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OSCILLATION OF NONLINEAR IMPULSIVE PARABOLIC DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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ABSTRACT. In this paper, oscillatory properties of solutions for certain nonlinear impulsive parabolic partial differential equation of neutral type are investigated and a series of new sufficient conditions and a necessary and sufficient condition for oscillation of the equations are established.

1. Introduction. The theory of delay partial differential equations can be applied to many fields, such as to biology, population growth, engineering, generic repression, control theory and climate model. In the last few years, the fundamental theory of partial differential equations with deviating argument has undergone intensive development. The qualitative theory of this class of equations, however, is still in an initial stage of development. A few papers have been published on oscillation theory of partial differential equations with delay. Many have been done under the assumption that the state variables and system parameters change continuously. However, one may easily visualize situations in nature where abrupt change such as shock and disasters may occur. These phenomena are short-time perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is, in the form of impulses. In 1991, the first paper [**8**] on this class of equations was published. But, for instance, on oscillation theory of impulsive partial differential equations only a few of papers have been published. Recently, Bainov, Minchev, Luo and Liu [**3, 4, 7, 18, 20, 22**] investigated the oscillation of solutions of impulsive partial differential

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equations with or without deviating argument. But there is a scarcity in the study of oscillation theory of nonlinear impulsive parabolic partial differential equations of neutral type.

In this paper, we discuss oscillatory properties of solutions for a class of nonlinear impulsive parabolic partial differential equations with several delays (1), (2), under the boundary condition (3).

(1)
$$\frac{\partial}{\partial t}(u(t,x)+q(t)u(t-\mu,x)) = a(t)h(u)\Delta u + \sum_{i=1}^{m} a_i(t)h_i(u(t-\tau_i,x))\Delta u(t-\tau_i,x) - \sum_{j=1}^{n} q_j(t,x)f_j(u(t-\sigma_j,x))$$
$$t \neq t_k, \quad (t,x) \in R_+ \times \Omega = G$$

(2)
$$u(t_k^+,x) - u(t_k^-,x) = b_k u(t_k,x),$$

with the boundary condition

(3)
$$u = 0, \qquad (t,x) \in R_+ \times \partial\Omega$$

and the initial condition $u(t,x) = \Phi(t,x)$, $(t,x) \in [-\delta,0] \times \Omega$ where $\Omega \subset R^N$ is a bounded domain with boundary $\partial\Omega$ smooth enough and n is a unit exterior normal vector of $\partial\Omega$, $\delta = \max\{\mu,\tau_i,\sigma_j\}$, $\Phi(t,x) \in C^2([-\delta,0]\times\Omega,R)$.

This article is organized as follows: Section 2 studies the oscillatory properties of solutions for problem (1)–(3). In Section 3, we, for the linear case, obtain a necessary and sufficient condition for oscillation of solutions.

Assume that the following conditions are fulfilled:

H_1) $a(t), a_i(t) \in PC(R_+,R_+), q(t) \in PC(R_+,(-1,0])$, $\mu,\tau_i,\sigma_j =$ const. > 0, $q_j(t,x) \in C(R_+ \times \overline{\Omega},(0,\infty))$, $i = 1,2,\dots,m$, $j = 1,2,\dots n$; where PC denotes the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$ and left continuous at $t = t_k$, $k = 1,2,\dots$. $q(t_k^+) = (1+b_k)q(t_k^-)$.

H_2) $h'(u), h_i'(u), f_j(u) \in C(R,R)$; $f_j(u)/u \geq C_j =$ const. > 0, for $u \neq 0$; $uh'(u) \geq 0$, $uh_i'(u) \geq 0$, $h(0) = 0$, $h_i(0) = 0$, $b_k, d_k =$

const. > -1, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$. $Q(t) = \prod_{t-\mu\le t_k<t}(1+b_k)^{-1}q(t) \in C(R_+,(-1,0])$.

H_3) $u(t,x)$ is piecewise continuous in t with discontinuities of the first kind only at $t = t_k$ and left continuous at $t = t_k$, $u(t_k,x) = u(t_k^-,x)$, $k = 1,2,\dots$.

We introduce the notations: $v(t) = \int_\Omega u(t,x)\,dx$ and $p_j(t) = \min q_j(t,x), x \in \overline{\Omega}$.

Definition 1.2. The solution $u \in C^2(\Gamma)\cap C^1(\overline{\Gamma})$ of problem (1)–(3) is called nonoscillatory in domain G if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

2. Oscillation properties of the problem (1)–(3). The following Theorem 2.2 is the main theorem of this paper. The proof of the theorem needs the following lemma [**24**].

Lemma 2.1. *Let $\rho = \text{const.} > 0$, $a_0(t), p(t) \in ([0,+\infty), R)$ be locally summable functions and $p(t) > 0$; $y(t_k) = y(t_k^-)$, $k = 1,2,\dots$. If the following condition is satisfied*

$$\liminf_{t\to\infty}\int_{t-\rho}^{t} p(s)\exp\left(\int_{s-\rho}^{s} a_0(r)\,dr\right)\prod_{s-\rho<t_k<s}(1+d_k)^{-1}\,ds > \frac{1}{e},$$

then the following differential inequality has no eventually positive solution.

$$\begin{gathered} y'(t) + a_0(t)y(t) + p(t)y(t-\rho) \le 0, \quad t \ge 0,\ t \ne t_k,\\ y(t_k^+) - y(t_k^-) = d_k y(t_k), \quad k = 1,2,\dots . \end{gathered}$$

Theorem 2.2. *Suppose that conditions H_1), H_2) and the following condition (4) hold for some $j \in \{1,\dots,n\}$,*

$$\liminf_{t\to\infty}\int_{t-\sigma_j}^{t} C_j p_j(s)\prod_{s-\sigma_j<t_k<s}(1+b_k)^{-1}\,ds > \frac{1}{e}. \tag{4}$$

Then every solution of the problem (1)–(3) oscillates in G.

Proof. Suppose that the assertion is not true and $u(t,x)$ is a nonoscillatory solution of problem (1)–(3). Without loss of generality, we may assume that there exists a $t_0 \geq T$ such that $u(t,x) > 0, u(t-\mu,x) > 0$, $u(t-\tau_i,x) > 0$, $i = 1,2,\dots,m$ and $u(t-\sigma_j,x) > 0$, $j = 1,2,\dots,n$ for any $(t,x) \in [t_0,\infty) \times \Omega$.

For $t \geq t_0$, $t \neq t_k$, $k = 1,2,\dots$, integrating (1) with respect to x over Ω yields

$$\begin{aligned}\frac{d}{dt}\left[\int_\Omega (u(t,x) + q(t)u(t-\mu,x))\,dx\right] &= a(t)\int_\Omega h(u)\Delta u\,dx \\ &+ \sum_{i=1}^m a_i(t)\int_\Omega h_i(u(t-\tau_i,x))\Delta u(t-\tau_i,x)\,dx \\ &- \sum_{j=1}^n \int_\Omega q_j(t,x) f_j(u(t-\sigma_j,x))\,dx \quad (t \geq t_0,\ t \neq t_k).\end{aligned}$$

By Green's formula and the boundary condition we have

$$\begin{aligned}\int_\Omega h(u)\Delta u\,dx &= \int_{\partial\Omega} h(u)\frac{\partial u}{\partial n}\,ds - \int_\Omega h'(u)|\operatorname{grad} u|^2\,dx \\ &\leq -\int_\Omega h'(u)|\operatorname{grad} u|^2\,dx \leq 0, \\ \int_\Omega h_i(u(t-\tau_i,x))&\Delta u(t-\tau_i,x)\,dx \leq 0.\end{aligned}$$

From condition H_2), we can easily obtain

$$\int_\Omega q_j(t,x) f_j(u(t-\sigma_j,x))\,dx \geq C_j p_j(t)\int_\Omega u(t-\sigma_j,x)\,dx.$$

Then $v(t) > 0$, and it follows that

$$\frac{d}{dt}[v(t) + q(t)v(t-\mu)] + \sum_{j=1}^n C_j p_j(t) v(t-\sigma_j) \leq 0, \quad (t \geq t_0,\ t \neq t_k).$$

Hence we obtain

$$\frac{d}{dt}[v(t) + q(t)v(t-\mu)] + C_j p_j(t) v(t-\sigma_j) \leq 0, \quad (t \geq t_0,\ t \neq t_k). \tag{5}$$

In inequality (5), set $w(t)=\prod_{t_0\le t_k<t}(1+b_k)^{-1}v(t)$. We can obtain the following results: (1) $w(t)$ is continuous on $[t_0,+\infty)$, (2) Inequality (5) has no eventually positive solution if and only if the following inequality (6) has no eventually positive solution.

$$(6)\quad \frac{d}{dt}[w(t)+Q(t)w(t-\mu)]+C_jP_j(t)w(t-\sigma_j)\le 0,\quad (t\ge t_0,\ t\ne t_k),$$

where

$$Q(t)=\prod_{t-\mu\le t_k<t}(1+b_k)^{-1}q(t),\ \ P_j(t)=\prod_{t-\sigma_j\le t_k<t}(1+b_k)^{-1}p_j(t).$$

In fact, $v(t)$ is continuous on each interval $(t_k,t_{k+1}]$, and in view of $v(t_k^+)=(1+b_k)v(t_k)$, it follows that for $t\ge t_0$,

$$w(t_k^+)=\prod_{t_0\le t_j\le t_k}(1+b_j)^{-1}v(t_k^+)=\prod_{t_0\le t_j<t_k}(1+b_j)^{-1}v(t_k)=w(t_k),$$

and for all $t\ge t_0$,

$$w(t_k^-)=\prod_{t_0\le t_j\le t_{k-1}}(1+b_j)^{-1}v(t_k^-)=\prod_{t_0\le t_j<t_k}(1+b_j)^{-1}v(t_k)=w(t_k),$$

which implies that $w(t)$ is continuous on $[t_0,+\infty)$. Moreover, we obtain that for almost everywhere $t\in[t_0,+\infty)$

$$\begin{aligned}
&\frac{d}{dt}[w(t)+Q(t)w(t-\mu)]+C_jP_j(t)w(t-\sigma_j)\\
&=\frac{d}{dt}\Bigg[\prod_{t_0\le t_k<t}(1+b_k)^{-1}v(t)\\
&\quad+\prod_{t-\mu\le t_k<t}(1+b_k)^{-1}q(t)\prod_{t_0\le t_k<t-\mu}(1+b_k)^{-1}v(t-\mu)\Bigg]\\
&\quad+C_j\prod_{t-\sigma_j\le t_k<t}(1+b_k)^{-1}p_j(t)\prod_{t_0\le t_k<t-\sigma_j}(1+b_k)^{-1}v(t-\sigma_j)\\
&=\prod_{t_0\le t_k<t}(1+b_k)^{-1}\left(\frac{d}{dt}[v(t)+q(t)v(t-\mu)]+C_jp_j(t)v(t-\sigma_j)\right)\le 0,
\end{aligned}$$

which implies that $w(t)$ is a positive solution.

Conversely, let $w(t)$ be an eventually positive solution and $w(t) > 0, w(t-\mu) > 0$ and $w(t-\sigma_j) > 0$ for $t \geq t_0$, set $v(t) = \prod_{t_0 \leq t_k < t}(1+b_k)w(t)$. As $w(t)$ is continuous on $[t_0, +\infty)$, $v(t)$ is continuous on each interval $(t_k, t_{k+1}], t_k \geq t_0$ and for almost everywhere $t \in [t_0, +\infty)$,

$$\frac{d}{dt}[v(t) + q(t)v(t-\mu)] + C_j p_j(t) v(t-\sigma_j) = \frac{d}{dt}\Bigg[\prod_{t_0 \leq t_k < t}(1+b_k)w(t) + q(t)\prod_{t_0 \leq t_k < t-\mu}(1+b_k)w(t-\mu)\Bigg] + C_j p_j(t) \times \prod_{t_0 \leq t_k < t-\sigma_j}(1+b_k)w(t-\sigma_j) = \prod_{t_0 \leq t_k < t}(1+b_k)\bigg(\frac{d}{dt}[w(t) + Q(t)w(t-\mu)] + C_j P_j(t) w(t-\sigma_j)\bigg) \leq 0.$$

On the other hand, for every $t_k \geq t_0$,

$$v(t_k^+) = \lim_{t \to t_k^+} \prod_{t_0 \leq t_j < t}(1+b_j)w(t) = \prod_{t_0 \leq t_j \leq t_k}(1+b_j)w(t_k),$$

and

$$v(t_k) = \prod_{t_0 \leq t_j < t_k}(1+b_j)w(t_k).$$

Thus, for every $t_k \geq t_0, k \in N$, we have

$$v(t_k^+) = (1+b_k)v(t_k).$$

Hence we obtain that $v(t)$ is a positive solution. This completes the proof of the claim.

Now in inequality (6), set

$$y(t) = w(t) + Q(t)w(t-\mu). \tag{7}$$

Hence we have

$$y'(t) + C_j P_j(t) w(t-\sigma_j) \leq 0, \quad (t \geq t_0,\ t \neq t_k), \tag{8}$$

From inequality (8) it is easy to see that $y(t)$ is non-increasing, so we obtain that $\lim_{t\to\infty} y(t) = L$.

Now we discuss L.

(1) If we suppose that $L = -\infty$, then $\lim_{t\to\infty} y(t) = -\infty$. From inequality (7), we can get that $w(t)$ is unbounded; consequently, there exists $\{s_k : k \to \infty, s_k \to \infty\}$, such that $y(s_k) < 0, w(s_k) = \max w(r)$, $r \in [t_0, s_k]$. Therefore $y(s_k) = w(s_k) + Q(s_k)w(s_k - \mu) \geq w(s_k)[1 + Q(s_k)] \geq 0$. This contradicts $y(s_k) < 0$.

(2) If we suppose that $L \neq 0$ is limited, then integrating inequality (8) from t_0 to t, we obtain

$$\int_{t_0}^{t} C_j P_j(t) w(t - \sigma_j) \leq y(t_0) - y(t),$$

or

$$\int_{t_0}^{+\infty} C_j P_j(t) w(t - \sigma_j) \leq y(t_0) - L.$$

This implies $w(t) \to 0$; hence, we have $y(t) \to 0$. This contradicts $L \neq 0$.

It follows that $L = 0$. Since $y(t)$ is non-increasing, $y(t) > 0$. Hence, from inequality (7), we get $y(t) < w(t)$ and from inequality (8), we obtain the following differential inequality

$$y'(t) + C_j P_j(t) y(t - \sigma_j) \leq 0, \quad (t \geq t_0,\ t \neq t_k). \tag{9}$$

For $t > t_0$, $t = t_k$, $k = 1, 2, \ldots$, since $w(t)$ is continuous on $[t_0, +\infty)$ and $Q(t_k^+) = Q(t_k^-)$, it is easy to verify that

$$y(t_k^+) = y(t_k). \tag{10}$$

Hence, we obtain that $y(t) > 0$ is an eventually positive solution of differential inequality (9), (10). But according to Lemma 2.1 (where $P_j(t) = \prod_{t-\sigma_j \leq t_k < t} (1 + b_k)^{-1} p_j(t), d_k = 0)$ and condition (4), the differential inequality (9), (10) has no eventually positive solution. This is a contradiction. This ends the proof of the theorem. □

Similarly, we can obtain the following theorems

Theorem 2.3. *Suppose that the conditions H_1), H_2) and the following condition hold for some $j \in \{1, \dots, n\}$,*

$$\lim_{t\to\infty} \sup \int_{t-\sigma_j}^{t} C_j p_j(s) \prod_{s-\sigma_j<t_k<s} (1+b_k)^{-1}\, ds > 1.$$

Then every solution of the problem (1)–(3) *oscillates in G.*

We introduce the following conditions.

h_1) There exists a constant $M > 0$ such that $0 \leq b_k \leq M$ for all $k \in N$.

h_2) There exists an integer m_1 such that $m_1(t_{k+1} - t_k) \geq \sigma_j, \tau_i$ for all $k \in N$ and for some $j \in \{1, \dots, n\}$, $i \in \{1, \dots, m\}$.

Theorem 2.4. *Suppose that the conditions H_1), H_2), h_1), h_2) and the following condition hold for some $j \in \{1, \dots, n\}$,*

$$\lim_{t\to\infty} \inf \int_{t-\sigma_j}^{t} C_j p_j(s)\, ds > \frac{(1+M)^{m_1}}{e}.$$

Then every solution of the problem (1)–(3) *oscillates in G.*

Theorem 2.5. *Suppose that the conditions H_1), H_2), h_1), h_2) and the following condition hold for some $j \in \{1, \cdots, n\}$,*

$$\lim_{t\to\infty} \sup \int_{t-\sigma_j}^{t} C_j p_j(s)\, ds > (1+M)^{m_1}.$$

Then every solution of the problem (1)–(3) *oscillates in G.*

More generally, we have the following Theorem 2.6.

Theorem 2.6. *Suppose that the conditions of Theorem* 2.2 *still hold and condition* (4) *is replaced by the differential inequality* (9), (10) *has no eventually positive solution. Then every solution of the problem* (1)–(3) *oscillates in G.*

The proofs are easy, we just omit it.

Making use of the following lemma of eigenvalue, we can obtain many results for problem (1)–(3). We suppose that $h(u)$, $h_i(u)$ are constants (suppose them all 1).

Lemma 2.7. *Suppose that λ_0 is the smallest eigenvalue of the problem*

$$\begin{gathered}\Delta\varphi(x) + \lambda\varphi(x) = 0, \quad x \in \Omega \\ \varphi(x) = 0, \quad x \in \partial\Omega.\end{gathered}$$

and $\varphi(x)$ is the corresponding eigenfunction of λ_0. Then $\lambda_0 > 0$, $\varphi(x) > 0$, $x \in \Omega$.

Theorem 2.8. *Suppose that the conditions H_1), H_2) and the following condition hold for some $j \in \{1, \dots, n\}$.*

$$\begin{aligned}(11) \quad \liminf_{t\to\infty} \int_{t-\sigma_j}^{t} C_j p_j(s) \exp\left(\int_{s-\sigma_j}^{s} \lambda_0 a(r)\, dr\right) \\ \times \prod_{s-\sigma_j < t_k < s} (1+b_k)^{-1}\, ds > \frac{1}{e}.\end{aligned}$$

Then every solution of the problem (1)–(3) oscillates in G.

Proof. Suppose that the assertion is not true and $u(t,x)$ is a nonoscillatory solution of problem (1)–(3). Without loss of generality, we may assume that there exists a $t_0 \ge T$ such that $u(t,x) > 0$, $u(t-\mu, x) > 0$, $u(t-\tau_i, x) > 0$, $i = 1, 2, \dots, m$ and $u(t-\sigma_j, x) > 0$, $j = 1, 2, \dots, n$ for any $(t,x) \in [t_0, \infty) \times \Omega$.

For $t \ge t_0$, $t \ne t_k$, $k = 1, 2, \dots$, multiplying equation (1) with $\varphi(x)$, which is the same as that in Lemma 2.7 and then integrating (1) with respect to x over Ω yields

$$\begin{aligned}\frac{d}{dt}\left[\int_\Omega (u(t,x)+q(t)u(t-\mu,x))\varphi(x)\,dx\right]\\ &= a(t)\int_\Omega \Delta u\varphi(x)\,dx + \sum_{i=1}^m a_i(t)\int_\Omega \Delta u(t-\tau_i,x)\varphi(x)\,dx\\ &\quad -\sum_{j=1}^n\int_\Omega q_j(t,x)f_j(u(t-\sigma_j,x))\varphi(x)\,dx\ (t\ge t_0,\ t\ne t_k).\end{aligned}$$

By Green's formula and the boundary condition we have

$$\int_\Omega u\Delta\varphi\,dx-\int_\Omega\varphi\Delta u\,dx=\int_{\partial\Omega}\frac{\partial\varphi}{\partial n}u\,ds-\int_{\partial\Omega}\frac{\partial u}{\partial n}\varphi\,ds=0.$$

It follows that

$$\begin{aligned}\int_\Omega \Delta u(t,x)\varphi(x)\,dx &= \int_\Omega \Delta\varphi(x)u(t,x)\,dx\\ &= -\lambda_0\int_\Omega \varphi(x)u(t,x)\,dx,\\ \int_\Omega \Delta u(t-\tau_j,x)\varphi(x)\,dx &= \int_\Omega \Delta\varphi(x)u(t-\tau_i,x)\,dx\\ &= -\lambda_0\int_\Omega \varphi(x)u(t-\tau_i,x)\,dx.\end{aligned}$$

From condition H_2), we can easily obtain

$$\begin{aligned}(12)\quad &\int_\Omega q_j(t,x)f_j(u(t-\sigma_j,x))\varphi(x)\,dx\\ &\qquad\ge C_jp_j(t)\int_\Omega u(t-\sigma_j,x)\varphi(x)\,dx\ (t\ge t_0,\ t\ne t_k).\end{aligned}$$

Denote $v(t)=\int_\Omega u(t,x)\varphi(x)\,dx$. Then $v(t)>0$, and it follows that

$$\begin{aligned}(13)\quad &\frac{d}{dt}[v(t)+q(t)v(t-\mu)]+\lambda_0a(t)v(t)+\lambda_0\sum_{i=1}^m a_i(t)v(t-\tau_i)\\ &\qquad+\sum_{j=1}^n C_jp_j(t)v(t-\sigma_j)\le 0\ (t\ge t_0,\ t\ne t_k).\end{aligned}$$

Hence we obtain a similar differential inequality as (5).

$$\frac{d}{dt}[v(t)+q(t)v(t-\mu)]+\lambda_0 a(t)v(t)+C_j p_j(t)v(t-\sigma_j)\le 0, \quad (t\ge t_0,\ t\ne t_k). \tag{14}$$

The following proof is similar to that used in Theorem 2.2. We omit it. This ends the proof of the theorem. □

The problem that papers [**7, 18**] discussed is a special case of Theorem 2.8 here.

Theorem 2.9. *Suppose that the conditions H_1), H_2) and the following condition hold for some $j\in\{1,\dots,n\}$,*

$$\limsup_{t\to\infty}\int_{t-\sigma_j}^{t} C_j p_j(s)\exp\left(\int_{s-\sigma_j}^{s}\lambda_0 a(r)\,dr\right)\prod_{s-\sigma_j<t_k<s}(1+b_k)^{-1}\,ds>1. \tag{15}$$

Then every solution of the problem (1)–(3) *oscillates in G.*

Theorem 2.10. *Suppose that the conditions H_1), H_2) and the following condition hold for some $a_i(t)$,*

$$\liminf_{t\to\infty}\int_{t-\tau_i}^{t}\lambda_0 a_i(s)\exp\left(\int_{s-\tau_i}^{s}\lambda_0 a(r)\,dr\right)\prod_{s-\tau_i<t_k<s}(1+b_k)^{-1}\,ds>\frac{1}{e}. \tag{16}$$

Then every solution of the problem (1)–(3) *oscillates in G.*

Proof. From differential inequality (13) we can obtain

$$\frac{d}{dt}[v(t)+q(t)v(t-\mu)]+\lambda_0 a(t)v(t)+\lambda_0 a_i(t)v(t-\tau_i)\le 0, \quad (t\ge t_0,\ t\ne t_k). \tag{17}$$

The following proof is the same as that used in Theorem 2.8. We just omit it. This ends the proof of Theorem 2.10. □

It should be noted that the criteria in this theorem only depends on diffusion coefficient $a_i(t)$.

Theorem 2.11. *Suppose that the conditions* H_1), H_2) *and the following condition hold for some* $a_i(t)$,

$$\limsup_{t\to\infty} \int_{t-\tau_i}^{t} \lambda_0 a_i(s) \exp\left(\int_{s-\tau_i}^{s} \lambda_0 a(r)\, dr\right) \prod_{s-\tau_i<t_k<s} (1+b_k)^{-1}\, ds > 1.$$

Then every solution of the problem (1)–(3) *oscillates in* G.

3. Necessary and sufficient condition. In this section, we will establish a necessary and sufficient condition for oscillation of impulsive parabolic partial differential equation with several delays. We consider the following linear problem.

$$\frac{\partial}{\partial t}(u(t,x) + q(t)u(t-\mu,x)) = a(t)\Delta u + \sum_{i=1}^{m} a_i(t)\Delta u(t-\tau_i,x)$$

$$- \sum_{j=1}^{n} p_j(t)u(t-\sigma_j,x), \quad t \neq t_k,\ (t,x) \in R_+ \times \Omega = G \tag{18}$$

$$u(t_k^+,x) - u(t_k^-,x) = b_k u(t_k,x), \quad k = 1,2,\ldots \tag{19}$$

with boundary condition (3).

Theorem 3.1. *Every solution of the problem* (3), (18), (19) *is oscillatory in domain* G *if and only if every solution of the following impulsive delay differential equation* (20), (21) *is oscillatory.*

$$\frac{d}{dt}[v(t) + q(t)v(t-\mu)] + a(t)\lambda_0 v(t) + \lambda_0 \sum_{i=1}^{m} a_i(t)v(t-\tau_i) + \sum_{j=1}^{n} p_j(t)v(t-\sigma_j) = 0, \tag{20}$$

$$v(t_k^+) - v(t_k^-) = b_k v(t_k), \quad k = 1, 2, \ldots . \tag{21}$$

Proof. Sufficiency. Using reduction to absurdity, let $u(t,x)$ be a nonoscillatory solution of the problem (3), (18), (19). Without loss of generality, we may assume that there exists a $t_0 \geq T$ such that $u(t,x) > 0$, $u(t-\mu, x) > 0$, $u(t-\tau_i, x) > 0$ and $u(t-\sigma_j, x) > 0$, $i = 1, \ldots, m$; $j = 1, \ldots, n$ for any $(t,x) \in [t_0, +\infty) \times \Omega$.

For $t \geq t_0$, $t \neq t_k$, $k = 1, 2, \ldots$, multiplying equation (18) with $\varphi(x)$, which is the same as that in Lemma 2.7, then integrating (18) with respect to x over Ω yields

$$\begin{aligned} \frac{d}{dt}\int_\Omega [u(t,x) + q(t)u(t-\mu)]\varphi(x)\,dx &= a(t)\int_\Omega \Delta u(t,x)\varphi(x)\,dx \\ &+ \sum_{i=1}^m a_i(t)\int_\Omega \Delta u(t-\tau_i, x)\varphi(x)\,dx \\ &- \sum_{j=1}^n \int_\Omega p_j(t)u(t-\sigma_j, x)\varphi(x)\,dx. \end{aligned} \tag{22}$$

By Green's formula, we have

$$\int_\Omega u\Delta\varphi(x)\,dx - \int_\Omega \varphi(x)\Delta u\,dx = \int_{\partial\Omega} u\frac{\partial\varphi(x)}{\partial n}\,ds - \int_{\partial\Omega}\varphi(x)\frac{\partial u}{\partial n}\,ds.$$

Since boundary condition (3) is the first kind of boundary condition, the right side of the above equality vanishes. It follows that

$$\int_\Omega \varphi(x)\Delta u\,dx = \int_\Omega u\Delta\varphi(x)\,dx = -\lambda_0\int_\Omega \varphi(x)u(t,x)\,dx$$

$$\begin{aligned} \int_\Omega \varphi(x)\Delta u(t-\tau_i, x)\,dx &= \int_\Omega u(t-\tau_i, x)\Delta\varphi(x)\,dx \\ &= -\lambda_0\int_\Omega \varphi(x)u(t-\tau_i, x)\,dx. \end{aligned}$$

Denote $v(t) = \int_\Omega \varphi(x)u(t,x)\,dx$. Then $v(t) > 0$. It follows that we can easily obtain

$$\begin{aligned} \frac{d}{dt}[v(t) + q(t)v(t-\mu)] + a(t)\lambda_0 v(t) + \lambda_0\sum_{i=1}^m a_i(t)v(t-\tau_i) \\ + \sum_{j=1}^n p_j(t)v(t-\sigma_j) = 0. \end{aligned} \tag{23}$$

For $t > t_0$, $t = t_k$, $k = 1, 2, \dots$, we have

$$\int_\Omega u(t_k^+, x)\, dx - \int_\Omega u(t_k^-, x)\, dx = b_k \int_\Omega u(t_k, x)\, dx.$$

This implies

$$v(t_k^+) - v(t_k^-) = b_k v(t_k). \tag{24}$$

Hence we obtain that $v(t) > 0$ satisfies equation (20), (21). This means that impulsive delay differential equation (20), (21) has a nonoscillatory solution. A contradiction. This ends the proof of sufficient condition.

Necessity. Still using reduction to absurdity, let $v(t)$ be a nonoscillatory solution of the equation (20), (21). Without loss of generality, we may assume that there exists a t_1 large enough such that $v(t) > 0$, $v(t-\mu)$, $v(t-\tau_i) > 0$ and $v(t-\sigma_j) > 0$, $i = 1, \dots, m$; $j = 1, \dots, n$ for any $t \in [t_1, +\infty)$.

For $t \geq t_1$, $t \neq t_k$, $k = 1, 2, \dots$, set $u(t,x) = v(t)\varphi(x)$; we have $u(t,x) > 0$ and we can easily obtain

$$\begin{aligned}\Delta u(t,x) &= \Delta[v(t)\varphi(x)] = v(t)\Delta\varphi(x) = -\lambda_0 v(t)\varphi(x)\\ \Delta u(t-\tau_i, x) &= \Delta[v(t-\tau_i)\varphi(x)] = v(t-\tau_i)\Delta\varphi(x)\\ &= -\lambda_0 v(t-\tau_i)\varphi(x).\end{aligned}$$

Making use of these results, from equation (20), we obtain

$$\begin{aligned}&\frac{d}{dt}[(v(t) + q(t)v(t-\mu))\varphi(x)] + a(t)\lambda_0 v(t)\varphi(x)\\ &\quad + \lambda_0 \sum_{i=1}^{m} a_i(t) v(t-\tau_i)\varphi(x) + \sum_{j=1}^{n} p_j(t) v(t-\sigma_j)\varphi(x) = 0.\end{aligned} \tag{25}$$

This means that $u(t,x) = v(t)\varphi(x)$ satisfies equation (18).

For $t \geq t_1$, $t = t_k$, $k = 1, 2, \dots$, from condition (21), it is easy to see that function $u(t,x) = v(t)\varphi(x)$ satisfies (19). And, because $\varphi(x) = 0$, $x \in \partial\Omega$, we have that $u(t,x) = v(t)\varphi(x)$ also satisfies boundary condition (3). This indicates that problem (3), (18) and (19) has a nonoscillatory solution. This is a contradiction. This ends the proof of Theorem 3.1. □

4. Remarks and examples. The results of this paper, from a practical standpoint, are very convenient because these criteria only depend on the coefficients of the equations, impulsive term and the time-delays; from the theoretical viewpoint, they uncovered the essential difference between partial differential equations with impulses, functional arguments and partial differential equations without impulses, functional arguments. The results of this paper improve the results in the papers [**3, 7, 15, 18**, etc.]. For example paper [**18**] only discussed the case of $b_k = b =$ Const., and in this paper that b_k may be different for every k.

The following are examples that justify the applicability of the conditions.

Example 1.

$$\frac{\partial}{\partial t}\left(u(t,x)+q(t)u\left(t-\frac{\pi}{2},x\right)\right) = u^2\Delta u + u^2\left(t-\frac{\pi}{2},x\right)\Delta u\left(t-\frac{\pi}{2},x\right) - u\left(t-\frac{3\pi}{2},x\right)e^{[u(t-(3\pi/2),x)]^2}$$

$$t \neq t_k, \quad (t,x)\in R_+\times\Omega = G$$

$$u(t_k^+,x)-u(t_k^-,x) = -\frac{1}{2}u(t_k,x),$$

with the boundary condition

$$u = 0, \quad (t,x)\in R_+\times\partial\Omega,$$

where $a(t) = 1, a_1(t) = 1$, $\mu = \pi/2$; $h(u) = u^2$, $h_1(u) = u^2$; $f(u) = ue^{u^2}$, $q(t,x) = 1$; $b_k = -1/2$; $q(t) = -(1/2)^k$, $t \in [k-1,k)$, $k = 1,\dots$. It is easy to verify that the conditions of Theorem 2.2 are satisfied. Hence the all solutions are oscillating.

Example 2.

$$\frac{\partial}{\partial t}\left(u(t,x)+q(t)u\left(t-\frac{\pi}{2},x\right)\right) = u^2\Delta u + u^2\left(t-\frac{\pi}{2},x\right)\Delta u\left(t-\frac{\pi}{2},x\right) - u\left(t-\frac{3\pi}{2},x\right)e^{[u(t-(3\pi/2),x)]^2}$$

$$t \neq t_k, \quad (t,x)\in R_+\times\Omega = G$$

$$u(t_k^+, x) - u(t_k^-, x) = -\frac{k}{k+1} u(t_k, x),$$

with the boundary condition

$$u = 0, \quad (t, x) \in R_+ \times \partial\Omega,$$

where $a(t) = 1$, $a_1(t) = 1$, $\mu = \pi/2$; $h(u) = u^2$, $h_1(u) = u^2$; $f(u) = ue^{u^2}$, $q(t,x) = 1$; $b_k = -k/(k+1)$; $q(t) = -(1/2)1/((k+1)!)$, $t \in [k, k+1)$, $k = 1, \dots$. It is easy to verify that the conditions of Theorem 2.2 are satisfied. Hence the all solutions are oscillating.

Example 3.

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(u(t,x) + q(t)u\left(t - \frac{3\pi}{2}, x\right)\right) \\
&\quad = u^2 \Delta u + u^2\left(t - \frac{\pi}{2}, x\right) \Delta u\left(t - \frac{\pi}{2}, x\right) \\
&\qquad - u\left(t - \frac{\pi}{2}, x\right) e^{(\cos t \sin x)^2} - u\left(t - \frac{3\pi}{2}, x\right) e^{[u(t-(3\pi/2),x)]^2} \\
&\qquad\qquad t \neq t_k, \quad (t, x) \in R_+ \times \Omega = G
\end{aligned}$$

$$u(t_k^+, x) - u(t_k^-, x) = -\frac{1}{2^k} u(t_k, x),$$

with the boundary condition

$$u = 0, \quad (t, x) \in R_+ \times \partial\Omega,$$

where $a(t) = 1$, $a_1(t) = 1$, $\mu = (3\pi)/2$; $h(u) = u^2$, $h_1(u) = u^2$; $f_1(u) = u$, $q_1(t,x) = e^{(\cos t \sin x)^2}$, $f_2(u) = ue^{u^2}$, $q_2(t,x) = 1$; $b_k = -1/(2^k)$; $q(t) = -(1/2)[(2^k - 1)/(2^k)][(2^{k-1} - 1)/((2^{k-1}))] \cdots 1/2$, $t \in [k, k+1)$, $k = 1, \dots$. It is easy to verify that the conditions of Theorem 2.2 are satisfied. Hence all of the solutions are oscillating.

REFERENCES

1. D.D. Bainov, Z. Kamont and E. Minchev, *Monotone iterative methods for impulsive hyperbolic differential-functional equations*, J. Comput. Appl. Math. **70** (1996), 329–347.

2. D.D. Bainov, V. Lakshmikantham and P.S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.

3. D.D. Bainov and E. Minchev, *Oscillation of the solutions of impulsive parabolic equations*, J. Comput. Appl. Math. **69** (1996), 207–214.

4. ———, *Forced oscillation of solutions of impulsive nonlinear parabolic differential-difference equations*, J. Korean Math. Soc. **35** (1998), 881–890.

5. D.D. Bainov and P.S. Simeonov, *Impulsive differential equations*: *Periodic solutions and applications*, Longman, London, 1993.

6. L. Berezansky and E. Braverman, *Oscillation of a linear delay impulsive differential equations*, Comm. Appl. Nonlinear Anal. **3** (1996), 61–77.

7. L.H. Deng and W.G. Ge, *Oscillation of the solutions of parabolic equations with impulses*, Acta Math. Sinica **44** (2001), 501–506.

8. L. Erbe, H. Freedman, X. Liu and J.H. Wu, *Comparison principles for impulsive parabolic equations with application to models of single species growth*, J. Austral. Soc. **32** (1991), 382–400.

9. I. Gyori and G. Ladas, *Oscillation theory of delay differential equation with applications*, Clarendon Press, Oxford, 1991.

10. M.X. He and S.C. Gao, *Oscillation of hyperbolic functional differential equations with deviating arguments*, Chinese Sci. Bulletin **38** (1993), 10–14.

11. M.X. He and A.P. Liu, *Oscillation of hyperbolic partial functional differential equations*, Appl. Math. Comput. **142** (2003), 205–224.

12. G. Ladas, *Sharp conditions for oscillation caused by delays*, Appl. Anal. **9** (1979), 93–98.

13. G. Ladas, V. Lakshmikantham and B.G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Marcel Dekker, New York, 1987.

14. A.P. Liu, *Oscillations of certain hyperbolic delay differential equations with damping term*, Math. Appl. **9** (1996), 321–324.

15. ———, *Oscillations of the solutions of parabolic partial differential equations of neutral type*, Acta Anal. Funct. Appl. **2** (2000), 376–381.

16. A.P. Liu and S.C. Cao, *Oscillations of the solutions of hyperbolic partial differential equations of neutral type*, Chinese Quart. J. Math. **17** (2002), 7–13.

17. A.P. Liu and M.X. He, *Oscillations of the solutions of nonlinear delay hyperbolic partial differential equations of neutral type*, Appl. Math. Mech. **23** (2002), 678–685.

18. A.P. Liu, Q.X. Ma and M.X. He, *Oscillation of nonlinear impulsive parabolic equations of neutral type*, Rocky Mountain J. Math. **36** (2006), 1011–1026.

19. A.P. Liu, L. Xiao and T. Liu, *Oscillation of nonlinear impulsive hyperbolic equations with delays*, Electr. J. Differential Equations **2004** (2004), 1–6.

20. A.P. Liu, L. Xiao, T. Liu and Y.A. Li, *Forced oscillations of solutions impulsive nonlinear parabolic differential equations with delay*, Dynam. Cont. Discrete Impulsive Systems, Mathematical Analysis **13** (2006), 668–674.

21. A.P. Liu and W.H. Yu, *Necessary and sufficient conditions for oscillations of nonlinear parabolic partial differential equations*, Pure Appl. Math. **18** (2001), 86–89.

22. J.W. Luo, *Oscillation of hyperbolic partial differential equations with impulses*, Appl. Math. Comput. **133** (2002), 309–318.

23. D.P. Mishev and D.D. Bainov, *Oscillation of the solutions of parabolic differential equations of neutral type*, Appl. Math. Comput. **28** (1988), 97–111.

24. J.R. Yan and C.H. Kou, *Oscillation of solutions of delay impulsive differential equations*, J. Math. Anal. Appl. **254** (2001), 358–370.

25. N. Yoshida, *Oscillation of nonlinear parabolic equations with functional arguments*, Hiroshima Math. J. **16** (1986), 305–314.

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