \leq 1\}$. Saito et al. [**18**] considered the set Ψ_n of all continuous and convex functions $\psi : \Delta_n \to \mathbf{R}$ which satisfy the conditions

$$(A_0)\colon \psi(0,0,\dots,0) = \psi(0,1,0,\dots,0) = \cdots = \psi(0,\dots,0,1) = 1$$

$$(A_1)\colon \psi(s_1,\dots,s_{n-1}) \geq (s_1+\cdots+s_{n-1}) \times \psi\left(\frac{s_1}{s_1+\cdots+s_{n-1}},\dots,\frac{s_{n-1}}{s_1+\cdots+s_{n-1}}\right)$$

$$(A_2)\colon \psi(s_1,\dots,s_{n-1}) \geq (1-s_1)\psi\left(0,\frac{s_2}{1-s_1},\dots,\frac{s_{n-1}}{1-s_1}\right)$$

$$\vdots$$

$$(A_n)\colon \psi(s_1,\dots,s_{n-1}) \geq (1-s_{n-1})\psi\left(\frac{s_1}{1-s_{n-1}},\dots,\frac{s_{n-2}}{1-s_{n-1}},0\right),$$

and they proved the following theorem.

Theorem 2.1. i) *If $\psi \in \Psi_n$, then $\|\cdot\|_\psi \in AN_n$, where*

$$\|(z_1,\dots,z_n)\|_\psi = \begin{cases} (\sum_{i=1}^n |z_i|)\psi\left(\frac{|z_2|}{\sum_{i=1}^n |z_i|},\dots,\frac{|z_n|}{\sum_{i=1}^n |z_i|}\right) & \text{if } (z_1,\dots,z_n) \neq (0,\dots,0) \\ 0 & \text{if } (z_1,\dots,z_n) = (0,\dots,0). \end{cases}$$

ii) *If* $\|\cdot\| \in AN_n$*, then the function* $\psi : \Delta_n \to \mathbf{R}$*, defined by* $\psi(s) = \|(1 - s_1 - \cdots - s_{n-1}, s_1, \ldots, s_{n-1})\|$ *for* $s = (s_1, \ldots, s_{n-1}) \in \Delta_n$*, belongs to* Ψ_n *and* $\|\cdot\| = \|\cdot\|_\psi$.

Typical examples of these norms are ℓ_p-norms, for $1 \le p \le \infty$, but there are plenty of examples of such norms which are non-ℓ_p norms ([**8, 16, 17, 18**]).

The following two lemmas are proved in [**18**].

Lemma 2.2. *Let* $\psi \in \Psi_n$*. Then*

i) $\|\cdot\|_\infty \le \|\cdot\|_\psi \le \|\cdot\|_1$*, and*

ii) $1/n \le \psi_\infty(s_1, \ldots, s_{n-1}) \le \psi(s_1, \ldots, s_{n-1}) \le 1$ *for all* $(s_1, \ldots, s_{n-1}) \in \Delta_n$*, where* ψ_∞ *is the function which is defined by* $\|\cdot\|_\infty$.

Lemma 2.3. *Let* $\psi \in \Psi_n$ *and* $z = (z_1, \ldots, z_n)$*,* $w = (w_1, \ldots, w_n) \in \mathbf{C}^n$.

i) *If* $|z_i| \le |w_i|$ *for every* $i = 1, \ldots, n$*, then* $\|z\|_\psi \le \|w\|_\psi$.

ii) *If* ψ *is strictly convex and* $|z_i| < |w_i|$ *for some* i*, then* $\|z\|_\psi < \|w\|_\psi$.

Mitani et al. defined in [**12**] the dual function ψ^* of $\psi \in \Psi_n$ as

$$\psi^*(s_1, \ldots, s_{n-1}) = \sup_{(t_1, \ldots, t_{n-1}) \in \Delta_n} \frac{(1-t_1-\cdots-t_{n-1})(1-s_1-\cdots-s_{n-1})+t_1 s_1+\cdots+t_{n-1}s_{n-1}}{\psi(t_1, \ldots, t_{n-1})}$$

for $(s_1, \ldots, s_{n-1}) \in \Delta_n$. They also proved that $\psi^* \in \Psi_n$, $(\psi^*)^* = \psi$ and $\|\cdot\|_\psi^* = \|\cdot\|_{\psi^*}$. For ψ and ψ^* the following generalized Hölder inequality is proved.

Proposition 2.4. *Let* $\psi \in \Psi_n$*. Then we have* $| < x, y > | \le \|x\|_\psi \|y\|_{\psi^*}$ *for any* $x, y \in \mathbf{C}^n$.

Let $\psi \in \Psi_n$ and $X_1, \ldots, X_n$ be Banach spaces. The ψ-direct sum of these spaces is the space $\prod_{i=1}^n X_i$ equipped with the norm

$\|(x_1,\dots,x_n)\| = \|(\|x_1\|,\dots,\|x_n\|)\|_\psi$, for $(x_1,\dots,x_n) \in \prod_{i=1}^n X_i$. This space is denoted by $(\sum_{i=1}^n \bigoplus X_i)_\psi$.

About the dual space of the ψ-direct sum of a finite family, the following property is proved in [**12**].

Proposition 2.5. *For every $\psi \in \Psi_n$ the dual space of $(\sum_{i=1}^n \bigoplus X_i)_\psi$ is isometric to $(\sum_{i=1}^n \bigoplus X_i^*)_{\psi^*}$.*

We recall some geometrical notions of Banach spaces.

Let $(X,\|\cdot\|)$ be a Banach space with $\dim X \ge 2$, and let $B_X = \{x \in X : \|x\| \le 1\}$ be the unit ball of X.

For $0 \le \varepsilon \le 2$ the modulus of convexity $\delta_X(\varepsilon)$ of X is defined by

$$\delta_X(\varepsilon) = \inf\Big\{1 - \frac{\|x+y\|}{2} : x, y \in B_X, \|x-y\| \ge \varepsilon\Big\}.$$

It is known that $\delta_X(\varepsilon) \le 1-(1-(\varepsilon^2/4))^{1/2}$ for $0 \le \varepsilon \le 2$ [**11**]. Therefore,

$$\delta_X(\varepsilon) \le \frac{\varepsilon}{2} \quad \text{for } 0 \le \varepsilon \le 2. \tag{2.1}$$

The space X is called uniformly convex if $\delta_X(\varepsilon) > 0$ for every $0 < \varepsilon \le 2$.

The space X is called uniformly non-square, if there exists a $\delta > 0$ such that $\min\{\|(x-y)/2\|, \|(x+y)/2\|\} < 1-\delta$ for every $x, y \in B_X$. It is clear that X is uniformly non-square if and only if there exists $0 < \varepsilon < 2$, such that $\delta_X(\varepsilon) > 0$.

For $\tau > 0$ the modulus of smoothness $\rho_X(\tau)$ of X is defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\right\}.$$

The space X is called uniformly smooth if $\lim_{\tau\to 0}(\rho_X(\tau))/\tau = 0$.

The following duality result is proved in [**5**] (see [**11**]).

Proposition 2.6. *For every Banach space X we have*

i) $\rho_{X^*}(\tau) = \sup\{(\tau\epsilon/2) - \delta_X(\varepsilon) : 0 \le \varepsilon \le 2\}$, *for every $\tau > 0$, and*

ii) *X is uniformly convex if and only if X^* is uniformly smooth.*

Let $\mathcal{X} = (X_i)_{i\in I}$ be a family of Banach spaces. For $0 \le \varepsilon \le 2$ the modulus of convexity $\delta_{\mathcal{X}}(\varepsilon)$ of $\mathcal{X}$ is defined by $\delta_{\mathcal{X}}(\varepsilon) = \inf_{i\in I}\delta_{X_i}(\varepsilon)$. The family $\mathcal{X}$ is called uniformly convex (respectively uniformly non-square) if $\delta_{\mathcal{X}}(\varepsilon) > 0$ for every (respectively for some) $0 < \varepsilon < 2$. For $\tau > 0$ the modulus of smoothness $\rho_{\mathcal{X}}(\tau)$ of $\mathcal{X}$ is defined by $\rho_{\mathcal{X}}(\tau) = \sup_{i\in I}\rho_{X_i}(\tau)$. The family $\mathcal{X}$ is called uniformly smooth if $\lim_{\tau\to 0}(\rho_{\mathcal{X}}(\tau))/\tau = 0$.

We put $\varepsilon(\mathcal{X}) = \sup\{0 \le \varepsilon \le 2 : \delta_{\mathcal{X}}(\varepsilon) = 0\}$. If $\mathcal{X} = \{X\}$ we write $\varepsilon(X)$.

It is clear that

i) $\varepsilon(\mathcal{X}) = 0$ if and only if the family $\mathcal{X}$ is uniformly convex, and

ii) $0 \le \varepsilon(\mathcal{X}) < 2$ if and only if the family $\mathcal{X}$ is uniformly non-square.

Using Proposition 2.6 we obtain the following result for a family of Banach spaces, which is proved in [**15**].

Proposition 2.7. *Let $\mathcal{X} = (X_i)_{i\in I}$ be a family of Banach spaces and $\mathcal{X}^* = (X_i^*)_{i\in I}$ be the family of the dual spaces. The family $\mathcal{X}$ is uniformly convex if and only if the family $\mathcal{X}^*$ is uniformly smooth.*

For more details about these geometrical notions see [**1, 11**].

3. ℓ_ψ spaces and infinite ψ-direct sums. Let $\Delta^{<\omega} = \{s = (s_n)_{n\in\mathbf{N}} \in [0,1]^{\mathbf{N}} : \{n \in \mathbf{N} : s_n \ne 0\}$ *is finite and* $\sum_{n=1}^{\infty} s_n \le 1\}$ and $\psi : \Delta^{<\omega} \to \mathbf{R}$.

We say that the sequence $(\psi_n)_{n=2}^{\infty}$, where $\psi_n : \Delta_n \to \mathbf{R}$ is defined by $\psi_n(s_1,\dots,s_{n-1}) = \psi(s_1,\dots,s_{n-1},0,0,\dots)$ for $n = 2,3,\dots,$ is *associated with* ψ.

Let $s = (s_n)_{n\in\mathbf{N}} \in \Delta^{<\omega}$, such that $s \ne 0$. We put $n_s = \max\{n : s_n \ne 0\}$. Then $\psi(s) = \psi_n(s_1,\dots,s_{n-1})$ for every $n > n_s$.

It is also clear that:

i) $\psi_n(s_1,\dots,s_{n-1}) = \psi_{n+1}(s_1,\dots,s_{n-1},0)$ for every $n \ge 2$ and $(s_1,\dots,s_{n-1}) \in \Delta_n$.

ii) ψ is strictly convex if and only if ψ_n is strictly convex for every $n \ge 2$.

iii) If ψ is continuous, then ψ_n is continuous for every $n \geq 2$.

Let $\Psi_\omega = \{\psi : \Delta^{<\omega} \to \mathbf{R} : \psi_n \in \Psi_n \textit{ for every } n = 2, 3, \dots\}$. It is easy to see that a $\psi \in \Psi_\omega$ is characterized by its associated sequence $(\psi_n)_{n=2}^\infty$ in the sense of the following lemma.

Lemma 3.1. *Let $(\psi_n)_{n=2}^\infty$ be a sequence of functions such that $\psi_n \in \Psi_n$ and $\psi_n(s_1, \dots, s_{n-1}) = \psi_{n+1}(s_1, \dots, s_{n-1}, 0)$ for every $(s_1, \dots, s_{n-1}) \in \Delta_n$ and $n = 2, 3, \dots$. The function $\psi : \Delta^{<\omega} \to \mathbf{R}$, defined by*

$$\psi(s) = \begin{cases} \psi_{n_s+1}(s_1, \dots, s_{n_s}) & \textit{if } s \neq 0 \\ 1 & \textit{if } s = 0, \end{cases}$$

belongs to Ψ_ω, and the sequence $(\psi_n)_{n=2}^\infty$ is its associated sequence.

An element of Ψ_ω is not necessarily continuous. For example, if $\psi(s) = \sup\{1 - \sum_{n=1}^\infty s_n,\ s_1,\ s_2, \dots\}$ for $s = (s_n)_{n \in \mathbf{N}} \in \Delta^{<\omega}$, then $\psi \in \Psi_\omega$, but this is not continuous.

Typical examples of elements of Ψ_ω are the functions

$$\psi_p(s) = \begin{cases} \left[\left(1 - \sum_{n=1}^\infty s_n\right)^p + \sum_{n=1}^\infty s_n^p\right]^{1/p} & \text{if } 1 \leq p < \infty \\ \sup\left\{1 - \sum_{n=1}^\infty s_n,\ s_1, s_2, \dots\right\} & \text{if } p = \infty, \end{cases}$$

for $s = (s_n)_{n \in \mathbf{N}} \in \Delta^{<\omega}$.

Using Lemma 2.2 (ii), we obtain the following corollary.

Corollary 3.2. *If $\psi \in \Psi_\omega$, then $\psi_\infty(s) \leq \psi(s) \leq 1$ for $s \in \Delta^{<\omega}$.*

Let $c_{00} = \{z = (z_n)_{n \in \mathbf{N}} \in \mathbf{C}^\mathbf{N} : \{n \in \mathbf{N} : z_n \neq 0\} \textit{ be finite}\}$, and let $(e_n)_{n \in \mathbf{N}}$ be the usual Hamel basis of c_{00}. For $z = (z_n)_{n \in \mathbf{N}} \in \mathbf{C}^\mathbf{N}$ we denote $|z| = (|z_n|)_{n \in \mathbf{N}}$.

Definition 3.3. A norm $\|\cdot\|$ on c_{00} is called absolute if $\|z\| = \||z|\|$ for every $z \in c_{00}$ and is called normalized if $\|e_n\| = 1$ for every $n = 1, 2, \dots$.

We denote by AN_ω the set of all absolute and normalized norms on c_{00}. For $\psi \in \Psi_\omega$, we define

$$\|(z_1, \dots, z_n, \dots)\|_\psi = \begin{cases} \left(\sum_{i=1}^\infty |z_i|\right) \psi \left(\frac{|z_2|}{\sum_{i=1}^\infty |z_i|}, \dots, \frac{|z_n|}{\sum_{i=1}^\infty |z_i|}, \dots \right) & \text{if } (z_1, \dots, z_n, \dots) \neq (0, \dots, 0) \\ 0 & \text{if } (z_1, \dots, z_n, \dots) = (0, \dots, 0) \end{cases}$$

for every $(z_1, \dots, z_n, \dots) \in c_{00}$.

It is obvious that $\psi(s) = \|(1 - \sum_{n=1}^\infty s_n, s_1, s_2, \dots)\|_\psi$ for $s = (s_n)_{n\in\mathbf{N}} \in \Delta^{<\omega}$. From Theorem 2.1 we obtain the following corollary.

Corollary 3.4. i) *If $\psi \in \Psi_\omega$, then $\|\cdot\|_\psi \in AN_\omega$.*

ii) *For any $\|\cdot\| \in AN_\omega$, the function $\psi : \Delta^{<\omega} \to \mathbf{R}$, defined by $\psi(s) = \|(1 - \sum_{n=1}^\infty s_n, s_1, s_2, \dots)\|$, belongs to Ψ_ω, and $\|(z_1, \dots, z_n, \dots)\| = \|(z_1, \dots, z_n, \dots)\|_\psi$ for every $(z_1, \dots, z_n, \dots) \in c_{00}$.*

For $n = 1, 2, \dots$, we denote by P_n the projection $P_n : \mathbf{C}^\mathbf{N} \to \mathbf{C}^n$ defined by $P_n(z_1, \dots, z_n, \dots) = (z_1, \dots, z_n)$ for $(z_1, \dots, z_n, \dots) \in \mathbf{C}^\mathbf{N}$. If $\psi \in \Psi_\omega$ and $(\psi_n)_{n=2}^\infty$ is its associated sequence, then for every $z \in c_{00}$ the sequence $(\|P_n(z)\|_{\psi_n})_{n=2}^\infty$ is non-decreasing.

Definition 3.5. Let $\psi \in \Psi_\omega$.

i) We define ℓ_ψ to be the completion of $(c_{00}, \|\cdot\|_\psi)$. The extended norm is denoted also by $\|\cdot\|_\psi$.

ii) We define $\ell_{\psi,\infty} = \{z \in \mathbf{C}^\mathbf{N} : \sup_n \|P_n(z)\|_{\psi_n} < +\infty\}$ equipped with the norm $\|z\|_{\psi,\infty} = \sup_n \|P_n(z)\|_{\psi_n}$ for $z \in \ell_{\psi,\infty}$.

Using Lemma 2.2 (i), we obtain $\sup_n |z_n| \leq \|z\|_{\psi,\infty} \leq \sum_{n=1}^\infty |z_n|$ for every $z = (z_n)_{n\in\mathbf{N}} \in \ell_{\psi,\infty}$ (the last sum may be infinite).

Proposition 3.6. *$\ell_{\psi,\infty}$ is a Banach space, for every $\psi \in \Psi_\omega$.*

Proof. Let $(w_n)_n$ be a Cauchy sequence in $\ell_{\psi,\infty}$. For every k the sequence $(P_k(w_n))_n$ is a Cauchy sequence in the Banach space

$(\mathbf{C}^k, \|\cdot\|_{\psi_k})$, and let $\lim_n P_k(w_n) = (v_1^k, \dots, v_k^k)$. Then $v_i^k = v_i^l$ for $1 \le i \le k$ and $l \ge k$. We set $w = (v_1^1, \dots, v_k^k, \dots)$ and we get $\|P_k(w_n) - P_k(w)\|_{\psi_k} \to 0$ for every $k = 2, 3, \dots$.

For any $\varepsilon > 0$ there exists an $n_0 \in \mathbf{N}$, such that $\|w_n - w_m\|_\psi < \varepsilon$ for every $n, m \ge n_0$. Therefore, $\|P_k(w_n) - P_k(w_m)\|_{\psi_k} < \varepsilon$ for $n, m \ge n_0$ and $k \ge 2$. Thus, $\|P_k(w_n) - P_k(w)\|_{\psi_k} < \varepsilon$ for every $n \ge n_0$ and $k \ge 2$. From this we obtain $w \in \ell_{\psi,\infty}$ and $\|w_n - w\| \to 0$.

Proposition 3.7. *Let $\psi \in \Psi_\omega$.*

i) *ℓ_ψ is a closed subspace of $\ell_{\psi,\infty}$.*

ii) *The sequence $(e_n)_{n\in\mathbf{N}}$ is a monotone and unconditional basis of ℓ_ψ.*

iii) *$(e_n)_{n\in\mathbf{N}}$ is a boundedly complete basis of ℓ_ψ if and only if $\ell_\psi = \ell_{\psi,\infty}$.*

iv) *If $(e_n)_{n\in\mathbf{N}}$ is a shrinking basis of ℓ_ψ, then $\ell_{\psi,\infty}$ is isometric to the second dual of ℓ_ψ.*

v) *$(e_n)_{n\in\mathbf{N}}$ is a shrinking and boundedly complete basis of ℓ_ψ if and only if ℓ_ψ is reflexive.*

Proof. (i), (ii) and (iii) are obvious from Definition 3.5. (iv) and (v) follow from James' theorems about unconditional bases and reflexivity [**7**] (see also in [**10**]).

Examples. i) The spaces ℓ_p, for $1 \le p < \infty$, are typical examples of ℓ_ψ spaces with corresponding functions ψ_p, $1 \le p < \infty$. For $p = \infty$, the space $\ell_{\psi_p,\infty}$ is identified to ℓ_∞ and its subspace ℓ_{ψ_p} is identified to c_0.

ii) Let $p \ge 1$ and $w = (w_n)_{n\in\mathbf{N}} \in c_0 \setminus \ell_1$, be such that $1 = w_1 \ge w_2 \ge \cdots \ge w_n \ge \cdots$. The Lorentz sequence space $d(w,p)$ is the space $d(w,p) = \{z = (z_n)_{n\in\mathbf{N}} : \sum_{n=1}^\infty w_n {z_n^*}^p < \infty\}$, where $(z_n^*)_{n\in\mathbf{N}}$ is a non-increasing rearrangement of $(|z_n|)_{n\in\mathbf{N}}$, equipped with the norm $\|z\|_{(w,p)} = (\sum_{n=1}^\infty w_n {z_n^*}^p)^{1/p}$. Since the restriction of $\|\cdot\|_{(w,p)}$ on c_{00} belongs to AN_ω, the function $\psi_{(w,p)}(s) =$

$\|(1-\sum_{n=1}^{\infty} s_n, s_1, s_2, \ldots, s_n, \ldots)\|_{(w,p)}$ for $s = (s_n)_{n\in\mathbf{N}} \in \Delta^{<\omega}$ belongs to Ψ_ω. The space $\ell_{\psi_{(w,p)}}$ is the Lorentz sequence space $d(w,p)$.

iii) An Orlicz function is a continuous, convex and non-decreasing function $M : [0,\infty) \to \mathbf{R}$ such that $M(0) = 0$ and $\lim_{t\to+\infty} M(x) = +\infty$. To any Orlicz function M we associate the Orlicz sequence space $\ell_M = \{z = (z_n)_{n\in\mathbf{N}} :$ there exists $t > 0$ such that $\sum_{n=1}^{\infty} M(|z_n|/t) < \infty\}$ equipped with the norm $\|z\|_M = \inf\{t > 0 : \sum_{n=1}^{\infty} M(|z_n|/t) \leq 1\}$ for $z = (z_n)_{n\in\mathbf{N}} \in \mathbf{C}^{\mathbf{N}}$. Of particular interest is the subspace h_M of ℓ_M consisting of those $z = (z_n)_{n\in\mathbf{N}} \in \ell_M$ for which $\sum_{n=1}^{\infty} M(|z_n|/t) < \infty$ for every $t > 0$. It is proved that the unit vectors $(e_n)_{n\in\mathbf{N}}$ form a symmetric basis of h_M, and $\ell_M = h_M$ if and only if $\limsup_{t\to 0} M(27)/M(t) < \infty$ [**10**]. Let M be an Orlicz function with $M(1) = 1$. Since the restriction of $\|\cdot\|_M$ on c_{00} belongs to AN_ω, the function $\psi_M(s) = \|(1-\sum_{n=1}^{\infty} s_n, s_1, s_2, \ldots, s_n, \ldots)\|_M$ for $s = (s_n)_{n\in\mathbf{N}} \in \Delta^{<\omega}$ belongs to Ψ_ω. It is easy to see that $\ell_{\psi_M,\infty} = \ell_M$ and $\ell_{\psi_M} = h_M$.

There are also other classical sequence Banach spaces which are spaces of this type. We can deduce plenty other examples of such spaces from the examples in [**8, 16, 17, 18**], as well.

In order to define the dual function of $\psi \in \Psi_\omega$ we need the following lemma.

Lemma 3.8. *Let $\psi \in \Psi_\omega$, $(\psi_n)_{n\in\mathbf{N}}$ be its associated sequence and ψ_n^* the dual function of ψ_n for $n = 2, 3, \ldots$. Then $\psi_{n+1}^*(s_1, \ldots, s_{n-1}, 0) = \psi_n^*(s_1, \ldots, s_{n-1})$ for every $(s_1, \ldots, s_{n-1}) \in \Delta_n$ and $n = 2, 3 \ldots$.*

Proof. Let $n \geq 2$ and $(s_1, \ldots, s_{n-1}) \in \Delta_n$. Then we have

$$\begin{aligned}
&\psi_{n+1}^*(s_1, \ldots, s_{n-1}, 0) \\
&\qquad = \sup_{(t_1,\ldots,t_n)\in\Delta_{n+1}} \frac{(1-\sum_{i=1}^{n} t_i)(1-\sum_{i=1}^{n-1} s_i) - \sum_{i=1}^{n-1} t_i s_i}{\psi_{n+1}(t_1, \ldots, t_n)} \\
&\qquad \geq \sup_{(t_1,\ldots,t_{n-1},\,0)\in\Delta_{n+1}} \frac{(1-\sum_{i=1}^{n-1} t_i)(1-\sum_{i=1}^{n-1} s_i) - \sum_{i=1}^{n-1} t_i s_i}{\psi_{n+1}(t_1, \ldots, t_{n-1}, 0)} \\
&\qquad = \sup_{(t_1,\ldots,t_{n-1})\in\Delta_n} \frac{(1-\sum_{i=1}^{n-1} t_i)(1-\sum_{i=1}^{n-1} s_i) - \sum_{i=1}^{n-1} t_i s_i}{\psi_n(t_1, \ldots, t_{n-1})}
\end{aligned}$$

$$= \psi_n^*(s_1, \dots, s_{n-1}).$$

Using property (A_{n+1}) of ψ_{n+1}, we obtain

$$\begin{aligned}
&\psi_{n+1}^*(s_1, \dots, s_{n-1}, 0)\\
&= \sup_{(t_1,\dots,t_n)\in\Delta_{n+1}} \frac{(1-\sum_{i=1}^{n} t_i)(1-\sum_{i=1}^{n-1} s_i)-\sum_{i=1}^{n-1} t_i s_i}{\psi_{n+1}(t_1,\dots,t_n)}\\
&= \sup_{\substack{(t_1,\dots,t_n)\in\Delta_{n+1}\\ t_n\neq 1}} \frac{(1-\sum_{i=1}^{n} t_i)/(1-t_n)(1-\sum_{i=1}^{n-1} s_i)-\sum_{i=1}^{n-1}(t_i/1-t_n)s_i}{(\psi_{n+1}(t_1,\dots,t_n))/(1-t_n)}\\
&\leq \sup_{\substack{(t_1,\dots,t_n)\in\Delta_{n+1}\\ t_n\neq 1}} \frac{(1-\sum_{i=1}^{n-1}(t_i/1-t_n))(1-\sum_{i=1}^{n-1} s_i)-\sum_{i=1}^{n-1}(t_i/1-t_n)s_i}{\psi_{n+1}((t_1/1-t_n),\dots,(t_{n-1}/1-t_n),0)}\\
&= \sup_{(t_1,\dots,t_{n-1})\in\Delta_n} \frac{(1-\sum_{i=1}^{n-1} t_i)(1-\sum_{i=1}^{n-1} s_i)-\sum_{i=1}^{n-1} t_i s_i}{\psi_n(t_1,\dots,t_{n-1})}\\
&= \psi_n^*(s_1, \dots, s_{n-1})
\end{aligned}$$

Using Lemma 3.1 and Lemma 3.8 we define the dual function of a $\psi \in \Psi_\omega$ as follows.

Definition 3.9. Let $\psi \in \Psi_\omega$. If $(\psi_n)_{n\in\mathbf{N}}$ is the associated sequence of $\psi \in \Psi_\omega$ and ψ_n^* is the dual function of ψ_n for $n = 2, 3, \dots$, the dual function $\psi*$ of ψ is defined to be the function with associated sequence $(\psi_n^*)_{n=2}^{\infty}$.

From Proposition 2.4, we obtain the following generalized Hölder inequality.

Lemma 3.10. *If $\psi \in \Psi_\omega, z = (z_n)_{n\in\mathbf{N}} \in \ell_{\psi,\infty}$ and $w = (w_n)_{n\in\mathbf{N}} \in \ell_{\psi^*,\infty}$, then $\sum_{n=1}^{\infty} |w_n z_n| \leq \|w\|_{\psi^*,\infty} \|z\|_{\psi,\infty}$.*

In order to see the relation between the spaces ℓ_ψ and ℓ_{ψ^*}, we need the following two lemmas.

Lemma 3.11. *Let $\psi \in \Psi_n$. If $(s_1, \dots, s_{n-1}) \in \vartheta\Delta_n$, then $\psi^*(s_1, \dots, s_{n-1}) = \sup_{(t_1,\dots,t_{n-1})\in\vartheta\Delta_n} (\sum_{i=1}^{n-1} t_i s_i)/\psi(t_1, \dots, t_{n-1})$, where $\vartheta\Delta_n$ is the boundary of Δ_n.*

Proof. Let $(s_1, \dots, s_{n-1}) \in \vartheta\Delta_n$. Since $\vartheta\Delta_n = \{(t_1, \dots, t_{n-1}) \in [0,1]^{n-1} : \sum_{i=1}^{n-1} t_i = 1\}$ we get $\psi^*(s_1, \dots, s_{n-1}) \geq \sup_{(t_1,\dots,t_{n-1}) \in \vartheta\Delta_n} \times (\sum_{i=1}^{n-1} t_i s_i)/\psi(t_1, \dots, t_{n-1})$. Using property A_1 of ψ we obtain

$$\begin{aligned}
\psi^*(s_1, \dots, s_{n-1}) &= \sup_{\substack{(t_1,\dots,t_{n-1}) \in \Delta_n \\ \sum_{i=1}^{n-1} t_i \neq 0}} \frac{\sum_{j=1}^{n-1} (t_j/(\sum_{i=1}^{n-1} t_i)) s_j}{\psi(t_1, \dots, t_{n-1})/\sum_{i=1}^{n-1} t_i} \\
&\leq \sup_{\substack{(t_1,\dots,t_{n-1}) \in \Delta_n \\ \sum_{i=1}^{n-1} t_i \neq 0}} \frac{\sum_{j=1}^{n-1} (t_j/\sum_{i=1}^{n-1} t_i) s_j}{\psi(t_1/(\sum_{i=1}^{n-1} t_i), \dots, t_{n-1}/(\sum_{i=1}^{n-1} t_i))} \\
&= \sup_{(t_1,\dots,t_{n-1}) \in \vartheta\Delta_n} \frac{\sum_{i=1}^{n-1} t_i s_i}{\psi(t_1, \dots, t_{n-1})}.
\end{aligned}$$

Thus, $\psi^*(s_1, \dots, s_{n-1}) = \sup_{(t_1,\dots,t_{n-1}) \in \vartheta\Delta_n} (\sum_{i=1}^{n-1} t_i s_i)/\psi(t_1, \dots, t_{n-1})$.

Lemma 3.12. *Let $\psi \in \Psi_\omega$. If $(e_n)_{n \in \mathbf{N}}$ is a boundedly complete basis of ℓ_{ψ^*} and $x^* \in \ell_\psi^*$, then $w = \sum_{n=1}^{\infty} w_n e_n \in \ell_{\psi^*}$, where $w_n = x^*(e_n)$ for every $n \in \mathbf{N}$.*

Proof. For $n = 1, 2, \dots$ we put

$$c_n = \begin{cases} |w_n|/w_n & \text{if } w_n \neq 0 \\ 0 & \text{if } w_n = 0 \end{cases}.$$

For every $n \geq 2$, we obtain

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} w_i e_i \right\|_{\psi^*} \\
&= \left(\sum_{i=1}^{n} |w_i| \right) \psi_n^* \left(\frac{|w_2|}{\sum_{i=1}^{n} |w_i|}, \dots, \frac{|w_n|}{\sum_{i=1}^{n} |w_i|} \right) = \left(\sum_{i=1}^{n} |w_i| \right) \\
&\times \sup_{(t_1,\dots,t_{n-1}) \in \Delta_n} \frac{(1 - \sum_{i=1}^{n-1} t_i)|w_1|/(\sum_{i=1}^{n} |w_i|) + 1/(\sum_{i=1}^{n} |w_i|) \sum_{j=1}^{n-1} t_j |w_{j+1}|}{\psi_n(t_1, \dots, t_{n-1})} \\
&= \sup_{(t_1,\dots,t_{n-1}) \in \Delta_n} \frac{(1 - \sum_{i=1}^{n-1} t_i)|w_1| + \sum_{j=1}^{n-1} t_j |w_{j+1}|}{\psi_n(t_1, \dots, t_{n-1})}
\end{aligned}$$

$$= \sup_{(t_1,\dots,t_{n-1})\in\Delta_n} \frac{c_1(1-\sum_{i=1}^{n-1} t_i)x^*(e_1)+\sum_{j=1}^{n-1} c_j t_j x^*(e_{j+1})}{\psi_n(t_1,\dots,t_{n-1})}$$
$$= \sup_{(t_1,\dots,t_{n-1})\in\Delta_n} \frac{x^*(c_1(1-\sum_{i=1}^{n} t_i),\, c_2t_1,\dots,c_nt_{n-1},\, 0,\dots)}{\psi_n(t_1,\dots,t_{n-1})}$$
$$\leq \sup_{(t_1,\dots,t_{n-1})\in\Delta_n} \frac{\|x^*\|\left\|(1-\sum_{i=1}^{n-1} t_i,\, t_1,\dots,t_{n-1})\right\|_{\psi_n}}{\psi_n(t_1,\dots,t_{n-1})}$$
$$= \|x^*\|.$$

Since (e_n) is boundedly complete, we have $w = \sum_{i=1}^\infty w_i e_i \in \ell_{\psi^*}$.

Proposition 3.13. *For $\psi \in \Psi_\omega$, $(e_n)_{n\in\mathbf{N}}$ is a shrinking basis of ℓ_ψ if and only if $(e_n)_{n\in\mathbf{N}}$ is a boundedly complete basis of ℓ_{ψ^*}.*

Proof. We suppose that $(e_n)_{n\in\mathbf{N}}$ is a shrinking basis of ℓ_ψ. Let $(w_n)_{n\in\mathbf{N}} \in \ell_{\psi^*,\infty}$ and $M = \sup_n \|P_n(w)\|_{\psi_n^*}$. From Lemma 3.10, we have $\sum_{n=1}^\infty |w_n||z_n| \leq \|w\|_{\psi^*,\infty} \|z\|_{\psi,\infty}$ for every $z = (z_n)_{n\in\mathbf{N}} \in \ell_{\psi,\infty}$. Let $\varepsilon > 0$ and $x^* \in \ell_\psi^*$ defined by $x^*(z) = \sum_{n=1}^\infty |w_n| z_n$ for $z = (z_n)_{n\in\mathbf{N}} \in \ell_\psi$. Since $(e_n)_{n\in\mathbf{N}}$ is a shrinking basis of ℓ_ψ, there exists an $n_0 \in \mathbf{N}$ such that $\left\| x^*|_{[(e_i)_{i=n}^\infty]} \right\| < \varepsilon$ for every $n \geq n_0$. Using Lemma 3.11, for every $n > m \geq n_0$, we obtain

$$\begin{aligned} \left\|\sum_{i=m}^n w_i e_i\right\|_{\psi^*} &= \left(\sum_{i=m}^n |w_i|\right)\psi_n^*\left(0,\dots,0,\frac{|w_m|}{\sum_{i=m}^n |w_i|},\dots,\frac{|w_n|}{\sum_{i=m}^n |w_i|}\right) \\ &= \sup_{(t_1,\dots,t_{n-1})\in\vartheta\Delta_n} \frac{\sum_{i=m}^n t_{i-1}|w_i|}{\psi_n(t_1,\dots,t_{n-1})} \\ &= \sup_{(t_1,\dots,t_{n-1})\in\vartheta\Delta_n} \frac{x^*(0,\dots,0,t_{m-1},\dots,t_{n-1},0,\dots)}{\psi_n(t_1,\dots,t_{n-1})}. \end{aligned}$$

Using properties $(A_2),\dots,(A_{m-2})$ of ψ_n, we obtain for every $(t_1,\dots, t_{n-1}) \in \vartheta\Delta_n$,

$$\begin{aligned} &\|(0,\dots,0,t_{m-1},\dots,t_{n-1},0,\dots)\|_\psi \\ &\qquad = \left(\sum_{i=m-1}^{n-1} t_i\right)\psi_n\left(0,\dots,0,\frac{t_{m-1}}{\sum_{i=m-1}^{n-1} t_i},\dots,\frac{t_{n-1}}{\sum_{i=m-1}^{n-1} t_i}\right) \\ &\qquad \leq \psi_n(t_1,\dots,t_{n-1}). \end{aligned}$$

Thus, $\|\sum_{i=m}^{n} w_i e_i\|_{\psi^*} \le \left\| x^*|_{[(e_i)_{i=m}^{\infty}]} \right\| < \varepsilon$. Therefore, $w = \sum_{i=1}^{\infty} w_i e_i \in \ell_{\psi^*}$ and so, (e_n) is a boundedly complete basis of ℓ_{ψ^*}.

We suppose that $(e_n)_{n\in\mathbf{N}}$ is a boundedly complete basis of ℓ_{ψ^*}, and let $x^* \in \ell_\psi^*$. We put $w = (w_n)_{n\in\mathbf{N}}$, where $w_n = x^*(e_n)$ for $n = 1, 2, \dots$. From Lemma 3.12 we obtain $w = \sum_{i=1}^{\infty} w_i e_i \in \ell_{\psi^*}$. Let $\varepsilon > 0$. There exists an $n_0 \in \mathbf{N}$ such that $\|\sum_{i=n}^{\infty} w_i e_i\|_{\psi^*} < \varepsilon$ for every $n \ge n_0$. Let $n \ge n_0$ and $z = (z_i)_{i\in\mathbf{N}} \in \ell_\psi$ be such that $z_1 = \dots = z_n = 0$ and $\|z\|_\psi \le 1$. Then $|x^*(z)| = |\sum_{i=n+1}^{\infty} z_i w_i| \le \|z\|_\psi \left\|\sum_{i=n+1}^{\infty} w_i e_i\right\|_{\psi^*} \le \varepsilon$. Thus, $(e_n)_{n\in\mathbf{N}}$ is a shrinking basis of ℓ_ψ.

For $\psi \in \Psi_\omega$ we put $T_\psi : \ell_{\psi^*} \to \ell_\psi^*$ defined by $T_\psi(w)(z) = \sum_{n=1}^{\infty} w_n z_n$ for every $w = (w_n)_{n\in\mathbf{N}} \in \ell_{\psi^*}$ and $z = (z_n)_{n\in\mathbf{N}} \in \ell_\psi$. From Lemma 3.10 we obtain that T_ψ is a bounded operator and $\|T_\psi(w)\| \le \|w\|_{\psi^*}$ for every $w \in \ell_{\psi^*}$.

Lemma 3.14. *Let $\psi \in \Psi_\omega$. Then $\|T_\psi(w)\| = \|T_\psi(|w|)\|$ for every $w \in \ell_{\psi^*}$.*

Proof. Let $w = (w_n)_{n\in\mathbf{N}} \in \ell_{\psi^*}$. For every $\varepsilon > 0$ there exists $z = (z_n)_{n\in\mathbf{N}} \in \ell_\psi$ with $\|z\| \le 1$ such that $\|T_\psi(w)\| \le |T_\psi(w)(z)| + \varepsilon = |\sum_{n=1}^{\infty} w_n z_n| + \varepsilon \le \sum_{n=1}^{\infty} |w_n||z_n| + \varepsilon = (T_\psi(|w|))(|z|) + \varepsilon \le \|T_\psi(|w|)\| + \varepsilon$. Therefore, $\|T_\psi(w)\| \le \|T_\psi(|w|)\|$.

For $n = 1, 2, \dots$, we put

$$c_n = \begin{cases} |w_n|/w_n & \text{if } w_n \ne 0 \\ 0 & \text{if } w_n = 0. \end{cases}$$

For every $\varepsilon > 0$ there exists a $z = (z_n)_{n\in\mathbf{N}} \in \ell_\psi$ with $\|z\| \le 1$ such that $\|T_\psi(|w|)\| \le |T_\psi(|w|)(z)| + \varepsilon = |\sum_{n=1}^{\infty} |w_n| z_n| + \varepsilon = |\sum_{n=1}^{\infty} c_n w_n z_n| + \varepsilon$.

We put $z' = (z'_n)_{n\in\mathbf{N}}$, where $z'_n = c_n z_n$ for $n \in \mathbf{N}$. Since $|z'_n| \le |z_n|$, we have $\|z'\| \le \|z\| \le 1$. So $\|T_\psi(|w|)\| \le |\sum_{n=1}^{\infty} w_n z'_n| + \varepsilon = |T_\psi(w)(z')| + \varepsilon \le \|T_\psi(w)\| + \varepsilon$. Therefore, $\|T_\psi(|w|)\| \le \|T_\psi(w)\|$.

By the following theorem we characterized the dual space of ℓ_ψ, when $(e_n)_{n\in\mathbf{N}})$ is a shrinking basis of ℓ_ψ.

Theorem 3.15. *Let $\psi \in \Psi_\omega$.*

i) *T_ψ is an isometry and $T_\psi(\ell_{\psi^*}) = [(e_n^*)_{n\in\mathbf{N}}]$, where $(e_n^*)_{n\in\mathbf{N}}$ is the sequence of biorthogonal functionals of $(e_n)_{n\in\mathbf{N}}$.*

ii) *If* (e_n) *is a shrinking basis of* ℓ_ψ*, then the dual space of* ℓ_ψ *is isometric to* ℓ_{ψ^*}*.*

Proof. i) Let $w = (w_n)_{n\in\mathbf{N}} \in \ell_{\psi^*}$ and $\varepsilon > 0$. Then there exists an $n \in \mathbf{N}$ such that

$$\begin{aligned}
\|w\|_{\psi^*} &\leq (1+\varepsilon)\,\|P_n(w)\|_{\psi_n^*}\\
&= (1+\varepsilon)\left(\sum_{i=1}^{n}|w_i|\right)\psi_n^*\left(\frac{|w_2|}{\sum_{i=1}^{n}|w_i|},\dots,\frac{|w_n|}{\sum_{i=1}^{n}|w_i|}\right)\\
&= (1+\varepsilon)\left(\sum_{i=1}^{n}|w_i|\right)\sup_{(t_1,\dots,t_{n-1})\in\Delta_n}\\
&\quad\times\frac{\left(1-\sum_{i=1}^{n-1}t_i\right)|w_1|\Big/\left(\sum_{i=1}^{n}|w_i|\right)+1\Big/\left(\sum_{i=1}^{n}|w_i|\right)\sum_{j=1}^{n-1}t_j|w_{j+1}|}{\psi_n(t_1,\dots,t_{n-1})}\\
&= (1+\varepsilon)\sup_{(t_1,\dots,t_{n-1})\in\Delta_n}\frac{\left(1-\sum_{i=1}^{n-1}t_i\right)|w_1|+\sum_{j=1}^{n-1}t_j|w_{j+1}|}{\psi_n(t_1,\dots,t_{n-1})}.
\end{aligned}$$

For $(t_1,\dots,t_{n-1}) \in \Delta_n$ we put $z_1 = 1-\sum_{i=1}^{n-1}t_i$, $z_2 = t_1,\dots,z_n = t_{n-1}$ and $z_m = 0$ for every $m > n$. Then $z = (z_n)_{n\in\mathbf{N}} \in \ell_\psi$ and $\|z\|_\psi = \|P_n(z)\|_{\psi_n} = \psi_n(t_1,\dots,t_{n-1}) \leq 1$. Therefore,

$$\|w\|_{\psi^*} \leq (1+\varepsilon)\sup_{\|z\|\leq 1}\frac{|T(|w|)(z)|}{\|z\|_\psi} \leq (1+\varepsilon)\,\|T(|w|)\|\,.$$

So, from Lemma 3.14 we obtain $\|w\|_{\psi^*} \leq \|T_\psi(w)\|$. Thus, $\|T_\psi(w)\| = \|w\|_{\psi^*}$.

It is clear that $T_\psi(e_n) = e_n^*$ for every $n \in \mathbf{N}$. Therefore, $T_\psi(\ell_{\psi^*}) = [(e_n^*)_{n\in\mathbf{N}}]$.

ii) Because $(e_n)_{n\in\mathbf{N}}$ is a shrinking basis of ℓ_ψ, the sequence of biorthogonal functionals $(e_n^*)_{n\in\mathbf{N}}$ is a basis of ℓ_ψ^*, and the result follows from (i).

For $\psi \in \Psi_\omega$ it is clear that $\delta_{\ell_\psi}(\varepsilon) = \inf_{n\in\mathbf{N}} \delta_{(\mathbf{C}^n,\|\cdot\|_{\psi_n})}(\varepsilon)$ for every $0 \leq \varepsilon \leq 2$. It is proved in [**18**] that for $\psi \in \Psi_n$ the space $(\mathbf{C}^n, \|\cdot\|_\psi)$ is strictly convex and, because it has finite dimension, uniformly convex if and only if ψ is strictly convex. This result is not true in the infinite case. As the next example shows, there exists a $\psi \in \Psi_\omega$ which is strictly convex, and the space ℓ_ψ does not admit an equivalent uniformly convex norm.

Example. Let $\psi : \Delta^{<\omega} \to \mathbf{R}$, such that $\psi(s) = 1/\sqrt{2}[1 + (1 - \sum_{i=1}^{\infty} s_i)^2 + \sum_{i=1}^{\infty} s_i^2]^{1/2}$ for $s = (s_i) \in \Delta^{<\omega}$.

Let $n \geq 2$. Then, $\psi_n(s) = 1/\sqrt{2}[1 + (1 - \sum_{i=1}^{n-1} s_i)^2 + \sum_{i=1}^{n-1} s_i^2]^{1/2}$ for $s = (s_1, \dots, s_{n-1}) \in \Delta_n$.

It is easy to see that ψ_n is continuous and satisfies the properties $A_0, \dots, A_n$. We will prove that ψ_n is strictly convex.

Let $s = (s_1, \dots, s_{n-1})$, $w = (w_1, \dots, w_{n-1})$ be two non collinear elements of Δ_n. Then, using the Minkowski inequality, we obtain

$$\begin{aligned}
\psi_n\left(\frac{s+w}{2}\right) &= \frac{1}{2\sqrt{2}}\left[4 + \left(2 - \sum_{i=1}^{n-1}(s_i + w_i)\right)^2 + \sum_{i=1}^{n-1}(s_i + w_i)^2\right]^{1/2} \\
&= \frac{1}{2\sqrt{2}}\left[(1+1)^2 + \left(\left(1 - \sum_{i=1}^{n-1} s_i\right) + \left(1 - \sum_{i=1}^{n-1} w_i\right)\right)^2 \right. \\
&\qquad \left. + \sum_{i=1}^{n-1}(s_i + w_i)^2\right]^{1/2} \\
&< \frac{1}{2\sqrt{2}}\left[\left(1 + \left(1 - \sum_{i=1}^{n-1} s_i\right)^2 + \sum_{i=1}^{n-1} s_i^2\right)^{1/2} \right. \\
&\qquad \left. + \left(1 + \left(1 - \sum_{i=1}^{n-1} w_i\right)^2 + \sum_{i=1}^{n-1} w_i^2\right)^{1/2}\right] \\
&= \frac{1}{2}[\psi_n(s) + \psi_n(w)].
\end{aligned}$$

Therefore, ψ_n is strictly convex for every $n = 2, \dots$. Thus, $\psi \in \Psi_\omega$ and ψ is strictly convex.

Let $z = (z_n)_{n\in\mathbf{N}} \in \ell_\psi$ and $z \neq 0$. Then,

$$\begin{aligned}\|P_n(z)\|_{\psi_n}^2 &= \Big(\sum_{i=1}^n |z_i|\Big)^2 \psi_n\Big(\frac{|z_2|}{\sum_{i=1}^n |z_i|}, \dots, \frac{|z_n|}{\sum_{i=1}^n |z_i|}\Big)^2 \\ &= \Big(\sum_{i=1}^n |z_i|\Big)^2 \frac{1}{2}\bigg[1 + \Big(1 - \frac{\sum_{i=2}^n |z_i|}{\sum_{i=1}^n |z_i|}\Big)^2 + \frac{\sum_{i=2}^n |z_i|^2}{(\sum_{i=1}^n |z_i|)^2}\bigg] \\ &= \frac{1}{2}\bigg[\Big(\sum_{i=1}^n |z_i|\Big)^2 + \sum_{i=1}^n |z_i|^2\bigg].\end{aligned}$$

From the above we obtain $1/\sqrt{2}\,\|z\|_1 \le \|z\|_\psi \le \|z\|_1$. Thus, $\|\cdot\|_\psi$ is equivalent to $\|\cdot\|_1$.

Therefore, the space ℓ_ψ is not reflexive and so, this space does not admit an equivalent uniformly convex norm.

Question: Is it possible to characterize strict convexity, uniform convexity, smoothness, uniform smoothness and uniform non-squareness of a space ℓ_ψ in terms of the function ψ?

Definition 3.16. Let $\psi \in \Psi_\omega$. For a sequence of Banach spaces $(X_n)_{n\in\mathbf{N}}$ we define the ψ-direct sum of $(X_n)_{n\in\mathbf{N}}$ to be the space

$$\Big(\sum_{n=1}^\infty \bigoplus X_n\Big)_\psi = \Big\{x = (x_n)_{n\in\mathbf{N}} \in \prod_{n=1}^\infty X_n : (\|x_n\|)_{n\in\mathbf{N}} \in \ell_\psi\Big\}$$

equipped with the norm $\|x\| = \|(\|x_n\|)_{n\in\mathbf{N}}\|_\psi$.

As in Proposition 3.6, we can prove that the ψ-direct sum of a sequence of Banach spaces is a Banach space for every $\psi \in \Psi_\omega$.

For $k = 2, 3, \dots$, let $Q_k : (\sum_{n=1}^\infty \bigoplus X_n)_\psi \to (\sum_{n=1}^k \bigoplus X_n)_{\psi_k}$ defined by $Q_k(x_1, \dots, x_n, \dots) = (x_1, \dots, x_k)$. It is clear that $\|Q_k(x)\| \le \|Q_{k+1}(x)\|$ for $k = 2, 3, \dots$ and $\|x\| = \sup_k \|Q_k(x)\|$. If $(e_n)_{n\in\mathbf{N}}$ is a boundedly complete basis of ℓ_ψ and $x \in \prod_{n=1}^\infty X_n$ such that $\sup_k \|Q_k(x)\| < \infty$, then $x \in (\sum_{n=1}^\infty \bigoplus X_n)_\psi$.

Examples of ψ-direct sums are the ℓ_p-direct sum of a sequence of Banach spaces for $\psi = \psi_p$, $1 \le p < \infty$, the Lorenz direct sum of a sequence of Banach spaces for $\psi = \psi_{(w,p)}$ and the Orlicz direct sum of a sequence of Banach spaces for $\psi = \psi_M$. Such types of spaces are considered also in interpolation theory [**2, 20**]. For $\psi \in \Psi_\omega$ and a sequence of Banach spaces $\mathcal{X} = (X_n)_{n\in\mathbf{N}}$ we denote by $T_{\psi,\mathcal{X}}$ the function

$$T_{\psi,\mathcal{X}} : \Big(\sum_{n=1}^{\infty} \bigoplus X_n^*\Big)_{\psi^*} \longrightarrow \Big[\Big(\sum_{n=1}^{\infty} \bigoplus X_n\Big)_{\psi}\Big]^*$$

defined by

$$(T_{\psi,\mathcal{X}}(z))(x) = \sum_{n=1}^{\infty} x_n^*(x_n)$$

for every $z = (x_n^*)_{n\in\mathbf{N}} \in (\sum_{n=1}^{\infty} \bigoplus X_n^*)_{\psi^*}$ and $x = (x_n)_{n\in\mathbf{N}} \in (\sum_{n=1}^{\infty} \bigoplus X_n)_{\psi}$. From Lemma 3.10 we obtain $\sum_{i=1}^{n} |x_n^*(x_n)| \leq \sum_{i=1}^{n} \|x_n^*\| \, \|x_n\| \leq \|z\| \, \|x\|$ and so, $\|T_{\psi,\mathcal{X}}(z)\| \leq \|z\|$ for every $z \in (\sum_{n=1}^{\infty} \bigoplus X_n^*)_{\psi^*}$.

Theorem 3.17. *Let $\psi \in \Psi_\omega$ and $\mathcal{X} = (X_n)_{n\in\mathbf{N}}$ be a sequence of Banach spaces. If $(e_n)_{n\in\mathbf{N}}$ is a shrinking basis of ℓ_ψ, then the operator $T_{\psi,\mathcal{X}}$ is an isometry of $(\sum_{n=1}^{\infty} \bigoplus X_n^*)_{\psi^*}$ onto $[(\sum_{n=1}^{\infty} \bigoplus X_n)_{\psi}]^*$.*

Proof. Let $z = (x_n^*)_{n\in\mathbf{N}} \in (\sum_{n=1}^{\infty} \bigoplus X_n^*)_{\psi^*}$ and $\varepsilon > 0$. For every n there exists an $x_n \in X_n$ with $\|x_n\| = 1$ such that $\|x_n^*\| \leq (1+\varepsilon) x_n^*(x_n)$. Then, for every $n \in \mathbf{N}$ we have

$$\begin{aligned}
\|Q_n(z)\| &= \Big(\sum_{i=1}^{n} \|x_i^*\|\Big)\psi_n^*\Big(\frac{\|x_2^*\|}{\sum_{i=1}^{n}\|x_i^*\|}, \dots, \frac{\|x_n^*\|}{\sum_{i=1}^{n}\|x_i^*\|}\Big) \\
&= \Big(\sum_{i=1}^{n} \|x_i^*\|\Big) \\
&\quad\times \sup_{(t_1,\dots,t_{n-1})\in\Delta_n} \frac{(1-\sum_{i=1}^{n-1} t_i)\frac{\|x_1^*\|}{\sum_{i=1}^{n}\|x_i^*\|} + \frac{1}{\sum_{i=1}^{n}\|x_i^*\|}\sum_{j=1}^{n-1} t_j \|x_{j+1}^*\|}{\psi_n(t_1,\dots,t_{n-1})} \\
&= \sup_{(t_1,\dots,t_{n-1})\in\Delta_n} \frac{(1-\sum_{i=1}^{n-1} t_i)\|x_1^*\| + \sum_{j=1}^{n-1} t_j \|x_{j+1}^*\|}{\psi_n(t_1,\dots,t_{n-1})} \\
&\leq (1+\varepsilon) \sup_{(t_1,\dots,t_{n-1})} \frac{x_1^*((1-\sum_{i=1}^{n-1} t_i)x_1) + \cdots + x_n^*(t_{n-1}x_n)}{\psi_n(t_1,\dots,t_{n-1})}
\end{aligned}$$

$$= (1+\varepsilon) \sup_{(t_1,\dots,t_{n-1})\in\Delta_n} \frac{T(z)((1-\sum_{i=1}^{n-1} t_i)x_1,\ t_1x_2,\dots,t_{n-1}x_n,0,\dots)}{\psi_n(t_1,\dots,t_{n-1})}.$$

For $(t_1,\dots,t_{n-1}) \in \Delta_n$ we put $x' = ((1-\sum_{i=1}^{n-1} t_i)x_1,\ t_1x_2,\dots,t_{n-1}x_n, 0,\dots)$. Since $\|x_i\| = 1$ for $i = 1,\dots,n$, we have $\|x'\| = \psi_n(t_1,\dots,t_{n-1})$. Therefore, $\|Q_n(z)\| \le (1+\varepsilon)\|T(z)\|$. So, $\|z\| \le \|T(z)\|$. Thus, $\|z\| = \|T(z)\|$.

Let $x^* \in [(\sum_{n=1}^\infty \bigoplus X_n)_\psi]^*$. For every $n \in \mathbf{N}$ we put $x_n^* : X_n \to \mathbf{R}$ defined by $x_n^*(x) = x^*(0,\dots,0,x,0\dots)$ for $x \in X_n$, and $z = (x_n^*)_{n\in\mathbf{N}}$. Let $w = (\|x_n^*\|)_{n\in\mathbf{N}}$. As above, we obtain $\|P_n(w)\|_{\psi_n^*} \le \|x^*\|$ for every $n \in \mathbf{N}$. So, $w \in \ell_{\psi^*}$, because $(e_n)_{n\in\mathbf{N}}$ is a boundedly complete basis of this space. Thus, $z \in (\sum_{n=1}^\infty \bigoplus X_n^*)_{\psi^*}$. It is easy to see that $T(z) = x^*$.

Corollary 3.18. *Let $\psi \in \Psi_\omega$. If $(e_n)_{n\in\mathbf{N}}$ is a shrinking and boundedly complete basis of ℓ_ψ, then the ψ-direct sum of a sequence of reflexive Banach spaces is a reflexive Banach space.*

Proof. Let $\mathcal{X} = (X_n)_{n\in\mathbf{N}}$ be a sequence of reflexive Banach spaces and $X = (\sum_{n=1}^\infty \bigoplus X_n)_\psi$. Using Theorem 3.17 and Proposition 3.13, we get that the operator $(T_{\psi,\mathcal{X}}^{-1})^* \circ T_{\psi^*,\mathcal{X}^*}$ is an isometry of X onto X^{**}, where $(T_{\psi,\mathcal{X}}^{-1})^*$ is the dual operator of $T_{\psi,\mathcal{X}}^{-1}$. It is easy to see that $(T_{\psi,\mathcal{X}}^{-1})^*(T_{\psi^*,\mathcal{X}^*}(x)) = x$ for every $x \in X$. So, X is reflexive.

4. Uniform convexity and smoothness of ψ-direct sums. Below we estimate the modulus of convexity of a finite ψ-direct sum. For this we need the following lemma.

Lemma 4.1. *Let $n \ge 2$, $\psi \in \Psi_n$, $\mathcal{X} = \{X_1,\dots,X_n\}$ be a finite family of Banach spaces, and let X be the ψ-direct sum of this family. If $\varepsilon(\mathcal{X}) < \varepsilon \le 2$ and $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n) \in B_X$, such that $\|x_i\| = \|y_i\|$ for $i = 1,\dots,n$ and $\|x-y\| \ge \varepsilon$, then*

$$\|x+y\| \le 2\left(1 - \delta_{\mathbf{C}^n}\left(\frac{\varepsilon-\varepsilon_0}{2}\delta_{\mathcal{X}}\left(\frac{\varepsilon+\varepsilon_0}{2}\right)\right)\right)$$

for every $\varepsilon(\mathcal{X}) \le \varepsilon_0 < \varepsilon$.

Proof. Let $\varepsilon(\mathcal{X}) \leq \varepsilon_0 < \varepsilon \leq 2$ and $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in B_X$, such that $\|x_i\| = \|y_i\|$ for every $i = 1, \dots, n$ and $\|x - y\| \geq \varepsilon$. We put $\varepsilon_1 = (\varepsilon + \varepsilon_0)/2$ and

$$M_1 = \left\{1 \leq i \leq n : x_i \neq 0 \text{ and } \frac{\|x_i - y_i\|}{\|x_i\|} \geq \varepsilon_1\right\}$$

and

$$M_2 = \{1, \dots, n\} \setminus M_1.$$

Then, we have $\|x_i + y_i\| \leq 2\|x_i\|(1 - \delta_{\mathcal{X}}(\varepsilon_1))$ for $i \in M_1$.

We put $z = (\|x_1\|, \dots, \|x_n\|)$, $w = (w_1, \dots, w_n)$, $v = (v_1, \dots, v_n)$, $u = (u_1, \dots, u_n) \in (\mathbf{C}^n, \|\cdot\|_\psi)$, where

$$w_i = \begin{cases} [1 - 2\delta_{\mathcal{X}}(\varepsilon_1)]\|x_i\| & \text{if } i \in M_1 \\ \|x_i\| & \text{if } i \in M_2 \end{cases},$$

$$v_i = \begin{cases} 0 & \text{if } i \in M_1 \\ \|x_i\| & \text{if } i \in M_2 \end{cases},$$

$$u_i = \begin{cases} \|x_i - y_i\| & \text{if } i \in M_1 \\ 0 & \text{if } i \in M_2 \end{cases}$$

for $i = 1, \dots, n$.

Since $\|x_i + y_i\| \leq \|x_i\| + w_i$ for $i = 1, \dots, n$, we obtain from Lemma 2.3

$$\|x + y\| \leq \|z + w\|_\psi. \tag{4.1}$$

Using also Lemma 2.3, we get

$$\varepsilon \leq \|x - y\| \leq \|\varepsilon v + u\|_\psi \leq \varepsilon_1 \|v\|_\psi + \|u\|_\psi \leq \varepsilon_1 + 2\|z - v\|_\psi.$$

Therefore, $\|z - v\|_\psi \geq (\varepsilon - \varepsilon_1)/2$. Since $z - w = 2\delta_{\mathcal{X}}(\varepsilon_1)(z - v)$, we have $\|z - w\|_\psi \geq \delta_{\mathcal{X}}(\varepsilon_1)(\varepsilon - \varepsilon_1)$. Thus,

$$\begin{aligned} \|z + w\|_\psi &\leq 2\left(1 - \delta_{\mathbf{C}^n}\left((\varepsilon - \varepsilon_1)\delta_{\mathcal{X}}(\varepsilon_1)\right)\right) \\ &= 2\left(1 - \delta_{\mathbf{C}^n}\left(\frac{\varepsilon - \varepsilon_0}{2}\delta_{\mathcal{X}}\left(\frac{\varepsilon + \varepsilon_0}{2}\right)\right)\right). \end{aligned} \tag{4.2}$$

From (4.1) and (4.2) we get

$$\|x+y\| \le 2\left(1-\delta_{\mathbf{C}^n}\left(\frac{\varepsilon-\varepsilon_0}{2}\delta_{\mathcal{X}}\left(\frac{\varepsilon+\varepsilon_0}{2}\right)\right)\right).$$

Theorem 4.2. *Let ψ be a strictly convex function in Ψ_n, and let $\mathcal{X}$, X be as in Lemma* 4.1. *If $\varepsilon(\mathcal{X}) < \varepsilon \le 2$, then*

$$\delta_X(\varepsilon) \ge \min\left\{\delta_{\mathbf{C}^n}\left(\frac{3\varepsilon-3\varepsilon_0}{8}\delta_{\mathcal{X}}\left(\frac{3\varepsilon+5\varepsilon_0}{8}\right)\right)-\frac{\varepsilon}{2\tau}, \delta_{\mathbf{C}^n}\left(\frac{\varepsilon}{\tau}\right)\right\}$$

for every $\varepsilon(\mathcal{X}) \le \varepsilon_0 < \varepsilon$ and $\tau > \varepsilon/(2\delta_{\mathbf{C}^n}((\varepsilon-\varepsilon_0)/4\delta_{\mathcal{X}}((\varepsilon+3\varepsilon_0)/4)))$.

Proof. Let $\varepsilon(\mathcal{X}) \le \varepsilon_0 < \varepsilon \le 2$ and $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in B_X$, with $\|x-y\| \ge \varepsilon$.

For $i = 1, \dots, n$ we put

$$z_i = \begin{cases} \|x_i\| / \|y_i\|\, y_i & \text{if } y_i \ne 0 \\ x_i & \text{if } y_i = 0. \end{cases}$$

Let $z = (z_1, \dots, z_n) \in X$ and $w = (\|x_1\|, \dots, \|x_n\|)$, $v = (\|y_1\|, \dots, \|y_n\|) \in \mathbf{C}^n$. From Lemma 2.3 we obtain $\|x+y\| \le \|w+v\|_\psi$. Since ψ is strictly convex, $(\mathbf{C}^n, \|\cdot\|_\psi)$ is uniformly convex [**18**]. Therefore, $\delta_{\mathbf{C}^n}((\varepsilon-\varepsilon_0)/4\delta_{\mathcal{X}}((\varepsilon+3\varepsilon_0)/4)) > 0$. We put $\tau_0 = \varepsilon/(2\delta_{\mathbf{C}^n}((\varepsilon-\varepsilon_0)/4\delta_{\mathcal{X}}((\varepsilon+3\varepsilon_0)/4)))$.

Let $\tau > \tau_0$. We distinguish two cases.

Case 1. $\|w-v\|_\psi > \varepsilon/\tau$. In this case we obtain

$$\|x+y\| \le \|w+v\|_\psi \le 2\left(1-\delta_{\mathbf{C}^n}\left(\frac{\varepsilon}{\tau}\right)\right).$$

Case 2. $\|w-v\|_\psi \le \varepsilon/\tau$. It is clear that $\|y-z\| = \|w-v\|_\psi$. Therefore,

$$\|x-z\| \ge \|x-y\| - \|y-z\| \ge \varepsilon - \|y-z\| \ge \varepsilon - \|w-v\|_\psi \ge \varepsilon - \frac{\varepsilon}{\tau}.$$

Since $\tau > \tau_0$, using inequality 2.1, we obtain

$$\frac{\varepsilon}{\tau} < \frac{\varepsilon}{\tau_0} = 2\delta_{\mathbf{C}^n}\left(\frac{\varepsilon-\varepsilon_0}{4}\delta_{\mathcal{X}}\left(\frac{\varepsilon+3\varepsilon_0}{4}\right)\right) \le \frac{\varepsilon-\varepsilon_0}{4}.$$

Thus, $\|x-z\| > (3\varepsilon+\varepsilon_0)/4$. So, from Lemma 4.1 we have

$$\|x+z\| \le 2\left(1-\delta_{\mathbf{C}^n}\left(\frac{3\varepsilon-3\varepsilon_0}{8}\delta_{\mathcal{X}}\left(\frac{3\varepsilon+5\varepsilon_0}{8}\right)\right)\right).$$

Since $\|x+y\| \le \|x+z\| + \|w-v\|$, we get

$$\|x+y\| \le 2\left(1-\left(\delta_{\mathbf{C}^n}\left(\frac{3\varepsilon-3\varepsilon_0}{8}\delta_{\mathcal{X}}\left(\frac{3\varepsilon+5\varepsilon_0}{8}\right)\right)-\frac{\varepsilon}{2\tau}\right)\right).$$

It is clear that

$$\delta_{\mathbf{C}^n}\left(\frac{3\varepsilon-3\varepsilon_0}{8}\delta_{\mathcal{X}}\left(\frac{3\varepsilon+5\varepsilon_0}{8}\right)\right) \ge \delta_{\mathbf{C}^n}\left(\frac{\varepsilon-\varepsilon_0}{4}\delta_{\mathcal{X}}\left(\frac{\varepsilon+3\varepsilon_0}{4}\right)\right) > \frac{\varepsilon}{2\tau}.$$

The result follows from the two cases.

Theorem 4.2 leads to the following results.

Corollary 4.3. *Let $\psi, \mathcal{X}$, X be as in Theorem 4.2. Then,*

i) $\delta_X(\varepsilon) \ge \delta_{\mathbf{C}^n}(\delta_{\mathbf{C}^n}((\varepsilon-\varepsilon_0)/4\delta_{\mathcal{X}}((\varepsilon+3\varepsilon_0)/4)))$ *for every* $\varepsilon(\mathcal{X}) \le \varepsilon_0 < \varepsilon \le 2$, *and*

ii) $\varepsilon(X) = \varepsilon(\mathcal{X})$.

Proof. i) We put $\tau = \varepsilon/(\delta_{\mathbf{C}^n}((\varepsilon-\varepsilon_0)/4\delta_{\mathcal{X}}((\varepsilon+3\varepsilon_0)/4)))$. It is clear that $\tau \ge \varepsilon/(\delta_{\mathbf{C}^n}((3\varepsilon-3\varepsilon_0)/8\delta_{\mathcal{X}}((3\varepsilon+5\varepsilon_0)/8)))$. Thus, using inequality 2.1 we obtain

$$\delta_{\mathbf{C}^n}\left(\frac{\varepsilon}{\tau}\right) \le \frac{\varepsilon}{2\tau} \le \delta_{\mathbf{C}^n}\left(\frac{3\varepsilon-3\varepsilon_0}{8}\delta_{\mathcal{X}}\left(\frac{3\varepsilon+5\varepsilon_0}{8}\right)\right) - \frac{\varepsilon}{2\tau}.$$

Therefore, the result follows from Theorem 4.2.

ii) It is clear that $\varepsilon(\mathcal{X}) \le \varepsilon(X)$. Since the space $\mathbf{C}^n$ is uniformly convex, the inequality $\varepsilon(\mathcal{X}) \ge \varepsilon(X)$ follows from (i).

In [**5, 8**] it was proved that if ψ is strictly convex then the ψ-direct of a finite family of uniformly convex Banach spaces is uniformly convex, but these proofs do not give an estimate of the modulus of convexity. Using Corollary 4.3 this result is improved by giving an estimate of the modulus of convexity. Also, in [**9**] it was proved that the ψ-direct sum of two non-square Banach spaces is non-square if and only if $\psi \neq \psi_1$ and $\psi \neq \psi_\infty$ and the question about finite ψ-direct sums was posed. Using Corollary 4.3 we obtain a result about the non-squareness of finite ψ-direct sums.

Corollary 4.4. *Let ψ be a strictly convex function in Ψ_n.*

i) *The ψ-direct sum X of a finite family of uniformly convex Banach spaces $\mathcal{X}$ is uniformly convex and $\delta_X(\varepsilon) \geq \delta_{\mathbf{C}^n}(\delta_{\mathbf{C}^n}((\varepsilon/4)\delta_{\mathcal{X}}(\varepsilon/4)))$ for every $0 \leq \varepsilon \leq 2$.*

ii) *The ψ-direct sum of a finite family of uniformly non-square Banach spaces is uniformly non-square.*

Using the estimate of Theorem 4.2, we obtain an estimate of modulus of convexity of the ψ-direct sums of a sequence of Banach spaces.

Theorem 4.5. *Let $\psi \in \Psi_\omega$ be such that the space ℓ_ψ is uniformly convex, and let X be the ψ-direct sum of a sequence of Banach spaces $\mathcal{X} = (X_n)_{n\in\mathbf{N}}$. If $\varepsilon(\mathcal{X}) < \varepsilon \leq 2$, then*

$$\delta_X(\varepsilon) \geq \min\left\{\delta_{\ell_\psi}\left(\frac{3\varepsilon - 3\varepsilon_0}{8}\delta_{\mathcal{X}}\left(\frac{3\varepsilon + 5\varepsilon_0}{8}\right)\right) - \frac{\varepsilon}{2\tau}, \delta_{\ell_\psi}\left(\frac{\varepsilon}{\tau}\right)\right\}$$

for every $\varepsilon(\mathcal{X}) \leq \varepsilon_0 < \varepsilon$ and $\tau > \varepsilon/(2\delta_{\ell_\psi}((\varepsilon - \varepsilon_0)/4\delta_{\mathcal{X}}((\varepsilon + 3\varepsilon_0)/4)))$.

Proof. Let $\varepsilon(\mathcal{X}) \leq \varepsilon_0 < \varepsilon \leq 2$, $\tau > \varepsilon/(2\delta_{\ell_\psi}((\varepsilon - \varepsilon_0)/4\delta_{\mathcal{X}}((\varepsilon + 3\varepsilon_0)/4)))$ and $x, y \in B_X$, such that $\|x - y\| > \varepsilon$. Then, there exists an $n_0 \in \mathbf{N}$ such that $\|Q_n(x) - Q_n(y)\| > \varepsilon$ for every $n \geq n_0$. Let $n \geq n_0$. We put $\mathcal{X}_n = (X_i)_{i=1}^n$. Since $\delta_{\mathcal{X}_n}(\epsilon) \geq \delta_{\mathcal{X}}(\epsilon)$ for every $0 \leq \epsilon \leq 2$ and $\varepsilon(\mathcal{X}_n) \leq \varepsilon(\mathcal{X})$, we have $\varepsilon(\mathcal{X}_n) \leq \varepsilon_0$ and

$\tau > \varepsilon/(2\delta_{\mathbf{C}^n}((\varepsilon-\varepsilon_0)/4\delta_{\mathcal{X}_n}((\varepsilon+3\varepsilon_0)/4)))$. From Theorem 4.2 we get

$$\begin{aligned}\|Q_n(x)+Q_n(y)\| &\\ &\leq 2\bigg(1-\min\bigg\{\delta_{\mathbf{C}^n}\bigg(\frac{3\varepsilon-3\varepsilon_0}{8}\delta_{\mathcal{X}_n}\bigg(\frac{3\varepsilon+5\varepsilon_0}{8}\bigg)\bigg)\\ &\qquad -\frac{\varepsilon}{2\tau},\delta_{\mathbf{C}^n}\bigg(\frac{\varepsilon}{\tau}\bigg)\bigg\}\bigg)\\ &\leq 2\bigg(1-\min\bigg\{\delta_{\ell_\psi}\bigg(\frac{3\varepsilon-3\varepsilon_0}{8}\delta_{\mathcal{X}}\bigg(\frac{3\varepsilon+5\varepsilon_0}{8}\bigg)\bigg)\\ &\qquad -\frac{\varepsilon}{2\tau},\delta_{\ell_\psi}\bigg(\frac{\varepsilon}{\tau}\bigg)\bigg\}\bigg)\end{aligned}$$

for every $n \geq n_0$. Thus,

$$\|x+y\| \leq 2\bigg(-\min\bigg\{\delta_{\ell_\psi}\bigg(\frac{3\varepsilon-3\varepsilon_0}{8}\delta_{\mathcal{X}}\bigg(\frac{3\varepsilon+5\varepsilon_0}{8}\bigg)\bigg)-\frac{\varepsilon}{2\tau},\ \delta_{\ell_\psi}\bigg(\frac{\varepsilon}{\tau}\bigg)\bigg\}\bigg).$$

From Theorem 4.5 we obtain the next corollary.

Corollary 4.6. *Let $\psi, \mathcal{X}, X$ be as in Theorem* 4.5*. Then,*

i) $\delta_X(\varepsilon) \geq \delta_{\ell_\psi}(\delta_{\ell_\psi}((\varepsilon-\varepsilon_0)/4\delta_{\mathcal{X}}((\varepsilon+3\varepsilon_0)/4)))$ *for every* $\varepsilon(\mathcal{X}) \leq \varepsilon_0 < \varepsilon \leq 2$, *and*

ii) $\varepsilon(X) = \varepsilon(\mathcal{X})$.

As in [**6**], using Proposition 3.7 it can be proved that the ψ direct sum of a sequence $(X_n)_{n\in\mathbf{N}}$ is strictly convex (respectively uniformly convex, locally uniformly convex, uniformly convex in every direction) if and only if ℓ_ψ and X_n, $n = 1, 2, \ldots$ are strictly convex (respectively uniformly convex, locally uniformly convex, uniformly convex in every direction). Using Corollary 4.6 the following results concerning the convexity and smoothness of ψ-directs sums of Banach space sequences, are proved.

Corollary 4.7. *Let $\psi \in \Psi_\omega$ be such that the space ℓ_ψ is uniformly convex.*

i) *The ψ-direct sum X of a uniformly convex sequence of Banach spaces $\mathcal{X}$ is uniformly convex and $\delta_X(\varepsilon) \geq \delta_{\ell_\psi}(\delta_{\ell_\psi}((\varepsilon/4)\delta_{\mathcal{X}}(\varepsilon/4)))$ for every $0 \leq \varepsilon \leq 2$.*

ii) *The ψ-direct sum of a uniformly non-square sequence of Banach spaces is uniformly non-square.*

Corollary 4.8. *Let $\psi \in \Psi_\omega$ be such that the space ℓ_ψ is uniformly smooth. Then the ψ-direct sum of a uniformly smooth sequence of Banach spaces is uniformly smooth.*

Proof. Because a uniformly smooth space is reflexive, from Proposition 3.7 we obtain that $(en)_{n\in\mathbf{N}}$ is a shrinking and boundedly complete basis of ℓ_ψ. Using Proposition 2.7, Theorem 3.17 and Corollary 4.7, we obtain the result.

Using Corollaries 4.7 and 4.8 we obtain the following results, which concern convexity and smoothness properties of ℓ_p, Lorentz and Orlicz direct sums of Banach spaces.

Corollary 4.9. *Let $1 < p < \infty$.*

i) *The ℓ_p-direct sum X of a uniformly convex sequence of Banach spaces $\mathcal{X}$ is uniformly convex and $\delta_X(\varepsilon) \geq \delta_{\ell_p}(\delta_{\ell_p}((\varepsilon/4)\delta_{\mathcal{X}}(\varepsilon/4)))$ for every $0 \leq \varepsilon \leq 2$.*

ii) *The ℓ_p-direct sum of a uniformly smooth sequence of Banach spaces is uniformly smooth.*

iii) *The ℓ_p-direct sum of a uniformly non-square sequence of Banach spaces is uniformly non-square.*

Corollary 4.10. *Let $1 < p < \infty$ and $w = (w_n)_{n\in\mathbf{N}} \in c_0 \setminus \ell_1$, with $1 = w_1 \geq w_2 \geq \cdots \geq 0$.*

I) *If the Lorentz space $d(w,p)$ is uniformly convex, then*

a) *the $\psi_{(w,p)}$-direct sum X of a uniformly convex sequence of Banach spaces $\mathcal{X}$ is uniformly convex and $\delta_X(\varepsilon) \geq \delta_{d(w,p)}(\delta_{d(w,p)}((\varepsilon/4)\delta_{\mathcal{X}}(\varepsilon/4)))$ for every $0 \leq \varepsilon \leq 2$, and*

b) *the $\psi_{(w,p)}$-direct sum of a uniformly non-square sequence of Banach spaces is uniformly non-square.*

II) *If the Lorentz space $d(w,p)$ is uniformly smooth, then the $\psi_{(w,p)}$-direct sum of a uniformly smooth sequence of Banach spaces is uniformly smooth.*

Corollary 4.11. *Let M be an Orlicz function with $M(1) = 1$.*

I) *If h_M is uniformly convex, then*

a) *the ψ_M-direct sum X of a uniformly convex sequence of Banach spaces $\mathcal{X}$ is uniformly convex and $\delta_X(\varepsilon) \geq \delta_{\ell_M}(\delta_{\ell_M}((\varepsilon/4)\delta_{\mathcal{X}}(\varepsilon/4)))$ for every $0 \leq \varepsilon \leq 2$, and*

b) *the ψ_M-direct sum of a uniformly non-square sequence of Banach spaces is uniformly non-square.*

II) *If h_M is uniformly smooth, then the ψ_M-direct sum of a uniformly smooth sequence of Banach spaces is uniformly smooth.*

Remark. As the referee of this paper pointed out Mitani and Saito [**13**] introduced spaces ℓ_ψ and $\ell_{\psi,\infty}$ in a different way. In that paper the norm structure of these spaces is studied. The results of Section 3 of the present paper complete this structure using the unconditional basis $(e_n)_{n\in\mathbf{N}}$ of the space ℓ_ψ. In [**13**] there is no mention of this basis. Using this basis Proposition 2.5 in [**13**] is clear. In [**13**] a function $\psi \in \Psi_\omega$ is called regular if $\ell_\psi = \ell_{\psi,\infty}$. From Proposition 3.7 of the present paper we obtain that ψ is regular if and only if the basis $(e_n)_{n\in\mathbf{N}}$ of ℓ_ψ is boundedly complete. Also, Proposition 2.8 in [**13**] is related to Theorem 3.15 of the present paper. Regarding the geometrical properties of these spaces, properties concerning strict convexity and uniform convexity were proved in [**13**] and the problems of characterizing strict convexity (respectively uniform convexity) of ℓ_ψ (respectively $\ell_{\psi,\infty}$) in terms of ψ were posed. In [**13**] there is no reference to ψ-direct sums of Banach spaces.

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