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PERIODIC AND HOMOCLINIC SOLUTIONS OF A CLASS OF FOURTH ORDER EQUATIONS

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ABSTRACT. In this paper, we investigate a class of fourth order equations, which include the extended Fisher-Kolmogorov Equations as a special case. Under different conditions, we obtain periodic solutions and homoclinic solutions respectively.

1. Introduction.

1.1. Background. In this paper we shall study a class of fourth order differential equations. The extended Fisher-Kolmogorov (EFK) equation is of this kind, for example, [**10**] studies the stationary solutions of EFK,

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \text{ on } \mathbf{R} \times (0, \infty),$$

where γ is a positive parameter. The problem of finding stationary solutions of EFK and relevant generalized equations has long drawn attention from both mathematicians and physicists. Various methods have been employed to attack these equations, from topological shooting, the maximum principle, Hamiltonian methods to variational methods. In [**9**], the authors carefully summarized most recent results and efforts in these directions. For results concerning variational approach, the readers could refer to [**2, 4, 7, 12**] and references therein. In particular, we shall mention [**7**]; in their paper, the authors successfully found a periodic minimizer of a second order functional whose Euler-Lagrange equation is just the above mentioned fourth order equation.

Our results have embraced [**2, 10, 14**] and part of generalizations in [**9**]. Our work is inspired by [**2, 14, 15**].

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1.2. Main results. In this paper, we focus on the nontrivial periodic solutions of the fourth order equation assuming the following form

$$(P_1) \qquad \begin{cases} u^{iv} - ku'' - c(x)\,u = f(x,u)\,,\ x \in (0,L)\,, \\ u(0) = u(L) = u''(0) = u''(L) = 0. \end{cases}$$

where $k > 0$ and $c(x)$ is a function satisfying

$$(A_1) \qquad c_1 \geqslant c(x) \geqslant c_2 > 0.$$

In what follows, $c(x)$ is always assumed to satisfy (A_1).

Under different conditions of $f(x,u)$, we investigate the existence and multiplicity of solutions. Let $f(x,u)$ be a continuous function and

$$F(x,u) = \int_0^u f(x,s)\, ds.$$

In addition, $f(x,u)$ and $F(x,u)$ are required to meet some of the following conditions as is needed.

There are a real number $p > 2$ and a constant $\widetilde{\alpha} > 0$ such that:

$$(C_1) \qquad \limsup_{|u| \to +\infty} \frac{F(x,u)}{|u|^p} \leqslant -\widetilde{\alpha}, \text{ uniformly in } x.$$

There are a real number $p > 2$ and a constant $\beta > 0$ such that:

$$(C_1^*) \qquad \limsup_{|u| \to +\infty} \frac{F(x,u)}{|u|^p} \leqslant \beta, \text{ uniformly in } x.$$

There are a real number $p > 2$ and a constant $\delta > 0$ such that:

$$(C_2) \qquad \liminf_{|u| \to +\infty} \frac{F(x,u)}{|u|^p} \geqslant \delta, \text{ uniformly in } x.$$

There are a real number $p > 2$ and a constant $\eta > 0$ such that:

$$(C_2^*) \qquad \liminf_{|u| \to +\infty} \frac{F(x,u)}{|u|^p} \geqslant -\eta, \text{ uniformly in } x.$$

There exist a constant $0 < \sigma < 1/2$ and $\rho > 0$ such that:

(C_3) $$|u| > \rho \Longrightarrow F(x,u) \leqslant \sigma u f(x,u), \text{ uniformly in } x.$$

(C_4) $$f(x,u) = o(|u|), \ |u| \to 0, \text{ uniformly in } x.$$

Note that when (C_1) and (C_2) are used together, we naturally require $\beta \geqslant \delta$. When (C_1) and (C_2) are used together, we require $\eta \geqslant \widetilde{\alpha}$. We set up our problem on the following Hilbert space with norm $\|\cdot\|$,

$$X = H^2(0,L) \cap H_0^1(0,L),$$

$$\|u\| = \left(\int_0^L \left[(u'')^2 + k(u')^2 \right] dx \right)^{1/2}.$$

The organization of this paper is as follows. In Section one, we first find the minimizer of

$(\min P_1)$ $$\min_{u \in X} \varphi(u),$$

where

$$\varphi(u) = \int_0^L \left\{ \frac{1}{2} \left[(u'')^2 + k(u')^2 - c(x) u^2 \right] - F(x,u) \right\} dx.$$

In the proof of existence of the minimizer, $f(x,u)$ is not required to be odd in u. However, if $F(x,u)$ is also even in u (equivalently $f(x,u)$ is odd in u), then we can further construct a periodic solution from the minimizer. In fact when $F(x,u)$ is even, we can construct $2L$-periodic solution from any nontrivial solution by extending it antisymmetrically to the real line.

Theorem 1. *Assume* (C_1) *is satisfied. Then* $(\min P1)$ *achieves its minimizer.*

Next, we study the influence of L on the existence of nontrivial solutions. By employing the abstract critical point theory from [**3**], we show that problem (P_1) has several numbers of geometrically distinct pairs of orbits according to the variance of L under the condition that $f(x,-u) = -f(x,u)$. Define

$$L_{1,2} = \pi\sqrt{\frac{k + \sqrt{k^2 + 4c_{1,2}}}{2c_{1,2}}}.$$

Theorem 2. *Suppose $f(x,u)$ is odd in u and satisfies (C_1), (C_2^*), (C_4). If $L > mL_2$ for some $m \in \mathbf{N}$, then problem (P_1) has m geometrically distinct periodic solutions.*

In Section two, we first prove an existence theorem in a general case.

Consider the following eigenvalue problem

$$\begin{cases} u^{iv} - ku'' + c(x)\,u = \lambda u,\ x \in (0, L)\,, \\ u(0) = u(L) = u''(0) = u''(L) = 0, \end{cases} \tag{1}$$

we will see that the principle eigenvalue $\lambda_1 > -\infty$ and then we denote the sequence of eigenvalues by

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant 0 < \lambda_{n+1} \leqslant \lambda_{n+2} \leqslant \cdots$$

where the eigenvalues repeat according to its multiplicity. We have

Theorem 3. *Suppose $F(x,u)$ satisfies (C_3), (C_4). Moreover*

$$F(x,u) \geqslant \frac{\lambda_n}{2}u^2. \tag{C_5}$$

Then the problem

$$\begin{cases} u^{iv} - ku'' + c(x)\,u = f(x,u)\,,\ x \in (0, L)\,, \\ u(0) = u(L) = u''(0) = u''(L) = 0, \end{cases} \tag{P_2}$$

has a nontrivial solution.

Next, we prove that

Theorem 4. *If $f(x,u)$ is odd in u and satisfies (C_3), (C_4), then problem (P_2) has infinitely many pairs of solutions, u_n and $-u_n$, which are critical points of $\varphi(u)$ and $\varphi(u_n) \to +\infty$, as $n \to +\infty$.*

In the last section, we find a homoclinic solution for (P_3).

$$(P_3) \qquad u^{iv} + ku'' + c(x)u = f(x,u),$$

where $c(x)$ is 1-periodic and satisfies (A_1), $2c_2^{1/2} > k$. We obtain the following

Theorem 5. *Suppose $f(x,u)$ is 1-periodic in x and meets (C_1^*), (C_2), (C_4). Moreover, there exist constants ϱ, $\tau > 0$, $\alpha > 2$ and two integers $q > \overline{q} > 1$, such that*

$$\frac{1}{2}uf(x,u) - F(x,u) \leqslant \varrho u^q + \tau u^{\overline{q}},$$
$$\frac{1}{\alpha}uf(x,u) - F(x,u) \geqslant 0.$$

Then there exists a homoclinic solution $u \in H^2(\mathbf{R})$ of problem (P_3).

Convention: The constants in this paper are mostly denoted by C, c, $\widetilde{C}$, $\widetilde{c}$, which are different from line to line.

2. Minimizer.

2.1. Minimizer.

Lemma 6. *Suppose u is a minimizer of $(\min P_1)$. Then u must solve problem (P_1).*

Proof. By the standard regularity theory, $u \in C^4([0,L])$. It is sufficient to show that u also satisfies the boundary conditions in (P_1).

In fact, since $u \in X$, $u(0) = u(L) = 0$. It remains to prove $u''(0) = u''(L) = 0$. Let $\eta \in C^2([0,L])$ and $\eta(0) = \eta(L) = 0$, because

u is a minimizer, so

$$\frac{d}{ds}\varphi(u+s\eta)\bigg|_{s=0}=0.$$

Precisely,

$$\int_0^L \left[u''\eta''+ku'\eta'-c(x)\,u\eta-f(x,u)\,\eta\right]\,dx=0.$$

By integration, we obtain

$$\begin{aligned}
&u''(L)\,\eta'(L)-u''(0)\,\eta'(0)\\
&\qquad\qquad+\int_0^L\left[u^{iv}\eta+ku''\eta-c(x)\,u\eta-f(x,u)\,\eta\right]\,dx\\
&\qquad=u''(L)\,\eta'(L)-u''(0)\,\eta'(0)\\
&\qquad=0.
\end{aligned}$$

Since $\eta'(0)$ and $\eta'(L)$ can be chosen arbitrarily, we must have $u''(0)=u''(L)=0$. □

Proof of Theorem 1. We first show $\varphi(u)$ is bounded from below, thus we can find a minimizing sequence.

According to (C_1), there is a constant $\rho>0$, whenever $|u|>\rho$ we have

$$-F(x,u)\geqslant\widetilde{\alpha}\,|u|^p\,,\ \text{uniformly in } x.$$

Therefore,

$$\begin{aligned}
\varphi(u)&=\int_0^L\left\{\frac{1}{2}\left[(u'')^2+k\,(u')^2-c(x)\,u^2\right]-F(x,u)\right\}dx\\
&=\int_0^L\frac{1}{2}\left[(u'')^2+k\,(u')^2+c(x)\,u^2\right]\,dx\\
&\quad+\int_{\{x\,|\,|u|>\rho\}\cup\{x\,|\,|u|\leqslant\rho\}}\left[-c(x)\,u^2-F(x,u)\right]\,dx\\
&\geqslant\int_0^L\frac{1}{2}\left[(u'')^2+k\,(u')^2+c(x)\,u^2\right]\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\{x||u|>\rho\}}\left[-c(x)u^2+\widetilde{\alpha}|u|^p\right]dx+C\\
&\geqslant\int_0^L\frac{1}{2}\left[(u'')^2+k(u')^2+c(x)u^2\right]dx+C\\
&\geqslant\int_0^L\frac{1}{2}\left[(u'')^2+k(u')^2\right]dx+C.
\end{aligned}
$$

Hence $\varphi(u)$ is bounded from below and the minimizing sequence of $\varphi(u)$ must be bounded in X. Going if necessary to a subsequence, we may assume that

$$u_n\rightharpoonup u,\ \text{in } H^2(0,L).$$

The functional $\varphi(u)$ is weakly lower semicontinuous on X (the main ingredients of the proof are Egorov's theorem and Lusin's theorem; for a detailed proof, see for example [**6**]). Therefore, following standard procedure, one obtains a minimizer of problem $(\min P_1)$ in X. □

2.2. The parameter L.

Lemma 8. *If $f(x,u)u<0$. Then, whenever $L\leqslant L_1$, the problem (P_1) has only a trivial solution.*

Proof. Suppose u is a nontrivial solution of problem (P_1); then we have

$$(\varphi'(u),u)=0,$$

that is,

$$\int_0^L\left[(u'')^2+k(u')^2-c(x)u^2-f(x,u)u\right]dx=0.$$

Therefore, employing the Poincaré and Hölder inequalities (cf. [**10**]), we obtain

$$
\begin{aligned}
0&>\int_0^L[f(x,u)u]\,dx\\
&=\int_0^L\left[(u'')^2+k(u')^2-c(x)u^2\right]dx\\
&\geqslant\int_0^L\left[\left(\frac{\pi}{L}\right)^4+k\left(\frac{\pi}{L}\right)^2-c_1\right]u^2dx,
\end{aligned}
$$

but the integrand in the last inequality is nonnegative whenever $L \leqslant L_1$, which is a contradiction. $\square$

Lemma 9. *If $L > L_2$, $f(x,u)$ satisfies (C_2^*) and (C_4), then problem (P_1) has a nontrivial solution.*

Proof. It suffices to show

$$\min_{u \in X} \varphi(u) < 0.$$

Let

$$\widetilde{u}(x) = \varepsilon \sin \frac{\pi x}{L}, \quad \varepsilon > 0.$$

Plugging the function into $\varphi(u)$ we obtain

$$\begin{aligned}
\varphi(\widetilde{u}) = & \frac{1}{2}\varepsilon^2 \int_0^L \left[\left(\frac{\pi}{L}\right)^4 \left(\sin\frac{\pi x}{L}\right)^2 \right. \\
& \left. + k\left(\frac{\pi}{L}\right)^2 \left(\cos\frac{\pi x}{L}\right)^2 - c(x)\left(\sin\frac{\pi x}{L}\right)^2 \right] dx \\
& - \int_0^L \left[F\left(x, \varepsilon \sin\frac{\pi x}{L}\right)\right] dx \\
\leqslant & \frac{L}{4}\varepsilon^2 \left[\left(\frac{\pi}{L}\right)^4 + k\left(\frac{\pi}{L}\right)^2 - c_2 \right] \\
& - \int_0^L \left[F\left(x, \varepsilon \sin\frac{\pi x}{L}\right)\right] dx
\end{aligned}$$

Using (C_4) and (C_2^*), we have, for all $\widetilde{\varepsilon}$, there exists $C_{\widetilde{\varepsilon}} > 0$,

$$F(x,u) \geqslant -\widetilde{\varepsilon}\,|u|^2 - C_{\widetilde{\varepsilon}}\,|u|^p. \tag{1}$$

Hence, using the above inequality with

$$\widetilde{\varepsilon} = \frac{1}{2}\left[\left(\frac{\pi}{L}\right)^4 + k\left(\frac{\pi}{L}\right)^2 - c_2 \right],$$

we arrive at

$$\begin{aligned}\varphi(\widetilde{u}) &\leqslant \frac{L}{4}\varepsilon^2\left[\left(\frac{\pi}{L}\right)^4 + k\left(\frac{\pi}{L}\right)^2 - c_2\right] + \frac{L}{2}\widetilde{\varepsilon}\varepsilon^2 \\ &\quad + C_{\widetilde{\varepsilon}}\varepsilon^p \int_0^L \left|\sin\frac{\pi x}{L}\right|^p dx \\ &= \frac{L}{2}\varepsilon^2\left[\left(\frac{\pi}{L}\right)^4 + k\left(\frac{\pi}{L}\right)^2 - c_2\right] + C_{\widetilde{\varepsilon}}\varepsilon^p \int_0^L \left|\sin\frac{\pi x}{L}\right|^p dx.\end{aligned}$$

It is easy to verify that

$$\left(\frac{\pi}{L}\right)^4 + k\left(\frac{\pi}{L}\right)^2 - c_2 < 0, \quad \forall L > L_2.$$

Therefore, for $\varepsilon > 0$ small enough, we have

$$\min_{u\in X}\varphi(u) \leqslant \varphi(\widetilde{u}) < 0,$$

which indicates there is a nontrivial minimizer of $\varphi(u)$. □

Remark. Note that $L_2 \geqslant L_1$.

2.3. Geometrically distinct periodic solutions.

Theorem 10 ([**3**] $P112$). *Assume that*

(f_1) $f \in \mathcal{C}^1(X, \mathbf{R})$ *is G-invariant, $G = \mathbf{Z}_2$ or $\mathbf{S}^1$.*

(f_2) *There exist two regular values $a < b$ such that (PS_c) hold for all $c \in [a, b]$.*

(f_3) *There exists G_d-invariant subspaces X_+ and X_- with*

$$j = \frac{1}{d}\operatorname{co}\dim X_+ < m = \frac{1}{d}\operatorname{co}\dim X_- < +\infty, \quad d = 1 \text{ or } 2,$$

where j and m are integers, such that

(1) $\mathrm{Fix}_G \subset X_+$, $\mathrm{Fix}_G \cap X_- = \{\theta\}$,

(2) $f(x) > a$, *for all $x \in X_+$,*

(3) *$f(x) < b$, for all $x \in X_- \cap X_\rho$ for some $\rho > 0$,*

(4) $\mathrm{Fix}_G \cap f^{-1}[a,b] = \{\theta\}$.

Then f has at least $m-j$ distinct critical orbits.

Proof of Theorem 2. To employ Theorem 10 to serve our purpose, we need to verify that: $\varphi(u)$ is even, $\varphi(0) = 0$ and $\varphi(u)$ is bounded from below by some constant a. These conditions are trivial in our situation. Apart from that we have to show $\varphi(u)$ meets the (PS) condition and there exists an m-dimensional subspace X_m, for $1 \leqslant m < \infty$, such that $\varphi(u)$ is bounded from above by a constant $b > a$ on $X_m \cap X_\rho$ for some $\rho > 0$. We divide the proof into two steps.

Step 1: (PS) condition.

Let $u_n \in X$ be a (PS) sequence, i.e.,

$$\varphi(u_n) \text{ is bounded and } \varphi'(u_n) \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

By similar argument as in Theorem 1, one easily verifies u_n is bounded in X. Hence we may assume

$$u_n \rightharpoonup u, \text{ in } X.$$

By the Sobolev imbedding theorem, we obtain

$$\begin{aligned}(\varphi'(u_n), u) &= \int_0^L [u_n'' u'' + k u_n' u' - c(x) u_n u - f(x, u_n) u]\, dx \\ &\longrightarrow \int_0^L \left[(u'')^2 + k(u')^2 - c(x) u^2 - f(x,u) u\right] dx \\ &= 0.\end{aligned}$$

Meanwhile, since $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to +\infty$, we deduce that

$$(\varphi'(u_n), u_n) \longrightarrow 0, \text{ as } n \longrightarrow +\infty.$$

Hence

$$\begin{aligned}\int_0^L \left[\left(u_n''\right)^2 + k\left(u_n'\right)^2\right]\,dx &= \left(\varphi'\left(u_n\right), u_n\right) + \int_0^L \left[c\left(x\right)u_n^2\right]\,dx \\ &\quad + \int_0^L \left[f\left(x, u_n\right)u_n\right]\,dx \\ &\longrightarrow \int_0^L \left[c\left(x\right)u^2 + f\left(x, u\right)u\right]\,dx.\end{aligned}$$

Thus we have

$$\left\|u_n\right\| \to \left\|u\right\|,$$

which together with $u_n \rightharpoonup u$ in X implies $\left\|u_n - u\right\| \to 0$, as $n \to +\infty$.

Step 2: Geometric condition.

Consider the subspace

$$X_m = \left\{\sin\frac{\pi x}{L}, \cdots, \sin\frac{m\pi x}{L}\right\}.$$

Any $\omega \in X_m$ can be written as

$$\omega = \sum_{k=1}^{m} \omega_i \sin\frac{i\pi x}{L}.$$

Using (1) with $\widetilde{\varepsilon}$ satisfying

$$\widetilde{\varepsilon}\int_0^L \left|\omega\right|^2 dx = \frac{1}{2}\sum_{i=1}^{m}\omega_i^2\left[\left(\frac{i\pi}{L}\right)^4 + k\left(\frac{i\pi}{L}\right)^2 - c\left(x\right)\right],$$

we have

$$\begin{aligned}\varphi\left(\omega\right) = \frac{1}{2}\sum_{i=1}^{m}\omega_i^2\int_0^L &\left[\left(\frac{i\pi}{L}\right)^4\left(\sin\frac{i\pi x}{L}\right)^2\right. \\ &\left. + k\left(\frac{i\pi}{L}\right)^2\left(\cos\frac{i\pi x}{L}\right)^2 - c\left(x\right)\left(\sin\frac{i\pi x}{L}\right)^2\right]dx \\ &- \int_0^L \left[F\left(x, \omega\right)\right]\,dx\end{aligned}$$

$$\leqslant \frac{L}{4}\sum_{i=1}^{m}\omega_i^2\left[\left(\frac{i\pi}{L}\right)^4 + k\left(\frac{i\pi}{L}\right)^2 - c(x)\right]$$
$$+\widetilde{\varepsilon}\int_0^L |\omega|^2\,dx + C_{\widetilde{\varepsilon}}\int_0^L |\omega|^p\,dx$$
$$= \frac{L}{2}\sum_{i=1}^{m}\omega_i^2\left[\left(\frac{i\pi}{L}\right)^4 + k\left(\frac{i\pi}{L}\right)^2 - c(x)\right] + C_{\widetilde{\varepsilon}}\int_0^L |\omega|^p\,dx.$$

Since $mL_2 < L$, there is a constant $\kappa > 0$ such that

$$\left[\left(\frac{i\pi}{L}\right)^4 + k\left(\frac{i\pi}{L}\right)^2 - c(x)\right] \leqslant -\kappa,\ \forall 1 \leqslant i \leqslant m.$$

Thus

$$\varphi(\omega) \leqslant -\frac{L}{2}\kappa\sum_{i=1}^{m}\omega_i^2 + C_{\widetilde{\varepsilon}}\int_0^L |\omega|^p\,dx.$$

Since any norms on a finite dimensional space are equivalent, for $\rho > 0$ small enough, we have

$$\varphi(\omega) < 0,\ \text{if}\ \|\omega\|_{X_m} < \rho. \qquad \square$$

Remark. This theorem remains valid if (C_1) is replaced with (C_3).

3. Nontrivial solutions of problem (P_2).

3.1. General case. In the general case, $F(x,u)$ is not assumed to be odd in u and typically assumed to satisfy (C_1^*), (C_3), (C_4), (C_5).

Lemma 11. *For $c(x)$ satisfying (A_1), $k > 0$, we have*

$$\lambda_1 = \inf_{u\in X,\|u\|_{L^2}=1}\int_0^L \left[(u'')^2 + k(u')^2 + c(x)u^2\right]dx > -\infty.$$

Proof. Apparently, the functional in question is bounded from below. Consider the minimizing sequence:

$$\int_0^L \left[(u_n'')^2 + k(u_n')^2\right]dx = 1,\ \frac{\int_0^L \left[(u_n'')^2 + k(u_n')^2 + c(x)u_n^2\right]dx}{\|u_n\|_{L^2}} \to \lambda_1.$$

Going if necessary to a subsequence, we may assume

$$u_n \rightharpoonup u, \text{ in } H^2(0, L).$$

According to the Rellich theorem, we have

$$\|u_n\|_{L^2} \longrightarrow \|u\|_{L^2}, \int_0^L \left[c(x) u_n^2\right] dx \longrightarrow \int_0^L \left[c(x) u^2\right] dx.$$

Since $\lambda_1 < +\infty$ and $u \neq 0$, we obtain

$$\lambda_1 \geqslant \frac{\int_0^L \left[(u'')^2 + k(u')^2 + c(x) u^2\right] dx}{\|u\|_{L^2}} > -\infty. \qquad \square$$

Remark 12. If $c(x)$ satisfies (A_1), then $\lambda_1 > 0$.

Let $e_1, e_2, e_3, \ldots$ be the corresponding orthonormal eigenfunctions of ( ) in $L^2(0, L)$.

Lemma 13. *Under the assumptions of the preceding lemma, if*

$$Y = \operatorname{span}\{e_1, e_2, \ldots, e_n\},$$
$$Z = \left\{u \in H^2(0, L) \cap H_0^1(0, L) \;\middle|\; \int_0^L uv = 0,\, v \in Y\right\}.$$

Then

$$\vartheta = \inf_{\substack{u \in Z \\ \|u\|=1}} \int_0^L \left[(u'')^2 + k(u')^2 + c(x) u^2\right] dx > 0.$$

Proof. By definition, on Z we have

$$\int_0^L \left[(u'')^2 + k(u')^2 + c(x) u^2\right] dx \geqslant \lambda_{n+1} \int_0^L u^2 dx.$$

Take a minimizing sequence $u_n \in Z$:

$$\int_0^L \left[\left(u_n''\right)^2 + k \left(u_n'\right)^2 \right] dx = 1,$$

$$\int_0^L \left[\left(u_n''\right)^2 + k \left(u_n'\right)^2 + c(x) u_n^2 \right] dx \longrightarrow \vartheta.$$

Going if necessary to a subsequence, we may assume

$$u_n \rightharpoonup u, \text{ in } H^2(0, L).$$

By the Rellich theorem,

$$\begin{aligned} \vartheta &= 1 + \int_0^L \left[c(x) u^2 \right] dx \\ &\geqslant \int_0^L \left[\left(u''\right)^2 + k \left(u'\right)^2 + c(x) u^2 \right] dx \\ &\geqslant \lambda_{n+1} \int_0^L u^2 dx. \end{aligned}$$

If $u = 0$, $\vartheta = 1$ and if $u \neq 0$, $\vartheta \geqslant \lambda_{n+1} \int_0^L u^2 dx > 0$. □

Proof of Theorem 3. *Step* 1: (PS) condition.

As usual we choose

$$\|u\|^2 = \int_0^L \left[\left(u''\right)^2 + k \left(u'\right)^2 \right] dx.$$

Let

$$\|u\|_*^2 = \int_0^L \left[\left(u''\right)^2 + k \left(u'\right)^2 + c(x) u^2 \right] dx,$$

then $\|u\|_*$ is an equivalent norm of X. From (C_3), we deduce there is a constant $c > 0$ such that

$$(C_3) \qquad c\left(|u|^{\sigma^{-1}} - 1 \right) \leqslant F(x, u).$$

Let u_n be a (PS) sequence, i.e.,

$$\varphi(u_n) \text{ is bounded and } \varphi'(u_n) \to 0 \text{ as } n \to +\infty.$$

Then by the preceding lemmas and (C_3), we obtain

$$\begin{aligned}\varphi(u_n) &\geqslant \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\int_0^L \left[c(x)\, u_n^2\right] dx \\ &\quad - \int_{\{x|u_n|>\rho\}} F(x,u_n)\, dx - C \\ &\geqslant \left(\frac{1}{2}-\sigma\right)\|u_n\|_*^2 + \sigma\left\{\|u_n\|_*^2 - \int_0^L f(x,u_n)\, u_n dx\right\} - C \\ &\geqslant \left(\frac{1}{2}-\sigma\right)\left(\vartheta\|z_n\|^2 + \lambda_1\|y_n\|_{L^2}\right) \\ &\quad + \sigma\left\{\|u_n\|_*^2 - \int_0^L f(x,u_n)\, u_n\, dx\right\} - C \\ &= \left(\frac{1}{2}-\sigma\right)\left(\vartheta\|z_n\|^2 + \lambda_1\|y_n\|_{L^2}\right) + \sigma\left(\varphi'(u_n), u_n\right) - C,\end{aligned}$$

where $u_n = z_n + y_n$, $y_n \in Y$, $z_n \in Z$. Since $\varphi'(u_n) \to 0$, we have

$$|(\varphi'(u_n), u_n)| \leqslant \|u_n\|, \text{ for } n \text{ large enough.}$$

Thus

$$\widetilde{C} \geqslant \varphi(u_n) \geqslant \left(\frac{1}{2}-\sigma\right)\left(\vartheta\|z_n\|^2 + \lambda_1\|y_n\|_{L^2}\right) - \sigma\|u_n\| - C,$$

which indicates $\|u_n\|$ must be bounded in X. Without loss of generality, we may assume that

$$u_n \rightharpoonup u, \text{ in } X.$$

Hence

$$\begin{aligned}\|u_n\|_* &= (\varphi'(u_n), u_n) + \int_0^L \left[f(x,u_n)\, u_n\right] dx \\ &\longrightarrow \int_0^L \left[f(x,u)\, u\right] dx \\ &= \|u\|_*,\end{aligned}$$

which together with $u_n \rightharpoonup u$ in X implies $\|u_n - u\| \to 0$, as $n \to +\infty$.

Step 2: *Linking geometry.* From condition (C_4), we have, for all $\widetilde{\varepsilon}$, there exists a $C_{\widetilde{\varepsilon}} > 0$,

$$F(x,u) \leqslant \widetilde{\varepsilon}|u|^2 + C_{\widetilde{\varepsilon}}|u|^p.$$

By virtue of the Sobolev imbedding theorem, we obtain

$$\varphi(u) \geqslant \left(\frac{1}{2} - \widetilde{\varepsilon}\widetilde{c}\right) \|u\|_*^2 - C_{\widetilde{\varepsilon}} \|u\|_*^p,$$

where $\widetilde{c}$ is Sobolev imbedding constant and $\widetilde{\varepsilon}$ satisfies $\widetilde{\varepsilon}\widetilde{c} < 1/2$. Whence $\varphi(u) > 0$ if $\|u\|_* = \rho$ is chosen small enough.

Besides, from (C_6), we have, on Y,

$$\varphi(u) \leqslant \int_0^L \left[\frac{\lambda_n}{2} u^2 - F(x, u)\right] dx \leqslant 0.$$

Let

$$z = \frac{\rho e_{n+1}}{\|e_{n+1}\|}.$$

Then using (C_3) we obtain

$$\varphi(u) \leqslant \frac{1}{2} \|u\|^2 + \frac{1}{2} \|a\|_{L^1} \|u\|_{L^\infty}^2 - c \|u\|_{L^{\sigma^{-1}}}^{\sigma^{-1}} + cL.$$

Since on the finite dimensional space $Y \oplus \mathbf{R}z$, all norms are equivalent, we have $\varphi(u) \leqslant 0$, if $u \in Y \oplus \mathbf{R}z$ and $\|u\|$ is large enough.

Thus by virtue of linking Theorem [**15**, P43], Problem (P_4) has a nontrivial solution. □

3.2. Periodic solutions–even functional case. In the case where

$$\varphi(u) = \int_0^L \left[(u'')^2 + k(u')^2 + c(x) u^2 - F(x, u)\right] dx$$

is an even functional and $c(x)$ satisfies (A_1), we can further obtain infinitely many solutions by using the following abstract critical point theorem.

Theorem 14 (Rabinowitz). *Let X be an infinite dimensional Banach space and X_n is the sequence of finite dimensional subspace of X with* $\dim X_n = n$*, and*

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset X, \quad \overline{\cup_{n=1}^{+\infty} X_n} = X.$$

$\varphi(u) \in \mathcal{C}^1(X, \mathbf{R})$ is an even functional, $\varphi(0) = 0$, and $\varphi(u)$ satisfies the (PS) condition. Suppose that

(i) *there are constants $\rho, \alpha > 0$ such that $\varphi(u)|_{\|u\|=\rho} \geqslant \alpha$, and*

(ii) *for every n there is a $\rho_n > 0$ such that $\varphi(u) \leqslant 0$ on $X_n \backslash B_{\rho_n}$.*

Then φ possesses infinitely many pairs of critical points with unbounded sequence of critical values.

Proof of Theorem 4. *Step* 1: (PS) condition.

Let $u_n \in X$ be a (PS) sequence; we claim that u_n is bounded in X. As before we use the equivalent norm

$$\|u\|_*^2 = \int_0^L \left[(u'')^2 + k(u')^2 + c(x)u^2\right] dx.$$

By condition (C_3),

$$\begin{aligned}
\varphi(u_n) &\geqslant \frac{1}{2}\|u_n\|_*^2 - \int_{\{x||u_n|>\rho\}} F(x, u_n)\, dx - C \\
&\geqslant \frac{1}{2}\|u_n\|_*^2 - \sigma \int_0^L f(x, u_n) u_n\, dx - C \\
&\geqslant \left(\frac{1}{2} - \sigma\right)\|u_n\|_*^2 + \sigma\left\{\|u_n\|_*^2 - \int_0^L f(x, u_n) u_n\, dx\right\} - C \\
&= \left(\frac{1}{2} - \sigma\right)\|u_n\|_*^2 + \sigma(\varphi'(u_n), u_n) - C.
\end{aligned}$$

Since $\varphi'(u_n) \to 0$, we have

$$|(\varphi'(u_n), u_n)| \leqslant \|u_n\|_*, \quad \text{for } n \text{ large enough.}$$

Thus

$$\widetilde{C} \geqslant \varphi(u_n) \geqslant \left(\frac{1}{2} - \sigma\right)\|u_n\|_*^2 - \sigma\|u_n\|_* - C,$$

which indicates $\|u_n\|_*$ and then $\|u_n\|$ must be bounded in X. Without loss of generality, we may assume that

$$u_n \rightharpoonup u, \quad \text{in } X.$$

Hence

$$\begin{aligned}\|u_n\|_* &= (\varphi'(u_n), u_n) + \int_0^L [f(x, u_n) u_n]\, dx \\ &\longrightarrow \int_0^L [f(x, u) u]\, dx \\ &= \|u\|_*,\end{aligned}$$

which together with $u_n \rightharpoonup u$ in X implies $\|u_n - u\| \to 0$, as $n \to +\infty$.

Step 2: *Geometric condition.* Consider the subspace

$$X_n = \left\{ \sin\frac{\pi x}{L} \cdots \sin\frac{n\pi x}{L} \right\}, \quad n \geqslant 1.$$

From condition (C_4), we have, for all $\widetilde{\varepsilon}$ that there exists a $C_{\widetilde{\varepsilon}} > 0$,

$$F(x, u) \leqslant \left(C\, |u|^{\sigma^{-1}} - \widetilde{C} \right).$$

By virtue of the Sobolev imbedding theorem, we obtain

$$\varphi(u) \geqslant \left(\frac{1}{2} - \widetilde{\varepsilon}\widetilde{c} \right) \|u\|_*^2 - C_{\widetilde{\varepsilon}} \|u\|_*^p,$$

where $\widetilde{c}$ is Sobolev imbedding constant and $\widetilde{\varepsilon}$ satisfies $\widetilde{\varepsilon}\widetilde{c} < 1/2$. Whence condition (i) holds if $\|u\|_* = \rho$ is chosen small enough.

Similarly, from condition (C_3), we have that there exists a C, $\widetilde{C} > 0$,

$$F(x, u) \geqslant \left(C\, |u|^{\sigma^{-1}} - \widetilde{C} \right).$$

Hence

$$\varphi(u) \leqslant C\, \|u\|_*^2 - \tilde{C}\, \|u\|_*^p, \quad u \in X.$$

Now we can choose $\widetilde{\rho} > 0$ large enough such that, for $u \in X_n$,

$$\varphi(u) \leqslant 0, \text{ if } \|u\|_{X_n} > \widetilde{\rho},$$

where we use again the fact that all norms on a finite dimensional space are equivalent. $\square$

4. Homoclinic solution. In this section, we consider the following functional

$$\varphi(u) = \int_{\mathbf{R}} \left\{ \frac{1}{2} \left[(u'')^2 - k(u')^2 - c(x)u^2 \right] - F(x,u) \right\} dx,$$

where $c(x)$ satisfies (A_1) and

$$2c_2^{1/2} > k.$$

Correspondingly, the space $H^2(\mathbf{R})$ becomes our protagonist with norm

$$\|u\| = \|u\|_{H^2(\mathbf{R})} = \int_{\mathbf{R}} \left[(u'')^2 + (u')^2 + u^2 \right] dx.$$

Lemma 16. *There exists a constant $c > 0$ such that*

$$\int_{\mathbf{R}} \left[(u'')^2 - k(u')^2 + c(x)u^2 \right] dx \geqslant c\|u\|^2.$$

Proof. The proof is not difficult and is left as an exercise. Alternatively, the readers could refer to [**2, 14**] for inspiration. □

Lemma 17. *Under conditions (C_1^*), (C_2) and (C_4), the functional $\varphi \in \mathcal{C}^1(H, \mathbf{R})$ and meets the following mountain pass conditions*:

(i) *There exists a $\rho > 0$ such that $\varphi(u) > 0$ on the sphere $\|u\| = \rho$.*

(ii) *There exists a $\widetilde{u} \in X$ such that $\|\widetilde{u}\| > \rho$ and $\varphi(\widetilde{u}) < 0$.*

Proof. As before, from conditions (C_1^*) and (C_4), we have, for all $\widetilde{\varepsilon}$ that there exists a $C_{\widetilde{\varepsilon}} > 0$,

$$F(x,u) \leqslant \widetilde{\varepsilon}|u|^2 + C_{\widetilde{\varepsilon}}|u|^p.$$

From the preceding lemma, we obtain, as long as $\|u\| = \rho$ is small,

$$\varphi(u) \geqslant \left(\frac{1}{2}c - \widetilde{\varepsilon}\widetilde{c} \right) \|u\|^2 - C_{\widetilde{\varepsilon}}\overline{c}\|u\|^p > 0,$$

where $\widetilde{c}$, $\overline{c}$, $\widetilde{\varepsilon}$ are constants satisfying

$$\overline{c}\,\|u\|^p \geqslant \int_{\mathbf{R}} |u(x)|^p\, dx,$$
$$\widetilde{c}\,\|u\|^2 \geqslant \int_{\mathbf{R}} |u(x)|^2\, dx, \quad \frac{1}{2}c - \widetilde{\varepsilon}\widetilde{c} > 0.$$

On the other hand, from conditions (C_2) and (C_4), we have that there exists a C, $\widetilde{C} > 0$,

$$F(x,u) \geqslant -C\,|u|^2 + \widetilde{C}\,|u|^p.$$

Fix any $\varpi \in H^2(\mathbf{R})$, $\varpi > 0$ on r. For $t > 0$ we achieve

$$\begin{aligned}\frac{\varphi(t\varpi)}{t^2} &= \frac{1}{2}\int_{\mathbf{R}} \left[(\varpi'')^2 - k(\varpi')^2 + c(x)\varpi^2\right] dx \\ &\quad - \frac{1}{t^2}\int_{\mathbf{R}} F(x, t\varpi)\, dx \\ &\leqslant \frac{1}{2}\int_{\mathbf{R}} \left[(\varpi'')^2 - k(\varpi')^2 + c(x)\varpi^2\right] dx \\ &\quad + C\int_{\mathbf{R}} \varpi^2 dx - \widetilde{C}t^{p-2}\int_{\mathbf{R}} |\varpi|^p\, dx \\ &\longrightarrow -\infty, \quad \text{as } t \to +\infty.\end{aligned}$$

Therefore, for some $\tilde{t} > \rho$, we have $\varphi(\widetilde{u}) < 0$ with $\widetilde{u} = \tilde{t}\varpi$. □

Proof of Theorem 5. By the preceding lemma, there is a sequence u_n in $H^2(\mathbf{R})$ such that

$$\varphi(u_n) \longrightarrow m > 0, \qquad \|\varphi'(u_n)\|_{H^2(\mathbf{R})^*} \longrightarrow 0,$$

where

$$m = \inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)),$$
$$\Gamma = \left\{\gamma \in \left(C[0,1], H^2(\mathbf{R})\right) \mid \gamma(0) = 0,\ \gamma(1) = \widetilde{u}\right\}.$$

As in the proof of Theorem 4, one easily deduces that u_n must be bounded in $H^2(\mathbf{R})$, whence $(\varphi'(u_n), u_n) \to 0$. Therefore

$$\int_{\mathbf{R}} \varrho u_n^q + \tau u_n^{\overline{q}} \geqslant \varphi(u_n) - \frac{1}{2}(\varphi'(u_n), u_n) \longrightarrow c > 0.$$

For notational simplicity, we denote $\int_j^{j+1} |\cdot|^s$ by $\|\cdot\|_{L_j^s}^s$. Then

$$
\begin{aligned}
c &\leqslant \int_{\mathbf{R}} \varrho u_n^q + \tau u_n^{\overline{q}} \\
&\leqslant \sum_i \int_j^{j+1} \left(\varrho \left| u_n \right|^q + \tau \left| u_n \right|^{\overline{q}} \right) \\
&\leqslant \sum_j \left(\|u_n\|_{L_j^q}^q + \|u_n\|_{L_j^{\overline{q}}}^{\overline{q}} \right) \\
&\leqslant \sup_j \max \left\{ \|u_n\|_{L_j^q}^{q-1},\ \|u_n\|_{L_j^{\overline{q}}}^{\overline{q}-1} \right\} \\
&\quad \times \sum_j \left(\|u_n\|_{L_j^q} + \|u_n\|_{L_j^{\overline{q}}} \right) \\
&\leqslant C \sup_j \max \left\{ \|u_n\|_{L_j^q}^{q-1},\ \|u_n\|_{L_j^{\overline{q}}}^{\overline{q}-1} \right\} \|u_n\|_{H^2(\mathbf{R})} \\
&\leqslant C \sup_j \max \left\{ \|u_n\|_{L_j^q}^{q-1},\ \|u_n\|_{L_j^{\overline{q}}}^{\overline{q}-1} \right\} \\
&\leqslant C \sup_j \max \left\{ \|u_n\|_{L_j^q}^{q-1},\ \|u_n\|_{L_j^q}^{\overline{q}-1} \right\},
\end{aligned}
$$

which leads to

$$0 < \widetilde{c} = \min \left\{ c^{1/(q-1)}, c^{1/(\overline{q}-1)} \right\} < \sup_j \|u_n\|_{L_j^q} .$$

Consequently,

$$\liminf_{n \to +\infty} \sup_j \int_j^{j+1} \left| u_n(x) \right|^q dx \geqslant c^{1/q} > 0,$$

and we are in a position to apply concentration compactness principle, cf. [**13**, P39]. Choose a sequence $i_n \in \mathbf{Z}$ such that

$$\liminf_{n \to +\infty} \int_0^1 \left| u_n(i_n + x) \right|^q dx > 0.$$

Then the sequence of functions $v_n(x) = u_n(i_n + x) \in H^2(\mathbf{R})$ is bounded and we may assume

$$v_n \rightharpoonup v, \quad \text{in } H^2(\mathbf{R}).$$

One easily sees that $v \neq 0$. By presumption $c(x)$ and $F(x,u)$ are 1-periodic functions with respect to x, we obtain

$$\varphi(u_n) = \varphi(v_n),$$

and

$$\|\varphi'(v_n)\|_{H^2(\mathbf{R})^*} \leqslant \|\varphi'(u_n)\|_{H^2(\mathbf{R})^*} \longrightarrow 0.$$

It follows that $\varphi'(v) = 0$, and the proof is complete. $\square$

Remark 18. From the proof, one may notice that the conclusion remains true if there exist constants $\varrho > 0$ and an integer $q > 1$ such that

$$F(x,u) - \frac{1}{2}uf(x,u) \leqslant \varrho u^q.$$

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