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AN EQUATION RELATED TO TWO-SIDED CENTRALIZERS IN PRIME RINGS

MAJA FOŠNER AND JOSO VUKMAN

ABSTRACT. The purpose of this paper is to prove the following result. Let m and n be positive integers, and let R be a prime ring with $\operatorname{char}(R) = 0$ or $m + n + 1 \le \operatorname{char}(R)$. Let $T : R \to R$ be an additive mapping satisfying the relation $T(x^{m+n+1}) = x^m T(x) x^n$ for all $x \in R$. In this case T is a two-sided centralizer.

This research is a continuation of our work [**6**]. Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n > 1$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. An additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ holds for all pairs $x, y \in R$. For a semiprime ring R all left centralizers are of the form $T(x) = qx$ for all $x \in R$, where q is an element of Martindale right ring of quotients Q_r of R (see [**3**, Chapter 2]). In case R has the identity element $T : R \to R$ is a left centralizer if and only if T is of the form $T(x) = ax$ for all $x \in R$ and some fixed element $a \in R$. The definition of a right centralizer should be self-explanatory. An additive mapping T is called a two-sided centralizer in case T is a left and a right centralizer. In case $T : R \to R$ is a two-sided centralizer, where R is a semiprime ring with extended centroid C, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ (see [**3**, Theorem 2.3.2]). An additive mapping $T : R \to R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$) holds for all $x \in R$. Zalar [**14**] has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Molnár [**8**] has proved that in case we have an additive mapping $T : A \to A$, where A is a semisimple H^*-algebra, satisfying the relation $T(x^3) = T(x)x^2$ ($T(x^3) = x^2T(x)$) for all $x \in A$, then T is a left (right) centralizer. For

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results concerning centralizers on rings and algebras we refer the reader to [**7, 12, 13**].

Motivated by the work of Brešar [**4**], Vukman [**11**] has proved the following result.

Theorem 1. *Let R be a 2-torsion free semiprime ring, and let $T : R \to R$ be an additive mapping satisfying the relation*

$$T(xyx) = xT(y)x \tag{1}$$

for all pairs $x, y \in R$. In this case T is a two-sided centralizer.

Since any two-sided centralizer on an arbitrary ring satisfies the relation (1), the result we have just mentioned above characterizes two-sided centralizers among all additive mappings on 2-torsion free semiprime rings.

Theorem 1 was the motivation for the result below proved by Vukman and Fošner [**6**].

Theorem 2. *Let R be a prime ring of characteristic different from two, and let $T : R \to R$ be an additive mapping satisfying the relation*

$$T(x^3) = xT(x)x \tag{2}$$

for all $x \in R$. Then T is a two-sided centralizer.

It is our aim in this paper to prove the following generalization of Theorem 2.

Theorem 3. *Let m and n be positive integers, and let R be a prime ring with $\operatorname{char}(R) = 0$ or $m + n + 1 \leq \operatorname{char}(R)$. Let $T : R \to R$ be an additive mapping satisfying the relation*

$$T(x^{m+n+1}) = x^m T(x) x^n \tag{3}$$

for all $x \in R$. Then T is a two-sided centralizer.

For the proof of Theorem 3 we need Theorem 4 below, which is of independent interest. Our result is obtained as an application of the theory of functional identities.

The theory of functional identities considers set-theoretic mappings on rings satisfying some identical relation. When treating such relations one usually concludes that the form of the mappings involved can be described, unless the ring is very special. We refer the reader to [**5**] for full treatment on the theory of functional identities and its applications.

Let R be a ring, and let X be a subset of R. By $C(X)$ we denote the set $\{r \in R \mid [r, X] = 0\}$. Let $m \in \mathbf{N}$, and let $E : X^{m-1} \to R$, $p : X^{m-2} \to R$ be arbitrary mappings. In the case when $m = 1$ this should be understood as that E is an element in R and $p = 0$. Let $1 \le i < j \le m$, and define E^i, p^{ij}, $p^{ji} : X^m \to R$ by

$$E^i(\overline{x}_m) = E(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m),$$
$$p^{ij}(\overline{x}_m) = p^{ji}(\overline{x}_m) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_m),$$

where $\overline{x}_m = (x_1, \dots, x_m) \in X^m$.

Let $I, J \subseteq \{1, \dots, m\}$, and for each $i \in I$, $j \in J$, let $E_i, F_j : X^{m-1} \to R$ be arbitrary mappings. Consider functional identities

$$\sum_{i \in I} E_i^i(\overline{x}_m) x_i + \sum_{j \in J} x_j F_j^j(\overline{x}_m) = 0, \quad (\overline{x}_m \in X^m) \tag{4}$$

$$\sum_{i \in I} E_i^i(\overline{x}_m) x_i + \sum_{j \in J} x_j F_j^j(\overline{x}_m) \in C(X) \quad (\overline{x}_m \in X^m). \tag{5}$$

A natural possibility when (4) and (5) are fulfilled is when there exist mappings $p_{ij} : X^{m-2} \to R$, $i \in I$, $j \in J$, $i \neq j$ and $\lambda_k : X^{m-1} \to C(X)$, $k \in I \cup J$, such that

$$(6)\qquad \begin{aligned} E_i^i(\overline{x}_m) &= \sum_{\substack{j \in J \\ j \neq i}} x_j p_{ij}^{ij}(\overline{x}_m) + \lambda_i^i(\overline{x}_m), \\ F_j^j(\overline{x}_m) &= -\sum_{i \in I, j \neq i} p_{ij}^{ij}(\overline{x}_m) x_i - \lambda_j^j(\overline{x}_m), \\ \lambda_k &= 0 \quad \text{if } k \notin I \cap J \end{aligned}$$

for all $\overline{x}_m \in X^m$, $i \in I$, $j \in J$. We shall say that every solution of the form (6) is a standard solution of (4) and (5).

The case when one of the sets I or J is empty is not excluded. The sum over the empty set of indexes should be simply read as zero. So, when $J = 0$, respectively $I = 0$, (4) and (5) reduce to

$$\sum_{i\in I} E_i^i(\overline{x}_m)x_i = 0 \qquad (7)$$

$$\left(\text{resp. } \sum_{j\in J} x_j F_j^j(\overline{x}_m) = 0\right) \quad (\overline{x}_m \in X^m),$$

$$\sum_{i\in I} E_i^i(\overline{x}_m)x_i \in C(X) \qquad (8)$$

$$\left(\text{resp. } \sum_{j\in J} x_j F_j^j(\overline{x}_m) \in C(X)\right) \quad (\overline{x}_m \in X^m).$$

In that case the (only) standard solution is

$$E_i = 0, \quad i \in I (\text{resp. } F_j = 0, \quad j \in J). \qquad (9)$$

A d-freeness of X will play an important role in this paper. For a definition of d-freeness we refer the reader to [**5**]. Under some natural assumptions one can establish that various subsets (such as ideals, Lie ideals, the sets of symmetric or skew symmetric elements in a ring with involution) of certain types of rings are d-free. We refer the reader to [**5**] for results of this kind. Let us mention that a prime ring R is a d-free subset of its maximal right ring of quotients, unless R satisfies the standard polynomial identity of degree less than $2d$.

Let R be a ring, and let

$$p(x_1, x_2, \dots, x_{m+n+1}) = \sum_{\pi\in S_{m+n+1}} x_{\pi(1)}x_{\pi(2)}\cdot\dots\cdot x_{\pi(m+n+1)}$$

be a fixed multilinear polynomial in noncommutative indeterminates $x_1, x_2, \dots, x_{m+n+1}$. Further, let L be a subset of R closed under p, i.e., $p(\overline{x}_{m+n+1}) \in L$ for all $x_1, x_2, \dots, x_{m+n+1} \in L$, where $\overline{x}_{m+n+1} =$

$(x_1, x_2, \dots, x_{m+n+1})$. We shall consider a mapping $T : L \to R$ satisfying

$$(10) \quad T(p(\overline{x}_{m+n+1})) = \sum_{\pi \in S_{m+n+1}} x_{\pi(1)} \cdots x_{\pi(m)} T(x_{\pi(m+1)}) x_{\pi(m+2)} \cdots x_{\pi(m+n+1)}$$

for all $x_1, x_2, \dots, x_{m+n+1} \in L$. Of course, every two-sided centralizer satisfies (10). Our goal is to show that under certain assumptions these are in fact the only mappings with this property. In the first step of the proof we derive a functional identity from (10). Let us mention that the idea of considering the expression $[p(\overline{x}_n), p(\overline{y}_n)]$ in its proof is taken from [**2**].

Theorem 4. *Let L be a $2(m+n+1)$-free Lie subring of R closed under p. If $T : L \to R$ is an additive mapping satisfying* (10)*, then there exist $p \in C(L)$ and $\lambda : L \to C(L)$ such that $T(x) = px + \lambda(x)$ for all $x \in L$.*

Proof. Let us write $k = m + n + 1$ for brevity. Note that for any $a \in R$ and $\overline{x}_k \in L^k$ we have

$$[p(\overline{x}_k), a] = \sum_{i=1}^{k} p(x_1, \dots, x_{i-1}, [x_i, a], x_{i+1}, \dots, x_k).$$

Using (10) it follows that

$$\begin{aligned} T[p(\overline{x}_k), a] &= \sum_{\pi \in S_k} [x_{\pi(1)}, a] x_{\pi(2)} \cdots x_{\pi(m)} T(x_{\pi(m+1)}) x_{\pi(m+2)} \cdots x_{\pi(k)} \\ &\quad + \sum_{\pi \in S_k} x_{\pi(1)} [x_{\pi(2)}, a] x_{\pi(3)} \cdots x_{\pi(m)} T(x_{\pi(m+1)}) x_{\pi(m+2)} \cdots x_{\pi(k)} \\ &\quad + \cdots + \sum_{\pi \in S_k} x_{\pi(1)} \cdots x_{\pi(m)} T[x_{\pi(m+1)}, a] x_{\pi(m+2)} \dots x_{\pi(k)} \\ &\quad + \cdots + \sum_{\pi \in S_k} x_{\pi(1)} \cdots x_{\pi(m)} T(x_{\pi(m+1)}) x_{\pi(m+2)} \cdots [x_{\pi(k)}, a]. \end{aligned}$$

Hence

(11) $T[p(\overline{x}_k), a]$

$$= \sum_{\pi \in S_k} [x_{\pi(1)} \cdots x_{\pi(m)}, a] T(x_{\pi(m+1)}) x_{\pi(m+2)} \cdots x_{\pi(k)}$$
$$+ \sum_{\pi \in S_k} x_{\pi(1)} \cdots x_{\pi(m)} T[x_{\pi(m+1)}, a] x_{\pi(m+2)} \cdots x_{\pi(k)}$$
$$+ \sum_{\pi \in S_k} x_{\pi(1)} \cdots x_{\pi(m)} T(x_{\pi(m+1)}) [x_{\pi(m+2)} \cdots x_{\pi(k)}, a]$$

In particular,

$$T[p(\overline{x}_k), p(\overline{y}_k)]$$
$$= \sum_{\pi \in S_k} [x_{\pi(1)} \cdots x_{\pi(m)}, p(\overline{y}_k)] T(x_{\pi(m+1)}) x_{\pi(m+2)} \cdots x_{\pi(k)}$$
$$+ \sum_{\pi \in S_k} x_{\pi(1)} \cdots x_{\pi(m)} T[x_{\pi(m+1)}, p(\overline{y}_k)] x_{\pi(m+2)} \cdots x_{\pi(k)}$$
$$+ \sum_{\pi \in S_k} x_{\pi(1)} \cdots x_{\pi(m)} T(x_{\pi(m+1)}) [x_{\pi(m+2)} \cdots x_{\pi(k)}, p(\overline{y}_k)]$$

for all $\overline{x}_k, \overline{y}_k \in L^k$. We also have (by (11))

(12)

$$\varphi(x_{\pi(m+1)}) = T[x_{\pi(m+1)}, p(\overline{y}_k)] = -T[p(\overline{y}_k), x_{\pi(m+1)}]$$
$$+ \sum_{\sigma \in S_k} [x_{\pi(m+1)}, y_{\sigma(1)} \ldots y_{\sigma(m)}] T(y_{\sigma(m+1)}) y_{\sigma(m+2)} \cdots y_{\sigma(k)}$$
$$+ \sum_{\sigma \in S_k} y_{\sigma(1)} \cdots y_{\sigma(m)} T[x_{\pi(m+1)}, y_{\sigma(m+1)}] y_{\sigma(m+2)} \ldots y_{\sigma(k)}$$
$$+ \sum_{\sigma \in S_k} y_{\sigma(1)} \cdots y_{\sigma(m)} T(y_{\sigma(m+1)}) [x_{\pi(m+1)}, y_{\sigma(m+2)} \cdots y_{\sigma(k)}]$$

for all $\overline{y}_n \in L^n$. Therefore (12) can be written as

$$T[p(\overline{x}_k), p(\overline{y}_k)]$$
$$= \sum_{\pi \in S_k} [x_{\pi(1)} \cdots x_{\pi(m)}, p(\overline{y}_k)] T(x_{\pi(m+1)}) x_{\pi(m+2)} \cdots x_{\pi(k)}$$
$$+ \sum_{\pi \in S_k} x_{\pi(1)} \cdots x_{\pi(m)} \varphi(x_{\pi(m+1)}) x_{\pi(m+2)} \ldots x_{\pi(k)}$$
$$+ \sum_{\pi \in S_k} x_{\pi(1)} \cdots x_{\pi(m)} T(x_{\pi(m+1)}) [x_{\pi(m+2)} \cdots x_{\pi(k)}, p(\overline{y}_k)]$$

for all $\overline{x}_k, \overline{y}_k \in L^k$. On the other hand, using $[p(\overline{x}_k), p(\overline{y}_k)] = -[p(\overline{y}_k), p(\overline{x}_k)]$, we get from the above identity

$$\begin{aligned} T[p(\overline{x}_k), p(\overline{y}_k)] &\\ = \sum_{\sigma\in S_k} [p(\overline{x}_k), y_{\sigma(1)}\cdots y_{\sigma(m)}]T(y_{\sigma(m+1)})y_{\sigma(m+2)}\cdots y_{\sigma(k)} &\\ + \sum_{\sigma\in S_k} y_{\sigma(1)}\cdots y_{\sigma(m)}\varphi'(y_{\sigma(m+1)})y_{\sigma(m+2)}\cdots y_{\sigma(k)} &\\ + \sum_{\sigma\in S_k} y_{\sigma(1)}\cdots y_{\sigma(m)}T(y_{\sigma(m+1)})[p(\overline{x}_k), y_{\sigma(m+2)}\cdots y_{\sigma(k)}] & \end{aligned}$$

for all $\overline{x}_k, \overline{y}_k \in L^k$, where

$$\begin{aligned} \varphi'(y_{\pi(m+1)}) &\\ = \sum_{\pi\in S_k} [x_{\pi(1)}\cdots x_{\pi(m)}, y_{\sigma(m+1)}]T(x_{\pi(m+1)})x_{\pi(m+2)}\cdots x_{\pi(k)} &\\ + \sum_{\pi\in S_k} x_{\pi(1)}\cdots x_{\pi(m)}T[x_{\pi(m+1)}, y_{\sigma(m+1)}]x_{\pi(m+2)}\cdots x_{\pi(k)} &\\ + \sum_{\pi\in S_k} x_{\pi(1)}\cdots x_{\pi(m)}T(x_{\pi(m+1)})[x_{\pi(m+2)}\cdots x_{\pi(k)}, y_{\sigma(m+1)}]. & \end{aligned}$$

Let $s : \mathbf{Z} \to \mathbf{Z}$ be a mapping defined by $s(i) = i - k$. For each $\sigma \in S_k$ the mapping $s^{-1}\sigma s : \{k+1, \ldots, 2k\} \to \{k+1, \ldots, 2k\}$ will be denoted by $\overline{\sigma}$. After comparing above identities and writing x_{k+i} instead of y_i, $i = 1, \ldots, k$, we arrive at

$$\begin{aligned} 0 = \sum_{\pi\in S_k}\sum_{\sigma\in S_k} \Big(& [x_{\pi(1)}\ldots x_{\pi(k)}, x_{\overline{\sigma}(k+1)}\cdots x_{\overline{\sigma}(k+m)}]T(x_{\overline{\sigma}(k+m+1)}) \\ & + x_{\overline{\sigma}(k+1)}\cdots x_{\overline{\sigma}(k+m)}\varphi'(x_{\overline{\sigma}(k+m+1)}) \\ & + x_{\overline{\sigma}(k+1)}\cdots T(x_{\overline{\sigma}(k+m+1)})x_{\pi(1)}\cdots x_{\pi(k)} \\ & - x_{\pi(1)}\cdots T(x_{\pi(m+1)})x_{\pi(m+2)}\cdots x_{\overline{\sigma}(k+m+1)}\Big)x_{\overline{\sigma}(k+m+2)}\cdots x_{\overline{\sigma}(2k)} \\ - \sum_{\pi\in S_k}\sum_{\sigma\in S_k} \Big(& [x_{\pi(1)}\cdots x_{\pi(m)}, x_{\overline{\sigma}(k+1)}\cdots x_{\overline{\sigma}(2k)}]T(x_{\pi(m+1)}) \\ & + x_{\pi(1)}\ldots x_{\pi(m)}\varphi(x_{\pi(m+1)}) \\ & + x_{\overline{\sigma}(k+1)}\cdots x_{\overline{\sigma}(k+m)}T(x_{\overline{\sigma}(k+m+1)})x_{\overline{\sigma}(k+m+2)}\cdots x_{\pi(m+1)} \\ & - x_{\pi(1)}\cdots x_{\pi(m)}T(x_{\pi(m+1)})x_{\overline{\sigma}(k+1)}\cdots x_{\overline{\sigma}(2k)}\Big)x_{\pi(m+2)}\cdots x_{\pi(k)}. \end{aligned}$$

Since L is $2k$-free, it follows that the so-obtained functional identity has only a standard solution. In particular,

$$0 = \sum_{\pi\in S_k}\sum_{\sigma\in S_{m+1}} \Big([x_{\pi(1)}\cdots x_{\pi(k)}, x_{\overline{\sigma}(k+1)}\cdots x_{\overline{\sigma}(k+m)}]T(x_{\overline{\sigma}(k+m+1)}) \\ + x_{\overline{\sigma}(k+1)}\cdots x_{\overline{\sigma}(k+m)}\varphi(x_{\overline{\sigma}(k+m+1)}) \\ + x_{\overline{\sigma}(k+1)}\cdots x_{\overline{\sigma}(k+m)}T(x_{\overline{\sigma}(k+m+1)})x_{\pi(1)}\ldots x_{\pi(k)} \\ - x_{\pi(1)}\ldots x_{\pi(m)}T(x_{\pi(m+1)})x_{\pi(m+2)}\cdots x_{\pi(k)}x_{\overline{\sigma}(k+1)}\cdots x_{\overline{\sigma}(k+m+1)}\Big).$$

Note that this is also a functional identity. It follows that

$$\sum_{\pi\in S_k}\sum_{\sigma\in S_{m+1}} x_{\pi(m+1)}\cdots x_{\overline{\sigma}(k+m)}T(x_{\pi(k+m+1)}) \\ - T(x_{\pi(m+1)})x_{\pi(m+2)}\ldots x_{\overline{\sigma}(k+m+1)} = 0, \tag{13}$$

where $\pi(i) = i$ for all $i = 1, \ldots, m$. After some steps we arrive at

$$T(x) = xp + \lambda(x) \tag{14}$$

for all $x \in L$, where $p \in R$ and $\lambda : R \to C(L)$. Similarly, by (13) we can prove

$$T(x) = qx + \mu(x) \tag{15}$$

for all $x \in L$. Comparing (14) and (15) we arrive at

$$xp - qx \in C(L)$$

for all $x \in L$. It follows that $p = q \in C(L)$ and $\lambda = \mu$. Thereby the proof is completed. □

We are now in a position to prove Theorem 3.

Proof. The complete linearization of (3) gives us (10).

First suppose that R is not a PI ring (satisfying the standard polynomial identity of degree less than $4k = 4(m + n + 1)$). According to Theorem 4 there exist $p \in C$ and $\lambda : R \to C$ such that

$$T(x) = px + \lambda(x)$$

for all $x \in R$. Using this with (10) we see that

$$\lambda(p(\overline{x}_k)) = \sum_{\pi \in S_k} x_{\pi(1)} \cdots x_{\pi(m)} \lambda(x_{\pi(m+1)}) x_{\pi(m+2)} \cdots x_{\pi(k)}$$

for all $x_1, x_2, \ldots, x_k \in R$. Since R is not a PI ring it follows that $\lambda(R) = 0$ and so $\lambda = 0$. Thus T is a two-sided centralizer, as desired.

Assume now that R is a PI ring. It is well-known that in this case R has a nonzero center (see [**10**]). Let c be a nonzero central element. Pick any $x \in R$, and set $x_1 = x_2 = \cdots = x_{k-1} = cx$ and $x_k = x$ in (10). We arrive at

$$\begin{aligned} T(k!c^{k-1}x^k) = (n+m)!c^{m+n-1}(&x^m T(cx) x^n n \\ &+ cx^m T(x) x^n + x^m T(cx) x^n m). \end{aligned}$$

On the other hand, setting $x_1 = x_2 = \cdots = x_{k-1} = c$ and $x_k = x^k$ in (10) we obtain

$$T(k!c^{k-1}x^k) = (n+m)!c^{m+n-1}(cT(x^k) + x^k T(c) m + T(c) x^k n).$$

Comparing the so-obtained relations we get

$$0 = (n+m)x^m T(cx) x^n - nT(c)x^k - mx^k T(c). \tag{16}$$

In the case when $x = c$ we have

$$T(c^2) = cT(c). \tag{17}$$

After the complete linearization of (16) and putting $x_1 = x$ and $x_2 = \cdots = x_k = c$ in the so obtained identity we get

$$(n+m)!c^{k-1}(-mxT(c) - nT(c)x + (n+m)T(cx)) = 0.$$

This yields

$$mxT(c) + nT(c)x = (n+m)T(cx) \tag{18}$$

for all $x \in R$. Now setting $x_1 = x$ and $x_2 = \cdots = x_k = c$ in (10) we obtain

$$T(c^{k-1}x) = c^{k-1}T(x)$$

for all $x \in R$. Using (18) we get

$$nT(c)x + mxT(c) = (n+m)cT(x) \tag{19}$$

for all $x \in R$. Thus

$$cT(x) = T(cx) \tag{20}$$

for all $x \in R$. Using (19) and (20) we arrive at

$$\begin{aligned}(n+m)T(xy)c &= n(T(c)x)y + mxyT(c) \\ &= -mxT(c)y + (n+m)T(x)cy + mxyT(c)\end{aligned} \tag{21}$$

for all $x, y \in R$. Multiplying (21) on the left by $z \in R$ we get

$$(n+m)zT(xy)c = -mzxT(c)y + (n+m)zT(x)cy + mzxyT(c). \tag{22}$$

But, on the other hand, putting zx instead of x in (21), we obtain

$$(n+m)T(zxy)c = -mzxT(c)y + (n+m)T(zx)cy + mzxyT(c). \tag{23}$$

Comparing (22) and (23) we get

$$c(T(zxy) - zT(xy)) = c(T(zx)y - zT(x)y)$$

for all $x, y, z \in R$. Since R is prime it follows that

$$T(zxy) = T(zx)y - zT(x)y + zT(xy) \tag{24}$$

for all $x, y, z \in R$. In particular,

$$T(x^{m+n+1}) = T(x^{m+1})x^n - x^mT(x)x^n + x^mT(x^{n+1})$$

for all $x \in R$. Thus

$$\begin{aligned}2(n+m)&x^mT(x)x^nc \\ &= (n+m)T(x^{m+1})x^nc + (n+m)x^mT(x^{n+1})c \\ &= mx^{m+1}T(c)x^n + nT(c)x^{m+n+1} \\ &\quad + mx^{m+n+1}T(c) + nx^mT(c)x^{n+1}\end{aligned}$$

by (19). On the other hand, we have

$$2(n+m)x^mT(x)x^nc = 2mx^{m+1}T(c)x^n + 2nx^mT(c)x^{n+1}.$$

Comparing the so-obtained identities we arrive at

$$nT(c)x^{m+n+1} + mx^{m+n+1}T(c) = mx^{m+1}T(c)x^n + nx^mT(c)x^{n+1},$$

for all $x \in R$. A complete linearization of this identity gives us (putting $x_1 = x_2 = x$, $x_3 = \cdots = x_{m+n+1} = c$ in the obtained relation)

$$[[T(c), x], x] = 0$$

for all $x \in R$. Using Posner's theorem [**9**] it follows that $[T(c), x] = 0$ for all $x \in R$. From (19) we get $T(x)c = T(c)x = xT(c)$ for all $x \in R$. Consequently, by (24) we get

$$T(zcy) = T(zc)y - zT(c)y + zT(cy),$$

which implies $T(zy) = T(z)y = zT(y)$ for all $z, y \in R$. Thus T is a two-sided centralizer. Thereby the proof is completed. □

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Faculty of logistics, University of Maribor, Mariborska cesta 7, 3000 Celje, Slovenia
Email address: maja.fosner@uni-mb.si

Department of Mathematics and Computer Science, Faculty of Sciences and Mathematics, University of Maribor, Koroška cesta 160, SI-2000 Maribor, Slovenia
Email address: joso.vukman@uni-mb.si