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NAGUMO CONDITIONS AND SECOND-ORDER QUASILINEAR EQUATIONS WITH COMPATIBLE NONLINEAR FUNCTIONAL BOUNDARY CONDITIONS

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Dedicated to the memory of Lloyd Jackson

ABSTRACT. We establish existence results for solutions to nonlinear functional boundary value problems for nonlinear second-order ordinary differential equations assuming there are lower and upper solutions and the right side satisfies a Nagumo growth bound. Our results contain as special cases many results for the p- and ϕ-Laplacians as well as many results where the boundary conditions depend on n-points or even functionals.

1. Introduction.

$$-\frac{d}{dt}\varphi(t,x,x(t),x'(t)) = f(t,x,x(t),x'(t)), \quad \text{for a.e. } t \in [0,1], \tag{1}$$

subject to general functional boundary conditions of the form

$$G(x(0),x(1),x,x'(0),x'(1)) = (0,0), \tag{2}$$

where $\varphi \in C([0,1] \times C[0,1] \times \mathbf{R}^2)$, $f : [0,1] \times C[0,1] \times \mathbf{R}^2 \to \mathbf{R}$ satisfies the Carathédory conditions and $G \in C(\mathbf{R}^2 \times C[0,1] \times \mathbf{R}^2; \mathbf{R}^2)$. Our assumptions on φ and f are due to Cabada and Pouso [**7**]. By a solution x we mean a function $x \in C^1[0,1]$ satisfying (2) such that $\varphi(t,x,x(t),x'(t))$ is absolutely continuous and satisfies (1) almost everywhere on $[0,1]$. We assume that there are ordered lower and upper solutions, α and β, respectively, for (1) and that the functional boundary conditions are compatible, in a sense defined below. The assumptions on φ are sufficiently general to apply to

$$(r(t)x' + q(t)x)' = f(t,x,x(t),x'(t))$$

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where $r > 0$ and q are continuous on $[0,1]$ and to the $p-$Laplacian

$$(|x'|^{p-2}x')' = f(t, x, x(t), x'(t))$$

where $p > 1$ is a constant. Moreover the functional dependence allowed in the differential equation and boundary conditions is sufficiently general to allow applications to a range of boundary value problems for higher order ordinary differential equations.

Existence requires a priori bounds on potential solutions and their derivatives as well as compatible boundary conditions. Here the lower and upper solutions provide the a priori bounds on solutions while two different Nagumo growth conditions, due to Cabada and Pouso [**7**] and to Cabada, O'Regan and Pouso [**6**], respectively, provide the derivative bounds. In [**17**], the authors used bounding surfaces to provide the derivative bounds.

We are particularly interested in problems with nonlinear boundary conditions. Some of the first works on nonlinear boundary conditions are due to Ako [**1, 2**], a student of Nagumo, and to Erbe [**9**], a student of Jackson. Erbe assumed that there are a pair of ordered lower and upper solutions α, $\beta \in C^2[a,b]$ for the ordinary differential equation

$$x'' = f(t, x, x') \tag{3}$$

and considered separated fully nonlinear boundary conditions of the form

$$g_a(y(a), y'(a)) = 0 = g_b(y(b), y'(b)) \tag{4}$$

where $g_a \in C([\alpha(a), \beta(a)] \times \mathbf{R})$ and $g_b \in C([\alpha(b), \beta(b)] \times \mathbf{R})$. He assumed that f is continuous and satisfies a Nagumo condition. In a very clever argument, Erbe essentially 'shoots' with boundary values to satisfy the boundary conditions one end at a time. Later Mawhin and Schmitt [**16**] (see also [**15**]) used Leray-Schauder degree theory combined with a new suitable modification of the equation and of the boundary conditions to give a more streamlined proof.

Independently of Erbe, Ako [**1, 2**] had considered semilinear boundary conditions of the type

$$x'(a) = g_1(x(a)), \qquad x'(b) = g_2(x(b)),$$

or

$$x'(a) = h_1(x(a), x(b)), \qquad x'(b) = h_2(x(a), x(b)),$$

with g, g_1, g_2, h_1 and h_2 continuous. Both Erbe and Ako imposed additional 'compatibility' assumptions on their boundary conditions which we briefly discuss later. Ako assumed that f is continuous on $(a, b) \times \mathbf{R}^2$ and considered cases where f either satisfies a Nagumo condition or there are bounding surfaces. He also imposed other assumptions which, in some cases, involves $\alpha(0) = \beta(0)$ and/or $\alpha(1) = \beta(1)$. We will consider these latter cases in a forthcoming paper. As Ako used semi-continuity properties of minimal and maximal solutions, it is difficult to see how his approach could be applied to general compatible boundary conditions.

Recently there has been a lot of interest in establishing the existence of solutions to (1) subjected to nonlinear boundary conditions of the form (2) where the equation and boundary conditions depend functionally on the solution. We refer to the Introduction of[**17**] for history and references.

Our theory incorporates a degree-based compatibility relationship between the boundary conditions and the lower and upper solutions. In the case of nonlinear two-point boundary conditions this relationship has been studied in Thompson [**20, 21**] and Thompson and Tisdell [**22**], and in the case of three point boundary conditions in Thompson and Tisdell [**23**].

In Section 2 we present the notation, define terms including compatibility, state assumptions and give preliminary results used in this paper. In Section 3 we recall the definition of compatible boundary conditions and state some background lemmas needed to prove that the boundary conditions considered in Erbe [**9**], Cabada and Pouso [**7**], Cabada et al. [**6**], and Fabry-Habets [**10**], are compatible with the lower and upper solutions. Moreover we indicate how the set valued boundary conditions considered by Bebernes and Wilhelmsen [**3**] and Bebernes and Fraker [**4**] can be replaced by compatible boundary conditions when proving existence for these set valued boundary conditions.

For further information about boundary value problems including the method of lower solutions, associated Nagumo growth conditions, bounding surfaces and historical notes see the books by Bernfeld and

Lakshmikantham [**5**], Gaines and Mawhin [**11**], Mawhin [**15**] and De Coster and Habets [**8**].

2. Definitions and preliminary results. We denote the boundary of a set U by ∂U and the closure of U by $\overline{U}$. We denote the space of continuous functions mapping from U to V by $C(U;V)$, omitting V if $V=\mathbf{R}$. We denote the space of absolutely continuous functions on an interval $[a,b]$ by $AC[a,b]$. If U is a bounded, open subset of $\mathbf{R}^n$, $q\in\mathbf{R}^n$, $f\in C(\overline{U};\mathbf{R}^n)$ and $q\notin f(\partial U)$ we denote the Brouwer degree of f on U at q by $d(f,U,q)$.

Our aim is to produce a very general existence result which applies to a very broad class of differential equations subject to very general nonlinear boundary conditions. Thus we study the φ-Laplacian differential equation introduced by Cabada and Pouso in [**7**], subjected to general nonlinear, compatible, boundary conditions.

As in [**7**], φ satisfies the following

Assumption $[\Phi]$. (a) $\varphi\in C([0,1]\times C[0,1]\times\mathbf{R}^2)$ *is such that for each* $(t,\xi,x)\in[0,1]\times C[0,1]\times\mathbf{R}$ *the mapping* $\varphi_{t,\xi,x}(\cdot)=\varphi(t,\xi,x,\cdot)$ *is increasing and such that* $\varphi_{t,\xi,x}(\mathbf{R})=\mathbf{R}$ *and, for every* $\kappa\in\mathbf{R}$ *the mappings* $\varphi(\cdot,\cdot,\cdot,\kappa)$ *and* $\phi(\cdot,\cdot,\cdot,\kappa)$ *are bounded on bounded subsets of* $[0,1]\times C[0,1]\times\mathbf{R}$.

(b) $\varphi(t,\xi,x,0)=0$ *for all* $(t,\xi,x)\in[0,1]\times C[0,1]\times\mathbf{R}$.

Here we define $\phi(t,\xi,x,z)=\varphi^{-1}_{t,\xi,x}(z)$ for all $(t,\xi,x,z)\in[0,1]\times C[0,1]\times\mathbf{R}^2$ while $\phi_{t,\xi,x}(\cdot)=\phi(t,\xi,x,\cdot)$ for each $(t,\xi,x)\in[0,1]\times C[0,1]\times\mathbf{R}$.

With $\alpha\le\beta\in C[0,\,1]$ we associate

(5)
$$\beta_M=\max_{t\in[0,1]}\{\beta(t)\},\qquad \alpha_m=\min_{t\in[0,1]}\{\alpha(t)\},$$

(6) $$\mu=\beta_M-\alpha_m,\qquad \nu=\max\{|\beta(1)-\alpha(0)|,|\beta(0)-\alpha(1)|\},$$

(7) $$\alpha_1=\alpha_m-1,\qquad \beta_1=\beta_M+1,$$

and

(8) $$\Delta=(\alpha(0),\beta(0))\times(\alpha(1),\beta(1)),$$

(9) $$\overline{\omega}=\{(t,x)\in[0,1]\times\mathbf{R}:\alpha(t)\le x\le\beta(t)\}$$

and

$$[\alpha, \beta] = \{x \in C[0,1] : \alpha \leq x \leq \beta\}. \tag{10}$$

We will call the pair α and β *non-degenerate* if $\Delta \neq \varnothing$. In a forthcoming paper we discuss the two cases $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$.

As in [**7**], f satisfies the following

Assumption [RHS]. *f is a Carathéodory function, that is,*

$$f(t,\cdot,\cdot,\cdot) \in C(C[0,1] \times \mathbf{R}^2) \quad \textit{for a.e. } t \in [0,1],$$

$f(\cdot,\xi,x,y)$ is measurable for all $(\xi,x,y) \in C[0,1] \times \mathbf{R}^2$; and for every $N > 0$ there exists an $r_N \in L^1[0,1]$ such that

$$|f(t,\xi,x,y)| \leq r_N(t),$$

for almost every $t \in [0,1]$, for all $\xi \in [\alpha,\beta]$, $x \in [\alpha(t),\beta(t)]$ and $|y| \leq N$.

To establish existence of solutions while relaxing the bounds on f of [**7**, Theorem 2.2] they introduce a variant of lower and upper solutions which involve inequalities relating to (1) as well as a second set of conditions relating to the boundary conditions [**7**, Definition 3.1]. In order to facilitate a comparison of our boundary conditions with several different sets of boundary conditions in the literature we will follow the more usual practice of defining lower and upper solutions to involve this first set of inequalities relating to (1). Consistent with Lloyd Jackson [**12**] we will refer to lower and upper solutions that satisfy these second set of inequalities under and over functions for the boundary value problem.

Definition 1. Two functions $\alpha,\ \beta \in W^{1,\infty}[0,1]$ such that $\alpha \leq \beta$ are said to be a *couple of lower and upper solutions* of (1) if the following conditions are satisfied.

(i) $D_-\alpha(t) \leq D^+\alpha(t)$ for all $t \in (0,1)$. Moreover, if $\tau \in (0,1)$ is such that $D_-\alpha(\tau) = D^+\alpha(\tau)$, then there exists an $\varepsilon > 0$ such that $\alpha \in$

$C^1[\tau-\varepsilon,\tau+\varepsilon]$ and for every $\xi\in[\alpha,\beta]$, $\varphi(\cdot,\xi,\alpha(\cdot),\alpha'(\cdot))\in AC[\tau,\tau+\varepsilon]$ and

$$-\frac{d}{dt}\varphi(t,\xi,\alpha(t),\alpha'(t))\le f(t,\xi,\alpha(t),\alpha'(t))$$

for almost every $t\in[\tau,\tau+\varepsilon]$.

(i′) $D^-\beta(t)\ge D_+\beta(t)$ for all $t\in(0,1)$. Moreover, if $\tau\in(a,b)$ is such that $D^-\beta(\tau)=D_+\beta(\tau)$, then there exists $\varepsilon>0$ such that $\beta\in C^1[\tau-\varepsilon,\tau+\varepsilon]$ and for every $\xi\in[\alpha,\beta]$, $\varphi(\cdot,\xi,\beta(\cdot),\beta'(\cdot))\in AC[\tau,\tau+\varepsilon]$ and

$$-\frac{d}{dt}\varphi(t,\xi,\beta(t),\beta'(t))\ge f(t,\xi,\beta(t),\beta'(t))$$

for almost every $t\in[\tau,\tau+\varepsilon]$.

We now describe the assumptions made in [**7**]. Cabada and Pouso consider the functional boundary value problem (P*), namely (1) subject to the possibly nonlinear boundary conditions

$$L_1(x(0),x(1),x'(0),x'(1),x)=0, \tag{11}$$

$$L_2(x(0),x(1))=0, \tag{12}$$

where L_i, $i=1,2$, satisfy the following

Assumption [BC2]. $L_1\in C(\mathbf{R}^4\times C[0,1])$ *is nondecreasing in the third variable, non-increasing in the fourth and nondecreasing in the fifth one.* $L_2\in C(\mathbf{R}^2)$ *is non-increasing with respect to its first variable.*

They also assume that α and β are under and over functions for (P^*):

Definition 2. Let $\alpha\le\beta$ be a couple of lower and upper solutions for (1). We call them a *couple of under and upper functions* for problem (P^*) if in addition

(i) $D^+\alpha(0)$, $D_-\alpha(1)\in\mathbf{R}$, $L_2(\alpha(0),\cdot)$ is injective,

$L_1(\alpha(0),\alpha(1),D^+\alpha(0),D_-\alpha(1),\alpha)\ge 0$ and $L_2(\alpha(0),\alpha(1))=0$.

(i′) $D_+\beta(0)$, $D^-\beta(1)\in\mathbf{R}$, $L_2(\beta(0),\cdot)$ is injective

$L_1(\beta(0),\beta(1),D_+\beta(0),D^-\beta(1),\beta)\ge 0$ and $L_2(\beta(0),\beta(1))=0$.

Cabada and Pouso [**7**, Definition 3.2] extend Kiguradze's Nagumo condition [**13**], to apply to (1). In particular they introduce the following definition [**7**, Definition 3.2]:

Definition 3. Let φ satisfy Assumption [Φ] and f satisfy Assumption [**RHS**]. Let $\alpha,\ \beta \in W^{1,\infty}[0,1]$ be a couple of lower and upper solutions for (1) on $[0,1]$. We say that f satisfies a *CP Nagumo condition with respect to* φ between α and β if there exist functions $k \in L^q[0,1]$, $1 \le q \le \infty$ and $h \in C([0,\infty);(0,\infty))$ such that

$$|f(t,\xi,x,y)| \le k(t)h(|\varphi(t,\xi,x,y)|),$$

for almost every $t \in [0,1]$, all $\xi \in [\alpha,\beta]$, $x \in [\alpha(t),\beta(t)]$ and $y \in \mathbf{R}$, and there exists an $N > 0$ satisfying the following relations:

(i) $N > \nu = \max\{|\beta(1)-\alpha(0)|,\ |\beta(0)-\alpha(1)|\}$;

(ii)

$$\int_{\sup_{[0,1]\times[\alpha,\beta]\times[\alpha_m,\beta_M]}\varphi(t,\xi,x,\nu)}^{\inf_{[0,1]\times[\alpha,\beta]\times[\alpha_m,\beta_M]}\varphi(t,\xi,x,N)} \frac{z^{(q-1)/q}}{h(z)}\,dz > \mu_1^{(q-1)/q}||k||_q$$

where

$$\mu_1 = \Big(||r_N||_1 + \sup_{[0,1]\times[\alpha,\beta]\times[\alpha_m,\beta_M]} \varphi(t,\xi,x,\nu)\Big),$$

and $r_N \in L^1[0,1]$ is the function given in [**RHS**].

(iii)

$$\int_{\sup_{[0,1]\times[\alpha,\beta]\times[\alpha_m,\beta_M]}\varphi(t,\xi,x,-N)}^{\inf_{[0,1]\times[\alpha,\beta]\times[\alpha_m,\beta_M]}\varphi(t,\xi,x,-\nu)} \frac{z^{(q-1)/q}}{h(z)}\,dz > \mu_2^{(q-1)/q}||k||_q$$

where

$$\mu_2 = \Big(||r_N||_1 + \sup_{[0,1]\times[\alpha,\beta]\times[\alpha_m,\beta_M]} \varphi(t,\xi,x,-\nu)\Big),$$

and $r_N \in L^1[0,1]$ is the function given in [**RHS**].

Any such $N > 0$ will be called a *Nagumo constant*.

This condition does not reduce to Kiguradze's Nagumo condition in the special case $\varphi(t,\xi,x,y) = y$.

We now describe the assumptions made by Cabada, O'Regan and Pouso [**6**], who considered the functional boundary value problem (P**), namely (1) subject to the possibly nonlinear boundary conditions

$$L_1(x(0), x'(0), x) = 0, \tag{13}$$

$$L_2(x(1), x'(1), x) = 0, \tag{14}$$

where the L_i, $i = 1, 2$, satisfy the following

Assumption [BC3]. *$L_i \in C(\mathbf{R}^2 \times C[0,1])$ are nondecreasing with respect to the last variable; that is, if $\xi_i \in [\alpha, \beta]$ satisfy $\xi_1(t) \le \xi_2(t)$ for all $t \in [0,1]$ then*

$$L_i(x, y, \xi_1) \le L_i(x, y, \xi_2) \text{ for all } (x, y) \in \mathbf{R}^2,$$

for $i = 1, 2$. Moreover for every $(x, \xi) \in \mathbf{R} \times [\alpha, \beta]$, $L_1(x, \cdot, \xi)$ is nondecreasing and $L_2(x, \cdot, \xi)$ is nonincreasing.

They also assume that α and β are under and over functions for (P^{**}).

Definition 4. Let $\alpha \le \beta$ be a couple of lower and upper solutions for (1). We call them a *couple of under and upper functions for problem* (P^{**}) if in addition

(ii) $D^+\alpha(0)$, $D_-\alpha(1) \in \mathbf{R}$,

$$L_1(\alpha(0), D^+\alpha(0), \alpha) \ge 0, \; L_2(\alpha(1), D_-\alpha(1), \alpha) \ge 0.$$

(ii′) $D_+\beta(0)$, $D^-\beta(1) \in \mathbf{R}$,

$$L_1(\beta(0), D_+\beta(0), \beta) \le 0, \; L_2(\beta(1), D^-\beta(1), \beta) \le 0.$$

In [**6**, Definition 3.3], Cabada, O'Regan and Pouso gave a variant of Cabada-Pouso's Nagumo condition which we now state.

Definition 5. Let φ and f satisfy Assumptions [Φ] and [**RHS**]. We say that f satisfies a *COP Nagumo condition with respect to* φ between

α and β if there exist $N > 0$ and functions $r \in L^q[0,1]$, $1 \le q \le \infty$ and $h : [0,\infty) \to (0,\infty)$ continuous, such that for almost every $t \in [0,1]$, all $(x,y) \in [\alpha(t),\beta(t)] \times \mathbf{R}$, and each $\xi \in [\alpha,\beta]$,

$$|f(t,\xi,x,y)| \le r(t)h(|y|), \tag{15}$$

(i) $N > \nu = \max\{|\beta(1) - \alpha(0)|,\ |\beta(0) - \alpha(1)\}$

(ii) For each $\xi \in [\alpha,\beta]$ and all t_1, $t_2 \in [0,1]$ we have

(16)
$$\int_{\varphi(t_1,\eta,\eta(t_1),\nu)}^{\varphi(t_2,\eta,\eta(t_2),N)} \frac{\inf_{[0,1]\times[\alpha_m,\beta_M]} |\phi(t,\xi,x,z)|^{(q-1)/q}}{\sup_{[0,1]\times[\alpha_m,\beta_M]} h(\phi(t,\xi,x,z))}\, dz > \mu^{(q-1)/q}||r||_q$$

and

(17)
$$\int_{\phi(t_1,\eta,\eta(t_1),-N)}^{\phi(t_2,\eta,\eta(t_2),-\nu)} \frac{\inf_{[0,1]\times[\alpha_m,\beta_M]} |\phi(t,\xi,x,z)|^{(q-1)/q}}{\sup_{[0,1]\times[\alpha_m,\beta_M]} h(\phi(t,\xi,x,z))}\, dz > \mu^{(q-1)/q}||r||_q.$$

Any such $N > \nu$ will be called a *Nagumo constant.*

We note as in [**6**] that in the special case $\varphi(t,\xi,x,y) = y$ and f is independent of ξ this Nagumo condition reduces to the Nagumo condition for $x'' = f(t,x,x')$ due to Kiguradze [**13**]. We introduce the following assumptions.

Assumption [N1]. *α and β are a couple of lower and upper solutions for* (1) *and that f satisfies the CP Nagumo condition* [**6, 13**] *with respect to φ between α and β.*

Assumption [N2]. *α and β are a couple of lower and upper solutions for* (1) *and that f satisfies the COP Nagumo condition* [**7, 13**] *with respect to φbetween α and β.*

We now state a Nagumo lemma, combining those of Cabada-Pouso [**6**] and Cabada et al. [**7**], and giving a priori bounds on $|x'|$ for solutions

$x \in C^1[0,1]$ of (1) which satisfy $\alpha \leq x \leq \beta$. Such a result is not explicitly stated in [**6, 7**], but proved as part of their existence proofs.

Lemma 1. *Let φ and f satisfy Assumptions* [Φ] *and* [**RHS**], *respectively, and let f satisfy Assumption* [**N1**] *or Assumption* [**N2**]. *Then any solution x of* (1) *with $\alpha \leq x \leq \beta$ satisfies $|x'| < N$, where N is a Nagumo constant.*

This allows us to state an existence theorem combining those of Cabada and Pouso [**7**] and Cabada et al. [**6**].

Theorem 1. *Assume φ and f satisfy Assumptions* [Φ] *and* [**RHS**] *and that f satisfies Assumption* [**N1**], *respectively, Assumption* [**N2**]. *Assume that the boundary conditions* (11) *and* (12), *respectively,* (13) *and* (14) *satisfy Assumption* [**BC2**], *respectively, Assumption* [**BC3**]. *Assume that $\alpha \leq \beta$ is a couple of under and over functions for problem* (P^*), *respectively* (P^{**}). *If N is a Nagumo constant such that $N \geq \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty, \nu\}$, then problem* (P^*), *respectively* (P^{**}), *has at least one solution $x \in [\alpha, \beta]$ such that $|x'(t)| < N$ for all $t \in [0,1]$.*

In a forthcoming paper we discuss the relationship between the various definitions of lower and upper solutions. In this paper we replace [**BC2**] and the assumptions of Definition 2, respectively, Assumption [**BC3**] and the assumptions of Definition 4, by the concept of compatibility of the boundary conditions with the couple of lower and upper solutions. We show that [**BC2**] and the assumptions of Definition 2, respectively, [**BC3**] and the assumptions of Definition 4, imply that Cabada and Pouso's, respectively, Cabada et al.'s boundary conditions are compatible with α and β. An examination of our proof shows that our compatibility assumptions on the boundary conditions are far less restrictive than the corresponding assumptions of Cabada and Pouso, respectively Cabada et a;.

Modification of f is common practice for existence proofs of boundary value problems and we will make the necessary modifications by using the following functions. Given two numbers or functions u and v let $u \vee v = \max\{u, v\}$, and let $u \wedge v = \min\{u, v\}$.

Definition 6. Let $\alpha \le \beta$ on $[0,1]$ and $N > 0$. Define $p : \mathbf{R} \to [\alpha(t), \beta(t)]$ and $p_N : \mathbf{R} \to [-N, N]$ by

$$p(x) = \beta \wedge x \vee \alpha, \quad p_N(x) = (-N) \vee x \wedge N.$$

We extend p and p_N to $x \in C[0,1]$ in the obvious way.

Let

$$\widetilde{\varphi}(t, \xi, x, y) = \varphi(t, p(\xi), p(x), y) \tag{18}$$

for $\xi \in C[0,1]$ and $(t, x, y) \in [0,1] \times \mathbf{R}^2$.

Definition 7. Let

$$l(t, \xi, x, y) = \begin{cases} |f(t, p(\xi), \beta(t), p_N(y))| & \text{for } \beta_1 \le x \\ f(t, p(\xi), p(x), p_N(y)) & \text{for } \alpha \le x \le \beta \\ -|f(t, p(\xi), \alpha(t), p_N(y))| & \text{for } x \le \alpha_1, \end{cases} \tag{19}$$

and extend l to $x \in [\alpha_1, \alpha] \cup [\beta, \beta_1]$ linearly as a continuous function.

For almost all $t \in [0,1]$ and all $(\xi, x, y) \in [\alpha, \beta] \times [\alpha(t), \beta(t)] \times [-N, N]$ we have

$$l(t, \xi, x, y) = f(t, \xi, x, y).$$

Thus it suffices to show that there is a solution of

$$-\frac{d}{dt}\widetilde{\varphi}(t, x, x(t), x'(t)) = l(t, x, x(t), p(x)'(t)) \tag{20}$$

together with boundary conditions (2) with $(x, x(t), x'(t)) \in [\alpha, \beta] \times [\alpha(t), \beta(t)] \times [-N, N]$ for $t \in [0,1]$.

Lemma 2. *Let l be given in Definition 7, and let x be a solution of*

$$-\frac{d}{dt}\widetilde{\varphi}(t, x, x(t), x'(t)) = \lambda l(t, x, x(t), p(x)'(t)) \tag{21}$$

subject to the boundary conditions

$$x(0) = C, \qquad x(1) = D, \tag{22}$$

where $0 \le \lambda \le 1$. *Then* l *satisfies Assumption* [**RHS**] *and* $|l| \le r_N \in L^1[0,1]$. *Moreover there is* $M \ge N$ *such that if* $\alpha_1 - 1 \le x \le \beta_1 + 1$ *then* $|x'| < M$.

Proof. Clearly l satisfies Assumption [**RHS**] and $|l| \le r_N \in L^1[0,1]$. Since $\phi_{t,p(\xi),p(x))}(y)$ is bounded for bounded y the existence of M follows immediately from the proof of [**7**, Theorem 2.2]. □

Clearly l satisfies Assumption [**RHS**] and Assumption [**N1**], respectively, Assumption [**N2**].

3. Nonlinear boundary conditions. We now recall the definition of compatible boundary conditions which is a simple, degree-based relationship between the boundary conditions and the lower and upper solutions. For more information we refer the reader to [**20, 22**] and [**17**, Definition 2].

Definition 8. We call the vector field $\Upsilon = (\Upsilon^0, \Upsilon^1) \in C(\overline{\Delta}; \mathbf{R}^2)$ *strongly inwardly pointing* on $\Delta = (\alpha(0), \beta(0)) \times (\alpha(1), \beta(1))$ if for all $(C, D) \in \partial\Delta$

$$\begin{aligned} \Upsilon^0(\alpha(0), D) &> D^+\alpha(0), \qquad \Upsilon^0(\beta(0), D) < D_+\beta(0), \\ \Upsilon^1(C, \alpha(1)) &< D_-\alpha(1), \qquad \Upsilon^1(C, \beta(1)) > D^-\beta(1). \end{aligned} \tag{23}$$

We call Υ *inwardly pointing* if the above inequalities are weak.

The following definition was introduced in [**15**, Definition 3]

Definition 9. Let $G \in C(\overline{\Delta} \times [\alpha, \beta] \times \mathbf{R}^2; \mathbf{R}^2)$. We say G is *strongly compatible with the pair* α, β if for all strongly inwardly pointing vector fields Υ on Δ and $\xi_0 \in [\alpha, \beta]$,

$$\mathcal{G}(C, D) \ne 0 \quad \text{for all } (C, D) \in \partial\Delta, \tag{24}$$

and

$$d(\mathcal{G}, \Delta, 0) \ne 0, \tag{25}$$

where

$$\mathcal{G}(C,D) = G(C,D,\xi_0,\Upsilon(C,D)). \tag{26}$$

We say G is *compatible with the pair* α, β if there is a sequence $G_j \in C(\overline{\Delta} \times [\alpha,\beta] \times \mathbf{R}^2;\mathbf{R}^2)$ strongly compatible with the pair α, β and converging uniformly to G on bounded subsets of $\overline{\Delta} \times [\alpha,\beta] \times \mathbf{R}^2$.

Remark 1. If $\Delta \neq \varnothing$ then strongly inwardly pointing vector fields Υ on Δ always exist. *Indeed we may choose*

$$\begin{aligned}
\Upsilon^0(C,D) &= (D_+\beta(0)-1)\frac{C-\alpha(0)}{\beta(0)-\alpha(0)} \\
&\quad + (D^+\alpha(0)+1)\frac{\beta(0)-C}{\beta(0)-\alpha(0)}, \\
\Upsilon^1(C,D) &= (D^-\beta(1)+1)\frac{D_-\alpha(1)}{\beta(1)-\alpha(1)} \\
&\quad + (D_-\alpha(1)-1)\frac{\beta(1)-D}{\beta(1)-\alpha(1)},
\end{aligned}$$

where $D^\pm$ *and* $D_\pm$ *are the Dini derivatives and* $\xi_0 = (\alpha+\beta)/2$, *when computing the Brouwer degree* (25).

The following results are [**17**, Lemma 1] and [**17**, Lemma 2] respectively.

Lemma 3. *If G is strongly compatible with the pair α, β then the Brouwer degree* (25) *is independent of the strongly inwardly pointing vector field Υ and $\xi_0 \in [\alpha,\beta]$.*

Lemma 4. *If G is strongly compatible with α and β then*

$$G(C,D,\xi,u,v) \neq (0,0)$$

if $(C,D) \in \partial\Delta$ and either $C=\alpha(0)$ and $u > D^+\alpha(0)$, or $C=\beta(0)$ and $u < D_+\beta(0)$, or $D=\alpha(1)$ and $v < D_-\alpha(1)$, or $D=\beta(1)$ and $v > D^-\beta(1)$.

The Picard (Dirichlet) boundary conditions, $x(0) = A,\ x(1) = B$ are compatible conditions if and only if $\alpha(0) \le A \le \beta(0)$, $\alpha(1) \le B \le \beta(1)$, the Neumann boundary conditions $x'(0) = A$, $x'(1) = B$ are compatible conditions if and only if $\alpha'(0) \ge A$, $\beta'(0) \le A$, $B \ge \alpha'(1)$, $B \le \beta'(1)$, and the periodic boundary conditions $x(0) = x(1)$, $x'(0) = x'(1)$ are compatible if and only if $\alpha(0) = \alpha(1)$, $\beta(0) = \beta(1)$, $\alpha'(0) \ge \alpha'(1)$, $\beta'(0) \le \beta'(1)$. These are the usual assumptions imposed on lower and upper solutions. See Thompson [**20**] for details.

We now show that the boundary conditions considered by Cabada and Pouso in [**7**, Theorem 3.1] and by Cabada et al. in [**6**, Theorem 3.1] are compatible. First we consider Cabada and Pouso's boundary conditions. We need the following notation and lemmas. Let

$$\begin{aligned}\partial\Delta_u = \{(C,D) \in \overline{\Delta} : (C = \alpha(0)\&D > \alpha(1))\\ \cup (C < \beta(0)\&D = \beta(1))\},\\ \partial\Delta_l = \{(C,D) \in \overline{\Delta} : (C = \beta(0)\&D < \beta(1))\\ \cup (C > \alpha(0)\&D = \alpha(1))\}.\end{aligned}$$

The following results are [**17**, Lemma 3] and [**17**, Corollary 1], respectively.

Lemma 5. *Let* $\mathcal{G} \in C(\overline{\Delta}; \mathbf{R}^2)$ *satisfy*

$$\mathcal{G}^0 < 0 \ \text{on} \ \partial\Delta_u \qquad \mathcal{G}^0 > 0 \ \text{on} \ \partial\Delta_l \tag{27}$$

$$\mathcal{G}^1(\beta(0), \beta(1)) > 0 > \mathcal{G}^1(\alpha(0), \alpha(1)). \tag{28}$$

Then

$$d(\mathcal{G}, \Delta, 0) = 1. \tag{29}$$

Corollary 1. *Let the boundary conditions be given by* (2), *where* $G \in C(\overline{\Delta} \times [\alpha, \beta] \times \mathbf{R}^2; \mathbf{R}^2)$, *and let* $\mathcal{G} \in C(\overline{\Delta}; \mathbf{R}^2)$ *given by* (26) *satisfy*

$$\mathcal{G}^0 \le 0 \ \text{on} \ \partial\Delta_u \qquad \mathcal{G}^0 \ge 0 \ \text{on} \ \partial\Delta_l \tag{30}$$

$$\mathcal{G}^1(\beta(0), \beta(1)) \ge 0 \ge \mathcal{G}^1(\alpha(0), \alpha(1)) \tag{31}$$

Then the boundary conditions are compatible.

The following result gives the compatibility of Cabada and Pouso's boundary conditions [**7**, Theorem 3.1].

Lemma 6. *Let* $G = (g^0, g^1) \in C(\overline{\Delta} \times [\alpha, \beta] \times \mathbf{R}^2; \mathbf{R}^2)$ *be given by*

$$\begin{aligned} g^0(C, D, x, u, v) &= -L_2(C, D) \\ g^1(C, D, x, u, v) &= -L_1(C, D, u, v, x), \end{aligned}$$

where L_1 *and* L_2 *satisfy Assumption* [**BC2**] *as well as* (i) *and* (i′) *of Definition* 2. *Then the boundary conditions given by*

$$G(x(0), x(1), x, x'(0), x'(1)) = 0$$

are compatible.

Proof. First let $\Upsilon \in C(\overline{\Delta}; \mathbf{R}^2)$ be strongly inwardly pointing and set $\mathcal{G}(C, D) = G(C, D, \xi_0, \Upsilon(C, D))$. In view of the above lemmas it suffices to show $\mathcal{G}$ satisfies (30) and (31). Now $\Upsilon^0(\alpha(0), D) > D^+\alpha(0)$, $\beta \geq \xi_0 \geq \alpha$ and $D \geq \alpha(1)$, so noting that

$$-L_1(\alpha(0), \alpha(1), D^+\alpha(0), D_-\alpha(1), \alpha) \leq 0$$

and using the monotonicity of L_1 we see that $\mathcal{G}^1(\alpha(0), \alpha(1)) \leq 0$. Similarly $\mathcal{G}^1(\beta(0), \beta(1)) \geq 0$. Now $-L_2(\alpha(0), \alpha(1)) \leq 0$ and using monotonicity and injectivity we see that

$$-L_2 \leq 0 \quad \text{on } \partial\Delta_u \quad \text{and} \quad -L_2 \geq 0 \quad \text{on } \partial\Delta_l.$$

Now $\mathcal{G}^0(C, D) = -L_2(C, D)$ so the boundary conditions are compatible. □

Now we consider the boundary conditions studied in Cabada et al. [**6**, Theorem 3.1]. We need the following result [**17**, Lemma 5].

Lemma 7. *If* $\mathcal{G}$ *satisfies*

$$\begin{aligned} \mathcal{G}^0(\alpha(0), D) < 0 < \mathcal{G}^0(\beta(0), D) \\ \mathcal{G}^1(C, \alpha(1)) < 0 < \mathcal{G}^1(C, \beta(1)). \end{aligned}$$

for all $(C,D) \in \overline{\Delta}$*, then*

$$d(\mathcal{G}, \Delta, 0) = 1.$$

Let $\gamma = (\alpha+\beta)/2$ and set $\overline{\eta}(C,D) = (C-\gamma(0), D-\gamma(1))$. It follows from the weighted sum formula for Brouwer degree that $d(\overline{\eta}, \Delta, 0) = 1$.

Lemma 8. *Let* $G = (g^0, g^1) \in C(\overline{\Delta} \times \mathbf{R}^2 \times [\alpha,\beta]; \mathbf{R}^2)$ *be given by*

$$g^0(C,D,\xi,u,v) = -L_2(C,u,\xi)$$
$$g^1(C,D,\xi,u,v) = -L_1(D,v,\xi),$$

where L_1 *and* L_2 *satisfy Assumption* [**BC3**] *as well as Assumptions* (ii) *and* (ii′) *of Definition* 4*. Then the boundary conditions given by*

$$G(x(0), x(1), x, x'(0), x'(1)) = 0$$

are compatible.

Proof. Let $G_n(C,D,\xi_0,u,v) = G(C,D,\xi_0,u,v) + \overline{\eta}(C,D)/n$. Suppose that $C = \alpha(0)$. Let Υ be a strongly inwardly pointing vector field on Δ. Thus $\Upsilon^0(\alpha(0),D) > D^+\alpha(0)$. Thus $0 \le L_2(\alpha(0), \Upsilon^0(\alpha(0),D), \xi_0) \le L_2(\alpha(0), \Upsilon^0(\alpha(0),D), \xi_0)$. Thus $\mathcal{G}_n^0(\alpha(0),D) > 0$ for all $(\alpha(0),D) \in \overline{\Delta}$. A similar argument for the cases $C = \beta(0)$, $D = \alpha(1)$ and $D = \beta(1)$ shows that $\mathcal{G}_n$ satisfies

$$\mathcal{G}_n^0(\alpha(0),D) < 0, \qquad \mathcal{G}_n^0(\alpha(1),D) > 0$$
$$\mathcal{G}_n^1(C,\alpha(1)) < 0, \qquad \mathcal{G}_n^1(C,\beta(1)) > 0.$$

It follows that $d(\mathcal{G}_n, \Delta, 0) = 1$. Thus the boundary conditions

$$G(x(0), x(1), x, x'(0), x'(1)) + \overline{\eta}(x(0), x(1))/n = 0$$

are strongly compatible with α and β. Since $\lim_{n\to\infty} G_n = G$ uniformly it follows that the boundary conditions $G(x(0), x(1), x, x'(0), x'(1)) = 0$ are compatible with α and β. □

In [**9**], for the differential equation (3) subject to the boundary conditions (4), Erbe assumed that $g_a(C,t)$ and $g_b(D,t)$ are nondecreasing with respect to t for $(C, D) \in [\alpha(a), \beta(a)] \times [\alpha(b), \beta(b)]$ and satisfy the inequalities

$$g_a(\alpha(a), \alpha'(a)) \geq 0 \geq g_b(\alpha(b), \alpha'(b)) \tag{32}$$

$$g_a(\beta(a), \beta'(a)) \leq 0 \leq g_b(\beta(b), \beta'(b)). \tag{33}$$

It is easy to see that Erbe's boundary conditions satisfy the assumptions of Cabada et al. and hence are compatible.

In [**3**], Bebernes and Fraker considered the boundary value problem (3) subject to the set-valued boundary conditions $(x(i), x'(i)) \in \mathcal{C}(i)$ where $\mathcal{C}(i) \subset \mathbf{R}^2$, for $i = 1, 2$. They assumed that f is continuous, there are lower and upper solutions and f satisfies a Nagumo growth condition. Moreover, they assumed compatibility conditions involving the $\mathcal{C}(i)$ containing connected components $\mathcal{J}(i)$ such that $\mathcal{J}(0) \cap \{S_0 \cup S_2\} \neq \varnothing$ and $\mathcal{J}(1) \cap \{S_1 \cup S_3\} \neq \varnothing$, where

$$\begin{aligned}
S_0 &= \{(x, -N)\, \alpha(0) \leq x \leq \beta(0)\} \cup \{(\alpha(0), y)\, \alpha'(0) \geq y \geq -N\},\\
S_2 &= \{(x, N)\, \alpha(0) \leq x \leq \beta(0)\} \cup \{(\beta(0), y)\, \beta'(0) \leq y \leq N\},\\
S_1 &= \{(x, -N)\, \alpha(1) \leq x \leq \beta(1)\} \cup \{(\beta(1), y)\, \beta'(1) \geq y \geq -N\},\\
S_3 &= \{(x, N)\, \alpha(1) \leq x \leq \beta(1)\} \cup \{(\alpha(1), y)\, \alpha'(1) \leq y \leq N\},
\end{aligned}$$

where N is a Nagumo constant. They used shooting with initial values combined with the Jordan separation theorem; see also Bebernes and Wilhelmsen [**4**] and their references. The second author in [**21**] showed how to construct compatible boundary conditions

$$G = (g^0(x(0), x'(0)), g^1(x(1), x'(1))$$

such that $\mathcal{J}(i)$ is the zero set of g^i. Thus Bebernes-Fraker's boundary conditions are compatible 'set valued' boundary conditions and their existence result follows from ours setting $r(t) = 1$ in COP Nagumo growth bound (15).

4. Existence of solutions. Before we present our main result we give preliminary ones needed in the proof. First we show that α and β are a couple of lower and upper solutions for (20).

Lemma 9. *Let φ and f satisfy Assumptions* $[\Phi]$ *and* $[\mathbf{RHS}]$. *Let α and β be a couple of lower and upper solutions for* (1) *and let $l(t, \xi, x, y)$*

be given in Definition 7 *where* $N > \nu$. *Then any solution,* x, *of* (20) *with* $(x(0), x(1)) \in \overline{\Delta}$ *satisfies* $\alpha \le x \le \beta$ *on* $[0,1]$.

Proof. Let α and β be a couple of lower and upper solutions and N be a Nagumo constant. Thus α and β are a couple of lower and upper solutions for (20). We argue by contradiction. Assume that $\alpha \le x$ fails at some point of $[0,1]$. Since $\alpha - x$ is continuous it attains its maximum on $[0,1]$ at some point τ so $0 < \alpha(\tau) - x(\tau)$. From our assumptions it follows that $\tau \in (0,1)$ and by continuity we may choose τ such that $\alpha(t) - x(t) < \alpha(\tau) - x(\tau)$ for $\tau < t < \tau_1 \le 1$. It follows from the definition of lower solution that $\alpha'(\tau) = x'(\tau)$ exists and there is an $\varepsilon > 0$ such that $\tau + \varepsilon \le \tau_1$, $\alpha \in C^1[\tau - \varepsilon, \tau + \varepsilon]$, and that $\widetilde{\varphi}(t, x, \alpha, \alpha')$ is absolutely continuous on $[\tau, \tau + \varepsilon]$. Moreover since $(x(0), x(1)) \in \overline{\Delta}$ it follows from the Mean Value theorem that $|x'(\tau)| \le \nu < N$. Thus we may choose $\varepsilon > 0$ small enough that $|x'(t)|, |\alpha'(t)| < N$ on $[\tau, \tau+\varepsilon]$. In view of this, since $\alpha(t) - x(t) > 0$ on $[\tau, \tau + \varepsilon]$ it follows that for almost every $t \in [\tau, \tau + \varepsilon]$

$$-\frac{d}{dt}\widetilde{\varphi}(t, x, \alpha, \alpha') \le l(t, x, \alpha, \alpha') = -\frac{d}{dt}\widetilde{\varphi}(t, x, x, x'). \tag{34}$$

Setting $v(t) = \varphi(t, x, \alpha(t), \alpha' t)) - \varphi(t, x, \xi, \alpha(t), x'(t))$ it follows that v is absolutely continuous on $[\tau, \tau + \varepsilon]$, $v(\tau) = 0$ and $v'(t) \ge 0$ almost everywhere on $[\tau, \tau + \varepsilon]$. Thus $\alpha' \ge x'$ almost everywhere on $[\tau, \tau + \varepsilon]$ and $\alpha(t) - x(t) \ge \alpha(\tau) - x(\tau)$ for $\tau \le t \le \tau + \varepsilon$, a contradiction. It follows that $x \ge \alpha$ on $[0,1]$. Similarly $x \le \beta$ on $[0,1]$ and the result follows. $\square$

Lemma 10. *Let* φ *and* f *satisfy Assumptions* $[\Phi]$ *and* $[\mathbf{RHS}]$. *For every* $x \in C^1[0,1]$ *there is a unique* $\tau = \tau_{x,E,\lambda} \in \mathbf{R}$ *such that*

$$\int_0^1 \left(\phi(t, x, x(t), \tau - \int_0^t \lambda l(s, x, x(s), p(x)'(s))\, ds) \right) dt = E. \tag{35}$$

Moreover there is a constant $k_1 > 0$ *which depends only upon* φ, ψ *and* k *such that* $|\tau_{x,E,\lambda}| \le k_1$ *for all* $x \in C^1[0,1]$, $\lambda \in [0,1]$ *and* $|E| \le k$. *Finally,* $\tau_{x,0,0} = 0$.

We now present one of our main existence theorems.

Theorem 2. *Let α and β be a couple of lower and upper solutions for* (1) *and $\Delta \neq \varnothing$. Let Assumptions* $[\Phi]$, **[RHS]** *and either Assumption* **[N1]** *or* **[N2]** *hold. If G is compatible with the pair α and β and N is a Nagumo constant, then problem* (1)–(2) *has a solution $x \in C^1[0,1]$ with $\varphi(t,x,x(t),x'(t))$ absolutely continuous, $\alpha \le x \le \beta$ and $|x'| < N$ on* $[0,1]$.

Proof. First consider the case that G is strongly compatible with the couple of lower and upper solutions, α and β, that $\Delta \neq \varnothing$ that Assumption **[N1]** holds, and that N is a Nagumo constant. Let $l(t,\xi,x,y)$ be given in Definition 7. Thus (21) is (20) when $\lambda = 1$. We show that a solution x of (20) and (2) with $(x(0),x(1)) \in \overline{\Delta}$ is the required solution to (1) and (2). From Lemma 9 we see that any solution, $x \in C^1[0,1]$, to (20) with $(x(0),x(1)) = (C,D) \in \overline{\Delta}$ satisfies $\alpha \le x \le \beta$ on $[0,1]$. Thus $p(x) = x$ so x is a solution of

$$(36)\quad -\frac{d}{dt}\varphi(t,x,x(t),x'(t)) = f(t,x,x(t),p_N(x'(t))),\ (\text{a.e. } t \in [0,1]).$$

It follows from Lemma 1 that $|x'| < N$ so that x is a solution to (1), as required. Thus it suffices to show that (21) and (2) has a solution when $\lambda = 1$. We use Leray-Schauder degree theory to prove there is a solution, through homotopies based on Lemma 9 and (2l) to compute the degree. It is easy to see that $\alpha_1 - 1$ and $\beta_1 + 1$ are a pair of lower and upper solutions for (21). It follows that if x is a solution of (21) with $(x(0),x(1)) \in (\alpha_1 - 1, \beta_1 + 1)^2$ then $\alpha_1 - 1 < x < \beta_1 + 1$ and it follows from Lemma 2 that $|x'| < M$ where $M \ge N$. Thus we let

$$\Omega = \{x \in C^1[0,1] :\ \alpha_1 < x < \beta_1,\ |x'| < M \text{ on } [0,1]\},$$

and let

$$\Gamma = \Omega \times \Delta.$$

Define $\mathcal{C} : C^1[0,1] \times \overline{\Delta} \times [0,1] \to C^1[0,1]$ by

$$(37)\quad \mathcal{C}(x,C,D,\lambda)(t) = \int_0^t \phi\bigg(u,x,x(u),\tau_{x,\lambda(D-C),\lambda} - \int_0^u \lambda l(s,x,x(s),p(x)'(s))\,ds\bigg)\,du + \lambda C + (1-\lambda)(\alpha_1+\beta_1)/2.$$

Since the conditions H_1, H_2 and H_3 of [**7**] are satisfied it follows as in [**7**, Theorem 2.2] that the problem (21) and boundary conditions (22) have a solution x for C, $D \in \mathbf{R}$ if and only if $x = \mathcal{C}(x, C, D, \lambda)$. Moreover it follows from the proof of [**7**, Theorem 2.2] that $\mathcal{C}$ is completely continuous. Further if $x = \mathcal{C}(x, C, D, 0)$ then $E = 0 = \lambda$ in (35) so $\tau = \tau_{x,0,0} = 0$ in (37) and it follows from $[\Phi]$**(b)** that $x = \mathcal{C}(x, C, D, 0) = (\alpha_1 + \beta_1)/2$.

Let Ψ be strongly inwardly pointing vector field on Δ, and let $\xi_0 = (\alpha + \beta)/2$. Define $\mathcal{H} : \overline{\Gamma} \times [0, 1] \to X$ by

$$\mathcal{H}(x, C, D, \lambda) = (x - \mathcal{C}(x, C, D, 2(\lambda - 1/2)), \mathcal{G}(C, D)),$$

for $0 \leq \lambda \leq 1/2$ and

$$\mathcal{H}(x, C, D, \lambda) = (x - \mathcal{C}(x, C, D, 1), \mathcal{S}(x, C, D, \lambda)),$$

for $1/2 \leq \lambda \leq 1$ where $\mathcal{S}$ is defined by

$$\begin{aligned} \mathcal{S} &= \mathcal{S}(x, C, D, \lambda) \\ &= G((C, D, 2(\lambda - 1/2)x + 2(1 - \lambda)\xi_0, \\ &\qquad 2(\lambda - 1/2)(x'(0), x'(1)) + 2(1 - \lambda)\Psi(C, D)). \end{aligned}$$

Now $\mathcal{H}$ is completely continuous since $\mathcal{C}$ is completely continuous. From earlier observations it easy to see that (x, C, D) is a solution of (20) with $(C, D) = (x(0), x(1)) \in \overline{\Delta}$, $(x, x(0), x(1)) \in \Gamma$ satisfying (2) if

$$\mathcal{H}(x, C, D, \lambda) = 0 \tag{38}$$

and $\lambda = 1$. Now if there is a solution with $(x, C, D) \in \partial\Gamma$ of (38) for $\lambda = 1$ then there is nothing to prove so we assume there is no solution in $\partial\Gamma$. We show $\mathcal{H}$ is an admissible homotopy for Leray-Schauder degree on Γ at 0, i.e., there are no solutions $(x, C, D) \in \partial\Gamma$ of (38) for $0 \leq \lambda < 1$. We argue by contradiction and assume that there is a solution of $\mathcal{H}(x, C, D, \lambda) = 0$ with $\lambda \in [0, 1)$ and $(x, C, D) \in \partial\Gamma$. We investigate the cases $\lambda \in [1/2, 1)$ and $[0, 1/2)$ separately.

Case (i). $\lambda \in [1/2, 1)$. Assume there is a solution (x, C, D) with $\lambda \in [1/2, 1)$ with $(C, D) \in \overline{\Delta}$ and $\alpha_1 - 1 \leq x \leq \beta_1 + 1$ on $[0, 1]$. Thus $(x(0), x(1)) = (C, D) \in \overline{\Delta}$. Assume $(C, D) \in \partial\Delta$. Consider the case $x(0) = C = \alpha(0)$. From Lemma 9 we see that $\alpha \leq x \leq \beta$ on $[0, 1]$

so $x'(0) \geq \alpha'(0)$. Since $\Psi_0(\alpha(0), D) > \alpha'(0)$ and $2(\lambda - 1/2) > 0$ it follows that $2(\lambda - 1/2)x'(0) + 2(1 - \lambda)\Psi_1(\alpha(0), D) > \alpha'(0)$ so $\mathcal{S}(x, \alpha(0), D, \lambda) \neq 0$ for $1/2 \leq \lambda < 1$, a contradiction. Similarly the other cases $(C, D) \in \partial\Delta$ lead to a contradiction so $(C, D) \notin \partial\Delta$. So assume $x \in \partial\Omega$. Since x satisfies (20) and Assumption [**N1**] holds it follows that $|x'| < N$ on $[0, 1]$. Thus $x(t) = \alpha_1 - 1$ or $x(t) = \beta_1 + 1$ for some $t \in [0, 1]$. Assume that $x(t) = \alpha_1 - 1$ for some $t \in [0, 1]$. Since $(C, D) = (x(0), x(1)) \in \Delta$ it follows that $t \in (0, 1)$ so $x'(t) = (\alpha_1 - 1)' = 0$ while $l(t, x, u, 0) > 0$ for $\alpha_1 - 1 \leq u \leq \alpha_1$, a contradiction so $x(t) > \alpha_1$ on $[0, 1]$. Similarly $x(t) < \beta_1 + 1$ on $[0, 1]$. Thus there are no solutions of $\mathcal{H}(x, C, D, \lambda) = 0$ with $\lambda \in [1/2, 1)$ and $(x, C, D) \in \partial\Gamma$.

Case (ii). $\lambda \in [0, 1/2)$. Assume first that $\lambda > 0$, and assume $(x, C, D) \in \partial\Gamma$. Since we are assuming that G is strongly compatible with α and β it follows that $(C, D) \notin \partial\Delta$. Assume that $x \in \partial\Omega$. The subcases $x(t) = \alpha_1 - 1$ and $x(t) = \beta_1 + 1$ lead to a contradiction in a similar way as in Case (i) after replacing l by λl and noting that $(x(0), x(1)) \in (\alpha_1 - 1, \beta_1 + 1)^2$. Since $|x'| < M$ by Lemma 2 it follows that $(x, C, D) \notin \partial\Gamma$. Consider the case $\lambda = 0$. Now $\mathcal{H}(x, C, D, 0) = (x - b, \mathcal{G}(C, D))$ where $b = (\alpha_1 + \beta_1)/2 \in \Omega$ so $\mathcal{H}(x, C, D, 0) \neq 0$ for all $(x, C, D) \in \partial\Gamma$. Thus $\mathcal{H}$ is an admissible homotopy for Leray Schauder degree on Γ at 0, and the product property of Leray-Schauder degree implies that

$$\begin{aligned} d(\mathcal{H}(\cdot, 1), \Gamma, 0) &= d(\mathcal{H}(\cdot, 0), \Gamma, 0), \\ &= d(\mathcal{I} - b, \Omega, 0) \times d(\mathcal{G}, \Delta, 0) \\ &= d(\mathcal{G}, \Delta, 0) \neq 0, \end{aligned}$$

where $\mathcal{I}$ is the identity on $C^1[0, 1]$. Thus by Leray-Schauder degree theory there is a solution $(x, C, D) \in \Gamma$ of $\mathcal{H}(x, C, D, 1) = 0$ and $x \in C^1[0, 1]$ is a solution of (37) when $\lambda = 1$. Thus $\varphi(t, x, x(t), x'(t))$ is absolutely continuous and x is a solution to (20) and (2). Since $\alpha \leq x \leq \beta$ and $|x'| < N$ on $[0, 1]$ it follows that x is a solution to (1) and (2), as required.

Suppose now that G is compatible with α and β. Then as in [**20**] there is a sequence $\{G_j\}$ strongly compatible with α and β which converges uniformly to G on compact subsets of $\mathbf{R}^2 \times [\alpha, \beta] \times \mathbf{R}^2$ to G. Let x_j be the corresponding solutions. By compactness, there is a subsequence of x_j which converges in $C^1[0,1]$ to the desired solution of the integral and hence differential equation and boundary conditions.

The case where Assumption [**N2**] can be treated similarly. □

Remark 2. Cabada et al. [**6**] do not assume that Assumption [Φ**(b)**] is satisfied. We did it to simplify an already complicated proof. This assumption can removed by setting

$$\begin{aligned}\zeta(t,\xi,x,x') &= \varphi(t,\xi,x,x') - \varphi(t,\xi,x,0)\\ &= \int_0^t f(s,\xi,x(s),x'(s))\,ds - \varphi(t,\xi,x(t),0)\end{aligned}$$

so that

$$x'(t) = \zeta^{-1}\bigg(t,\xi,x,\int_0^t f(s,\xi,x(s),x'(s))\,ds - \varphi(t,\xi,x(t),0)\bigg).$$

Then $\zeta(t,\xi,x,y)$ satisfies [Φ] including [Φ]**(b)**. Thus it follows from Lemma 10 that there exists $\tau_{x,E,\lambda}$ such that

$$\int_0^1 \zeta^{-1}\left(t,x,x(t),\tau-\lambda L(t)\right)\,dt = E,$$

where $L(t) = \int_0^t l(s,x,x(s),x'(s))\,ds - \widetilde{\varphi}(t,x,x(t),0)$. Thus existence follows from the proof of Theorem 2 after modifying $\mathcal{C}$ in the obvious way.

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