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SURJECTIVE LINEAR MAPS PRESERVING CERTAIN SPECTRAL RADII

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ABSTRACT. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space $\mathcal{H}$. In this paper, we prove that a surjective linear map ϕ from $\mathcal{L}(\mathcal{H})$ into itself preserves the spectral radius $r_1(.)$ if and only if ϕ is an automorphism multiplied by a unimodular scalar. We also consider the case when $\mathcal{H}$ is a finite dimensional Hilbert space and prove that a linear map ϕ from $M_n(\mathbf{C})$ into itself preserves the spectral radius $r_1(.)$ if and only if ϕ is either an automorphism or an anti-automorphism multiplied by a unimodular scalar. Finally, we use this result to show that a linear map ϕ from $M_n(\mathbf{C})$ into itself preserves the inner local spectral radius at nonzero fixed vector $x_0 \in \mathbf{C}^n$ if and only if there exist a unimodular scalar $\alpha \in \mathbf{C}$ and an invertible matrix $A \in M_n(\mathbf{C})$ such that $A(x_0) = x_0$ and $\phi(T) = \alpha ATA^{-1}$ for all $T \in M_n(\mathbf{C})$.

1. Introduction. Let X be a complex Banach space, and let $\mathcal{L}(X)$ denote the algebra of all bounded linear operators on X. For an operator $T \in \mathcal{L}(X)$, we denote the spectrum by $\sigma(T)$ and the approximate point spectrum by $\sigma_{ap}(T)$. We also denote as usual the spectral radius of T by $r(T) := \max\{|\lambda| : \lambda \in \sigma(T)\}$ which coincides, by Gelfand's formula for the spectral radius, with the limit of the convergent sequence $(\|T^n\|^{1/n})_n$. The minimum modulus of T is $m(T) := \inf\{\|Tx\| : \|x\| = 1\}$, and is positive precisely when T is injective and has a closed range. Note that the sequence $(m(T^n)^{1/n})_n$ converges and its limit, denoted by $r_1(T)$, coincides with its supremum. In [**12**], Makai and Zemánek proved, in fact, that $r_1(T)$ is nothing but the minimum modulus of $\sigma_{ap}(T)$.

The local resolvent of an operator $T \in \mathcal{L}(X)$ at a point $x \in X$, $\rho_T(x)$, is the set of all $\lambda \in \mathbf{C}$ for which there exists an open neighborhood U_λ

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of λ in $\mathbf{C}$ and an analytic function $f : U_\lambda \to X$ such that the equation $(T-\mu)f(\mu) = x$ holds for all $\mu \in U_\lambda$. The local spectrum of T at x, denoted by $\sigma_T(x)$, is given by

$$\sigma_T(x) := \mathbf{C} \backslash \rho_T(x),$$

and is a compact subset of $\sigma(T)$. While the (outer) local spectral radius of T at x is defined by

$$r_T(x) := \limsup_{n \to +\infty} \|T^n x\|^{1/n},$$

and coincides with the maximum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. Recall that T is said to have the single-valued extension property if for every open set U of $\mathbf{C}$, the equation

$$(T-\lambda)\phi(\lambda) = 0, \quad (\lambda \in U),$$

has no nontrivial analytic solution on U.

In [**6**], Bourhim and Miller characterized linear maps from the algebra of all $n \times n$ complex matrices, $M_n(\mathbf{C})$, into itself that preserve the local spectral radius at a fixed nonzero point. They showed that if $x_0 \in \mathbf{C}^n$ is a fixed nonzero vector, then a linear map $\phi : M_n(\mathbf{C}) \to M_n(\mathbf{C})$ satisfies $r_{\phi(T)}(x_0) = r_T(x_0)$ for all $T \in M_n(\mathbf{C})$ if and only if there exist a scalar $\alpha \in \mathbf{C}$ of modulus one and an invertible matrix $A \in M_n(\mathbf{C})$ such that $A(x_0) = x_0$ and $\phi(T) = \alpha A T A^{-1}$ for all $T \in M_n(\mathbf{C})$. The proof of this result uses an important theorem due to Brešar and Šemrl which characterizes linear maps from $\mathcal{L}(X)$ onto itself that preserve the spectral radius; see [**8**].

In this paper, we first characterize in Section 2 surjective linear maps preserving the spectral radius $r_1(.)$. We then use the obtained result to describe linear maps on $M_n(\mathbf{C})$ preserving the inner local spectral radius at a fixed nonzero point. For related results concerning additive or linear maps preserving the local spectra, we refer the interested reader to [**7, 10**].

2. Linear maps preserving the spectral radius $r_1(.)$. We state and prove the main result of this section. Its proof uses some ideas and arguments from [**5, 6**].

Theorem 2.1. *Assume that $\mathcal{H}$ is an infinite-dimensional complex Hilbert space. A surjective linear map ϕ from $\mathcal{L}(\mathcal{H})$ into itself preserves the spectral radius $r_1(.)$, i.e.,*

$$r_1(\phi(T)) = r_1(T), \quad (T \in \mathcal{L}(\mathcal{H})),$$

if and only if there exist a scalar $c \in \mathbf{C}$ of modulus one and an invertible operator $A \in \mathcal{L}(\mathcal{H})$ such $\phi(T) = cATA^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$.

Proof. The "only if" part is easily verified. So, assume that ϕ preserves the spectral radius $r_1(.)$, and let us see that ϕ is an automorphism of $\mathcal{L}(\mathcal{H})$ up to a multiplicative constant of modulus one. For every operator $T \in \mathcal{L}(\mathcal{H})$, we have $m(T) \leq r_1(T)$, and $m(T^n) \leq \|T^{n-1}\| m(T)$ for all $n \geq 1$. We thus note that $m(T) = 0$ if and only if $r_1(T) = 0$, and the map ϕ strongly preserves operators with minimum modulus zero. Equivalently, ϕ preserves the left invertibility in both directions.

Now, we shall show that $\phi(I)$ is invertible. This fact holds at once by applying [**4**, Theorem 2.2]. However, for the sake of completeness, we shall prove it here. Since $r_1(\phi(I)) = r_1(I) = 1$, we note that $m(\phi(I)) > 0$ and thus there is an operator $R \in \mathcal{L}(\mathcal{H})$ such that $R\phi(I) = I$. Set

$$\varphi(T) := R\phi(T), \quad (T \in \mathcal{L}(\mathcal{H})).$$

The map φ is unital and satisfies $m(\varphi(T)) \leq \|R\| m(\phi(T))$ for all $T \in \mathcal{L}(\mathcal{H})$. From this, we infer that $\sigma_{ap}(T) \subset \sigma_{ap}(\varphi(T))$ for all $T \in \mathcal{L}(\mathcal{H})$ and that

$$r(T) \leq r(\varphi(T))$$

for all $T \in \mathcal{L}(\mathcal{H})$. By [**6**, Lemma 2.6], the map φ is injective. As ϕ is surjective, there exists an operator $R_0 \in \mathcal{L}(\mathcal{H})$ such that $\phi(R_0) = I - \phi(I)R$. We have $\varphi(R_0) = R\phi(R_0) = R(I - \phi(I)R) = 0$, and $R_0 = 0$ by the injectivity of φ. This implies that $0 = \phi(R_0) = I - \phi(I)R$, and $\phi(I)$ is invertible.

Keep in mind that the map ϕ strongly preserves operators with minimum modulus zero and note that, since φ is a unital linear map for which $\|R^{-1}\|^{-1} m(\phi(T)) \leq m(\varphi(T)) \leq \|R\| m(\phi(T))$ for all $T \in \mathcal{L}(\mathcal{H})$, we have

$$\begin{aligned}\lambda \in \sigma_{ap}(\varphi(T)) &\iff m(\varphi(T) - \lambda I) = 0\\ &\iff m(\varphi(T - \lambda I)) = 0\\ &\iff m(\phi(T - \lambda I)) = 0\\ &\iff m(T - \lambda I) = 0\\ &\iff \lambda \in \sigma_{ap}(T)\end{aligned}$$

for all $T \in \mathcal{L}(\mathcal{H})$. In other words, we have $\sigma_{ap}(\varphi(T)) = \sigma_{ap}(T)$ for all $T \in \mathcal{L}(\mathcal{H})$. By [**9**, Corollary 3.1], there exists an invertible operator $A \in \mathcal{L}(\mathcal{H})$ such that $\varphi(T) = ATA^{-1}$for all $T \in \mathcal{L}(\mathcal{H})$.

It remains to show that $R = cI$ for certain $c \in \mathbf{C}$ of modulus one. To do so, it suffices to show that x and Rx are linearly dependent for all $x \in \mathcal{H}$. We first note that

$$r_1(R\phi(T)) = r_1(\varphi(T)) = r_1(T) = r_1(\phi(T))$$

for all $T \in \mathcal{L}(\mathcal{H})$. Since ϕ is surjective, we in fact have

$$r_1(T) = r_1(RT) \tag{2.1}$$

for all $T \in \mathcal{L}(\mathcal{H})$. Next, suppose by the way of contradiction that there is $x_0 \in \mathcal{H}$ such that x_0 and Rx_0 are linearly independent, and let W be the orthogonal complement of the linear subspace spanned by $\{x_0, Rx_0\}$ in $\mathcal{H}$. Fix a nonzero complex number α for which $|\alpha| < 1$, and define linearly the operator $Q \in \mathcal{L}(\mathcal{H})$ by

$$Qx := \begin{cases} x & \text{if } x \in W\\ \alpha^{-1}Rx_0 & \text{if } x = x_0\\ \alpha x_0 & \text{if } x = Rx_0.\end{cases}$$

It easy to check that $r_1(Q) = 1$ and that $RQ(Rx_0) = \alpha Rx_0$. These show that

$$r_1(RQ) \leq |\alpha| < 1 = r_1(Q),$$

and lead to a contradiction; see (2.1). Thus there exists $c \in \mathbf{C}$ such that $R = cI$, and $1 = r_1(I) = r_1(R) = |c|$. The proof is therefore complete. □

The following result shows that, unlike in the infinite dimensional case, even anti-automorphisms on $M_n(\mathbf{C})$ preserve the spectral radius $r_1(.)$.

Theorem 2.2. *A linear map ϕ from $M_n(\mathbf{C})$ into itself preserves the spectral radius $r_1(.)$ if and only if there exist a scalar $c \in \mathbf{C}$ of modulus one and an invertible matrix $A \in M_n(\mathbf{C})$ such that either $\phi(T) = cATA^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$ or $\phi(T) = cAT^{\mathrm{tr}}A^{-1}$ for all $T \in M_n(\mathbf{C})$. Here T^{tr} denotes the transpose of $T \in M_n(\mathbf{C})$.*

Proof. Note that, since $r_1(T) = \min\{|\lambda| : \lambda \in \sigma(T)\}$ for all $T \in M_n(\mathbf{C})$, we only need to prove the necessity. Assume that ϕ preserves the spectral radius $r_1(.)$, and note that ϕ preserves the invertibility in both directions. By either [**2**] or [**13**], there are invertible matrices A and B in $M_n(\mathbf{C})$ such that either $\phi(T) = ATB$ for all $T \in M_n(\mathbf{C})$ or $\phi(T) = AT^{\mathrm{tr}}B$ for all $T \in M_n(\mathbf{C})$.

To complete the proof, it suffices to show that $\phi(I) = AB = cI$ for some $c \in \mathbf{C}$ of modulus one. We note that

$$r_1(\phi(I)^{-1}\phi(T)) = r_1(T) = r_1(\phi(T))$$

for all $T \in M_n(\mathbf{C})$. Since ϕ is bijective, we in fact have

$$r_1(\phi(I)^{-1}T) = r_1(T) \tag{2.2}$$

for all $T \in M_n(\mathbf{C})$. Now, just as we did in the proof of the preceding theorem, we can prove using this identity, (2.2), that $\phi(I) = AB = cI$ for some scalar c of modulus one. □

3. Linear maps preserving the inner local spectral radius. We first review some notation and definitions from local spectral theory that we will need in the sequel. Our references are the papers of V.G. Miller, T.L. Miller and Neumann [**14**] and the remarkable books of Aiena [**1**] and of Laursen and Neumann [**11**].

For a closed subset $\mathfrak{S}$ of $\mathbf{C}$ and an operator $T \in \mathcal{L}(X)$, the corresponding global spectral subspace of T, denoted by $\mathcal{X}_T(\mathfrak{S})$, is the set of all $x \in X$ for which there is an analytic function $f : \mathbf{C} \setminus \mathfrak{S} \to X$ such that $(\lambda - T)f(\lambda) = x$ for all $\lambda \in \mathbf{C} \setminus \mathfrak{S}$. It is a hyperinvariant subspace of T generally non closed. Note that the outer local spectral radius of T at a point $x \in X$ is nothing but the radius of convergence of the series $F(\lambda) := -\sum_{n \geq 0} T^n x/\lambda^{n+1}$ which satisfies $(T - \lambda)F(\lambda) = x$ for all $|\lambda| > r_T(x)$. Thus

$$r_T(x) = \inf\{\delta \geq 0 : x \in \mathcal{X}_T(\nabla(0,\delta))\},$$

where $\nabla(0,\delta)$ is the closed disc of radius δ centered at the origin. The counterpart of the outer local spectral radius of T at x is the so-called inner local spectral radius of T at x which played a decisive role in [**14**]. It is defined by

$$\iota_T(x) := \sup\{\delta \geq 0 : x \in \mathcal{X}_T(\mathbf{C} \setminus V(0,\delta))\},$$

where $V(0,\delta)$ denotes the open disc of radius δ centered at 0. When T is injective and x is in the hyperrange, $T^\infty X := \cap_{n\geq 0} T^n X$, of T, then

$$\iota_T(x) = \liminf_{n\to+\infty} \|T^{-n}x\|^{-1/n},$$

which is the radius of convergence of the series $F(\lambda) = \sum_{n\geq 0} \lambda^n T^{-n-1}x$ that satisfies $(T-\lambda)F(\lambda) = x$ for all $|\lambda| < \iota_T(x)$. We refer the reader to [**14**] for more details.

We are now in a position to state the main result of this section which characterizes linear maps from $M_n(\mathbf{C})$ into itself that preserve the inner local spectral radius.

Theorem 3.1. *Let n be a positive integer, and let $x_0 \in \mathbf{C}^n$ be a fixed nonzero vector. A linear map ϕ from $M_n(\mathbf{C})$ into itself satisfies $\iota_{\phi(T)}(x_0) = \iota_T(x_0)$ for all $T \in M_n(\mathbf{C})$ if and only if there are a scalar $\alpha \in \mathbf{C}$ of modulus one and an invertible matrix $A \in M_n(\mathbf{C})$ such that $Ax_0 = x_0$, and $\phi(T) = \alpha ATA^{-1}$ for all $T \in M_n(\mathbf{C})$.*

The proof of this theorem uses several auxiliary lemmas. The first one, quoted from [**14**], identifies the outer and inner local spectral radii of an operator with Dunford's condition (C) at cyclic vectors. Recall that an operator $T \in \mathcal{L}(X)$ is said to have Dunford's condition (C) if the local subspace $X_T(\Omega) := \{x \in X : \sigma_T(x) \subseteq \Omega\}$ is closed for every closed subset Ω of the complex field $\mathbf{C}$. Recall also that an element $x \in X$ is said to be a cyclic vector for T if the linear span of the orbit $\{T^n x : n = 0, 1, 2, 3, \ dots\}$ is dense in X.

Lemma 3.2. *For an operator $T \in \mathcal{L}(X)$, the following hold.*

(i) *If T has the single-valued extension property, then*

$$\iota_T(x) = \min\{|\lambda| : \lambda \in \sigma_T(x)\}$$

for all $x \in X$.

(ii) *If T has the Dunford condition (C) and x is a cyclic vector for T, then either $\iota_T(x) = 0$ or $\iota_T(x) = r_1(T)$.*

The next two lemmas are simple, and their proofs are straightforward and are therefore left to the reader; see for instance [**6**].

Lemma 3.3. *Let x_0 be a fixed vector in X. For an invertible operator $A \in \mathcal{L}(X)$, the automorphism $\phi : T \in \mathcal{L}(X) \mapsto ATA^{-1} \in \mathcal{L}(X)$ preserves the inner local spectral radius at x_0 if and only if $Ax_0 = \lambda x_0$ for some $\lambda \in \mathbf{C}$.*

Let us first fix some more notation. For a Banach spaces X, let X^* be its dual. For $T \in \mathcal{L}(X)$ we will denote by T^* the adjoint of T.

Lemma 3.4. *Assume that X is a complex Banach space of dimension at least two, and let $x_0 \in X$ be a nonzero vector of X. If $A \in \mathcal{L}(X^*, X)$ is a bijective operator, then the anti-automorphism $\phi : T \mapsto AT^*A^{-1}$ does not preserve the inner local spectral radius at x_0.*

We have now collected all the necessary ingredients and are therefore in a position to prove the main result of this section.

Proof of Theorem 3.1. We only need to check the necessity. So, assume that ϕ preserves the inner local spectral radius at x_0, and let us first prove that ϕ is bijective. Let $R_0 \in M_n(\mathbf{C})$ such that $\phi(R_0) = 0$, and let us show that $R_0 = 0$. We have

$$\iota_{R_0+T}(x_0) = \iota_{\phi(R_0+T)}(x_0) = \iota_{\phi(T)}(x_0) = \iota_T(x_0) = 0 \tag{3.1}$$

for all nilpotent matrices $T \in M_n(\mathbf{C})$. By Lemma 3.2 (i), we see that $R_0 + T$ is not invertible for all nilpotent matrices $T \in M_n(\mathbf{C})$. By [**15**, Proposition 5.2], we have $R_0 = 0$, and ϕ is injective. Therefore, the map ϕ is, in fact, bijective; as desired.

Next, let us prove that $r_1(\phi(T)) = r_1(T)$ for all matrices $T \in M_n(\mathbf{C})$. To do so, let $T \in M_n(\mathbf{C})$ be a cyclic matrix for which x_0 is a cyclic vector. Note that, if $\iota_T(x_0) = 0$, then $0 \in \sigma_T(x_0) \subseteq \sigma(T) = \sigma_{ap}(T)$ and also $r_1(T) = 0$. Therefore by, Lemma 3.2 (ii), we have $r_1(T) = \iota_T(x_0)$.

Now, let T be an arbitrary matrix in $M_n(\mathbf{C})$ and note that there is a sequence $(T_k)_k$ of matrices in $M_n(\mathbf{C})$ having x_0 as a common cyclic vector and converging to T; see for instance [**6**, Lemma 2.5]. For every $k \geq 0$, we have

$$r_1(T_k) = \iota_{T_k}(x_0) = \iota_{\phi(T_k)}(x_0) \geq r_1(\phi(T_k)). \tag{3.2}$$

By [**3**, Corollary 3.4.5], the spectral function $r_1(.)$ is continuous, and

$$r_1(T) \geq r_1(\phi(T)) \tag{3.3}$$

for all $T \in M_n(\mathbf{C})$. Since ϕ^{-1} also preserves the inner local spectral radius at x_0, it follows that

$$r_1(T) \geq r_1(\phi^{-1}(T)) \tag{3.4}$$

for all $T \in M_n(\mathbf{C})$. Combining the above two inequalities (3.3) and (3.4), we get

$$r_1(\phi(T)) = r_1(T)$$

for all $T \in M_n(\mathbf{C})$. By Theorem 2.2 there are a complex number c of modulus one and an invertible matrix $A \in M_n(\mathbf{C})$ such that either

$$\phi(T) = cATA^{-1}, \quad (T \in M_n(\mathbf{C})),$$

or

$$\phi(T) = cAT^{\mathrm{tr}}A^{-1}, \; (T \in M_n(\mathbf{C})).$$

Lemma 3.3 and Lemma 3.4 state that ϕ takes only the first form with $Ax_0 = \lambda x_0$ for some nonzero $\lambda \in \mathbf{C}$. Replacing A by A/λ if necessary, we get the desired conclusion. □

Finally, we close the paper with the following natural conjecture which suggests itself.

Conjecture. *Assume that $\mathcal{H}$ is an infinite-dimensional Hilbert space, and let x_0 be a nonzero vector of $\mathcal{H}$. A linear map ϕ from $\mathcal{L}(\mathcal{H})$ onto itself preserves the inner local spectral radius at x_0 if and only if are a scalar $\alpha \in \mathbf{C}$ of modulus one and an invertible operator $A \in \mathcal{L}(\mathcal{H})$ such that $Ax_0 = x_0$, and $\phi(T) = \alpha ATA^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$.*

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