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BOUNDARY VALUE PROBLEMS FOR SINGULAR ELLIPTIC EQUATIONS

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To the memory of Lloyd K. Jackson

ABSTRACT. We study the existence of positive solutions to singular elliptic boundary value problems involving the p-Laplace operator. We establish a sub-supersolution theorem and use an eigenfunction of the p-Laplacian to construct sub- and super-solutions. Our assumptions on the singular term are more relaxed than in some previous papers, even for the case $p = 2$, as we allow for non-monotone singular terms with blowup controlled by a power. We also allow for a parameter dependent term and study how its growth affects our existence result.

1. Introduction. Let Ω be a smooth bounded domain in $\mathbf{R}^N$, $N \geq 1$, and $p > 1$. We are interested in the following singular elliptic problem

$$\begin{cases} -\Delta_p u = ag(u) + \lambda h(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

is the p-Laplace operator; λ is a nonnegative parameter;

$$a : \Omega \longrightarrow [1, \infty)$$

is in $L^\infty(\Omega)$;

$$g : (0, \infty) \longrightarrow \mathbf{R}$$

is continuous and satisfies

$$\lim_{s \to 0^+} g(s) = \infty;$$

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and,

$$h : [0, \infty) \longrightarrow \mathbf{R}$$

is continuous.

Lazer and McKenna [**10**] have proved that (1.1) has a unique classical solution when $\lambda = 0$, $g(s) = s^{-\gamma}$, $s \in (0, \infty)$, $\gamma > 0$ and Ω is in class $C^{2+\beta}$, $\beta > 0$. Lair and Shaker [**9**] and Zhang and Cheng [**21**] have obtained the results of Lazer and McKenna (in the case $0 < \gamma < 1$) deducing the existence of solutions of

$$\begin{cases} -\Delta u = ag(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where g is nonincreasing and satisfies

$$\int_0^1 g(s)\, ds < \infty.$$

Although Ω in [**9**] is either a bounded domain or the whole space $\mathbf{R}^N$, (while Ω in [**21**] is bounded) and the conditions on a in [**9**] are weaker than those in [**21**], the results of [**21**] cannot be deduced from those of [**9**]. An additional significant paper is the paper by Crandall, Rabinowitz and Tartar [**3**], where the existence of solutions to the more general problem

$$Lu = g(x, u) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

is studied, with L a linear second order elliptic operator which satisfies the maximum principle and g is positive and becomes singular as

$$u \longrightarrow 0 \text{ uniformly in } x.$$

Their techniques are also based on the use of sub-supersolution theorems.

In the case that the problem depends on the parameter, several papers [**2, 17, 18, 20**] studied (1.1) when g and h are of particular forms. In particular, Coclite and Palmieri have proved in [**2**] that if $\alpha \geq 1$, then

$$\begin{cases} -\Delta u = u^{-\gamma} + (\lambda u)^{\alpha} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has at least one solution when λ is small and (1.2) has no solution when λ is large. Using iteration techniques, the problem has also been studied by Sun and Wu [**17**] when $0 \le \alpha < 1$, $0 < \gamma < N^{-1}$. Cîrstea, Gherghu and Rădulescu [**1**] have considered (1.1) for g nonincreasing, h nondecreasing and $p = 2$ and have proved (with some additional technical assumptions on g and h) that the problem

$$\begin{cases} -\Delta u = ag(u) + \lambda h(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

has a unique solution u_λ for all $\lambda \ge 0$ and u_λ is increasing with respect to λ (i.e., $0 \le \lambda_1 \le \lambda_2$ implies $u_{\lambda_1} \le u_{\lambda_2}$ in Ω), provided that

$$\lim_{s\to\infty} \frac{h(s)}{s} = 0 \text{ (see [\mathbf{1}, Theorem 1]),}$$

and if

$$\lim_{s\to\infty} \frac{h(s)}{s} > 0,$$

then there exists $\lambda^* > 0$ such that (1.3) has a solution when $\lambda \in (0, \lambda^*)$ and has no solution when $\lambda \ge \lambda^*$ (see [**1**, Theorem 2]). We also draw the reader's attention to the papers [**4, 6**] in which the existence and nonexistence of solutions to singular elliptic problems depending on two parameters were studied.

When $p \in (1, \infty)$, by using a sub-supersolution approach and a mountain pass theorem, Giacomoni, Schindler and Takáč [**5**] have proved that

$$\begin{cases} -\Delta_p u = \lambda u^{-\delta} + u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\delta \in (0, 1)$, $q \in (p-1, p^* - 1)$ (p^* is the critical Sobolev exponent defined by p), has multiple weak solutions (depending on the certain value of the parameter λ).

All of the papers mentioned above needed a monotonicity condition on the singular term g. Thus the question arises whether or not the existence of solutions for (1.1) is still true when the monotonicity property is removed. Hai, [**7, 8**], has given affirmative answers to this question in the case that Ω is an annulus, by establishing existence results for radial solutions which are solutions of associated ordinary differential equations.

We approach to solve (1.1) by proving a version of a sub-supersolution theorem for singular elliptic problems and then finding such a well-ordered pair of sub-supersolutions for the specific singular problem under consideration. With this method, we can remove not only the monotonicity condition but also some technical conditions on the singular terms in the papers above.

There are many sub-supersolution results available and we refer to [**11**–**13**] for some recent results for nonsingular nonlinear elliptic problems. Such results, however, are not directly applicable to singular elliptic problems. We hence establish a sub-supersolution theorem which is suitable to study the existence of solutions for

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the case $f(\cdot, 0)$ is undefined. This is done in the next section.

2. A sub-supersolution theorem. The aim of this section is to establish a sub-supersolution theorem for

$$-\Delta_p u = f(x,u) \quad \text{in } \Omega, \tag{2.1}$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$ and f is a Carathéodory function defined on $\Omega \times (0,\infty)$; i.e., $f(x,\cdot)$ is continuous on $(0,\infty)$ for almost every $x \in \Omega$ and $f(\cdot, s)$ is measurable for all $s > 0$.

Definition 2.1. A function u in $W^{1,p}_{\text{loc}}(\Omega)$ is called a subsolution (supersolution) to (2.1) in the sense of distributions, if, and only if:

$$u(x) > 0, \quad x \in \Omega, \tag{2.2}$$

$$f(\cdot, u(\cdot)) \in L^1_{\text{loc}}(\Omega), \tag{2.3}$$

and for all nonnegative functions $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx \le (\ge) \int_\Omega f(x,u)\varphi \, dx. \tag{2.4}$$

Note that the definition of subsolution and supersolution here is different from that in [**11**–**13**]. In fact, the function u in this definition might not necessarily be an element of $W^{1,p}(\Omega)$ and, therefore, its trace on $\partial\Omega$ need not be well-defined.

Remark 2.2. If u is a subsolution or a supersolution of (2.1) and if u belongs to $W^{1,p}(\Omega)$ and $f(\cdot, u(\cdot)) \in (W_0^{1,p}(\Omega))^*$, then u is a subsolution or supersolution respectively of (2.1) in the classical sense (see, e.g., [**11**–**13**]).

Definition 2.3. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, is called a solution to (2.1) in the sense of distributions, if, and only if:

$$u(x) > 0, \quad x \in \Omega, \tag{2.5}$$

and for all functions $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx = \int_\Omega f(x,u)\varphi \, dx. \tag{2.6}$$

The following is the main result in this section.

Theorem 2.4. *Assume that problem* (2.1) *has a subsolution* $\underline{u}$ *and a supersolution* $\overline{u} \in L^\infty_{\mathrm{loc}}(\Omega)$ *in the sense of distributions such that*

$$0 < \underline{u}(x) \le \overline{u}(x), \quad a.e.\ x \in \Omega.$$

Assume further, there exists a function $c \in L^\infty_{\mathrm{loc}}(\Omega)$ *such that*

$$|f(x,s)| \le c(x), \quad a.e.\ x \in \Omega,\ \text{for all } s \in [\underline{u}(x), \overline{u}(x)]. \tag{2.7}$$

Then problem (2.1) *has a solution* u *in the sense of distributions and* u *satisfies*

$$\underline{u} \le u \le \overline{u}, \quad a.e.\ \text{in } \Omega. \tag{2.8}$$

Proof. Let $\{\Omega_n\}_{n\in\mathbf{N}}$ be a sequence of smooth subdomains of Ω such that

$$\overline{\Omega}_n \subset \Omega_{n+1}, \quad n = 1, 2, \dots, \cup_{n\in\mathbf{N}}\Omega_n = \Omega.$$

We proceed with the proof by establishing some auxiliary results.

Lemma 2.5. *There exists a sequence* $\{v_n\}_{n\in\mathbf{N}} \subset W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ *such that*:

(i) $0 \le v_1(x) \le v_2(x) \le \cdots \le \overline{u}(x) - \underline{u}(x)$, *almost everywhere* $x \in \Omega$ *and*

(ii) *for each* $n \in \mathbf{N}$, *the restriction of* v_n *to* Ω_n *is a weak solution of*

$$\begin{cases} -\Delta_p(v_n + \underline{u}) = f(x, v_n + \underline{u}) & \textit{in } \Omega_n, \\ v_n = 0 & \textit{on } \partial\Omega_n. \end{cases}$$

Proof. Fix $n \in \mathbf{N}$. Note that $\underline{v} := 0$ and $\overline{v} := \overline{u} - \underline{u} \ge 0$ are, respectively, a subsolution and a supersolution (in the classical sense, see [**11**–**13**]) of

$$\begin{cases} -\Delta_p(v_n + \underline{u}) = f(x, v_n + \underline{u}) & \text{in } \Omega_n, \\ v_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{2.9}$$

Since, for almost every $x \in \Omega_n$, all $s \in (\underline{v}(x), \overline{v}(x))$,

$$|f(x, s + \underline{u})| \le c(x),$$

we may apply Remark 1.5 in [**13**] to find a minimal solution v_n, with respect to the pair $(\underline{v}, \overline{v})$, of problem (2.9) satisfying

$$\underline{v}(x) \le v_n(x) \le \overline{v}(x), \quad \text{a.e. } x \in \Omega_n.$$

This means any other solution v_n' of (2.9), such that

$$\underline{v}(x) \le v_n'(x) \le \overline{v}(x), \quad \text{a.e. } x \in \Omega_n,$$

must satisfy

$$v_n(x) \le v_n'(x), \quad \text{a.e. } x \in \Omega_n.$$

Since $\overline{v} \in L^\infty(\Omega_n)$, so is v_n. This, together with the regularity results in [**14**], implies that v_n is Hölder continuous. We may therefore consider v_n as a function in $W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ by defining $v_n = 0$ in $\Omega \setminus \Omega_n$.

Next, we show

$$v_n(x) \le v_{n+1}(x), \quad x \in \Omega,\ n = 1, 2, \cdots .$$

This inequality is clearly true when $x \in \Omega \setminus \Omega_n$. Assume then that there exists $n \in \mathbf{N}$ such that the Lebesgue measure of the set

$$\{y \in \Omega_n : v_n(y) > v_{n+1}(y)\}$$

is positive. We note that

$$v_{n+1} \mid_{\partial\Omega_n} \ge 0,$$

and

$$\int_{\Omega_n} |\nabla(v_{n+1} + \underline{u})|^{p-2}\nabla(v_{n+1} + \underline{u}) \cdot \nabla\varphi \, dx$$
$$= \int_{\Omega} f(x, v_{n+1} + \underline{u})\varphi \, dx,$$

for all $\varphi \in W_0^{1,p}(\Omega_n)$. Hence, v_{n+1} is a supersolution to (2.9) in the classical sense. We may apply Remark 1.5 in [**13**] again to find a solution w_n satisfying

$$0 \le w_n(x) \le \min\{v_n(x), v_{n+1}(x)\} \text{ a.e. } x \in \Omega_n.$$

Consequently,

$$w_n(x) < v_n(x), \quad x \in \{y \in \Omega_n : v_n(y) > v_{n+1}(y)\}.$$

This, on the other hand, may not happen, because v_n is the minimal solution of (2.9). □

Let u_n denote $v_n + \underline{u}$ for all $n \in \mathbf{N}$. The monotonicity of the sequence $\{v_n\}$ shows that $\{u_n\}$ converges to a function u at every point in $\overline{\Omega}$. We need to show that u is a solution of (2.1) in the sense of distributions.

Lemma 2.6. *For all domains $U \subset \Omega$, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that*

$$u_{n_k} \longrightarrow u \text{ in } W^{1,p}(U).$$

Proof. Let $\varphi \in C_0^\infty(\Omega)$ be such that $0 \le \varphi \le 1$ in Ω and $\varphi = 1$ in U. Let K denote the support of φ. Without loss of generality, assume that $K \subset \Omega_n$ for all $n \in \mathbf{N}$. Since v_n is a solution of (2.9), applying Hölder's inequality and the product rule of differentiation, we obtain for $n = 1, 2, \dots$,

$$\frac{1}{2^p} \int_K |\nabla(\varphi u_n)|^p \, dx \le C_1 + C_2 \bigg(\int_K |\nabla(\varphi u_n)|^p \, dx + C_3 \bigg)^{(p-1)/p},$$

where

$$C_1 = \int_K |\overline{u} \nabla(\varphi)|^p \, dx + \int_K c\varphi^p \overline{u} \, dx,$$

$$C_2 = 2^p \bigg(\int_K |p\overline{u} \nabla \varphi|^p \, dx \bigg)^{1/p},$$

$$C_3 = \int_K |\overline{u} \nabla \varphi|^p \, dx.$$

Therefore, $\{\varphi u_n\}$ is bounded in $W_0^{1,p}(K)$ and hence $\{u_n\}$ is bounded in $W^{1,p}(U)$. This implies that there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that

$$u_{n_k} \rightharpoonup u \text{ in } W^{1,p}(U),$$

because $\{u_n\}$ converges to u pointwise in Ω. The process above may be applied again to find a subsequence of $\{u_{n_k}\}$, still called $\{u_{n_k}\}$, such that

$$u_{n_k} \rightharpoonup u \text{ in } W^{1,p}(K).$$

Next, we show

$$u_{n_k} \longrightarrow u \text{ in } W^{1,p}(U).$$

It is sufficient to show that

$$|\nabla u_{n_k}| \longrightarrow |\nabla u| \text{ in } L^p(U),$$

because it follows then from Lebesgue's convergence theorem, that

$$u_n \longrightarrow u \text{ in } L^p(U).$$

Since v_{n_k} is a solution of

$$\begin{cases} -\Delta_p(v_{n_k} + \underline{u}) = f(x, v_{n_k} + \underline{u}) & \text{in } \Omega_{n_k}, \\ v_{n_k} = 0 & \text{on } \partial\Omega_{n_k}, \end{cases}$$

we have

$$\int_K |\nabla u_{n_k}|^{p-2}\nabla u_{n_k} \cdot \nabla(\varphi(u_{n_k}-u))\,dx = \int_K f(x,u_{n_k})\varphi(u_{n_k}-u)\,dx.$$

It follows from Lebesgue's convergence theorem and condition (2.7), that

$$\lim_{k\to\infty}\int_K f(x,u_{n_k})\varphi(u_{n_k}-u)\,dx = 0.$$

Thus,

$$\lim_{k\to\infty}\int_K |\nabla u_{n_k}|^{p-2}\nabla u_{n_k}\cdot\nabla(\varphi(u_{n_k}-u))\,dx = 0.$$

On the other hand, applying Hölder's inequality, we obtain

$$\left|\int_K (u_{n_k}-u)|\nabla u_{n_k}|^{p-2}\nabla u_{n_k}\cdot\nabla\varphi\,dx\right| \le C_4\|u_{n_k}-u\|_{L^p(K)},$$

where

$$C_4 = \|\nabla\varphi\|_{L^\infty(K)}\sup_{k\in\mathbf{N}}\left\{\|\nabla u_{n_k}\|_{L^p(K)}^{p-1}\right\}.$$

This, together with Lebesgue's convergence theorem, implies

$$\lim_{k\to\infty}\int_K \varphi|\nabla u_{n_k}|^{p-2}\nabla u_{n_k}\cdot\nabla(u_{n_k}-u)\,dx = 0.$$

Hence,

$$\lim_{k\to\infty}\int_K \varphi(|\nabla u_{n_k}|^{p-2}\nabla u_{n_k} - |\nabla u|^{p-2}\nabla u)\cdot\nabla(u_{n_k}-u)\,dx = 0.$$

Since the integrand is nonnegative and $\varphi = 1$ in U,

$$\lim_{k\to\infty}\int_U (|\nabla u_{n_k}|^{p-2}\nabla u_{n_k} - |\nabla u|^{p-2}\nabla u)\cdot\nabla(u_{n_k}-u)\,dx = 0.$$

It follows that

$$u_{n_k}\longrightarrow u$$

in $W^{1,p}(U)$. $\square$

Let $\xi \in C_0^\infty(\Omega)$ and

$$V = \{x \in \Omega : \xi(x) \neq 0\}.$$

Since, for $n \gg 1$, $V \subset \Omega_n$, we have

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \xi \, dx = \int_\Omega f(x, u_n) \xi \, dx.$$

By Lemma 2.6, we may assume that $\{u_n\}$ converges to u in $W^{1,p}(V)$. Letting $n \to \infty$, we obtain the assertion of Theorem 2.4. □

Remark 2.7. If both $\underline{u}$ and $\overline{u}$ are in $C(\overline{\Omega})$ and their value on $\partial\Omega$ is identically zero, then inequality (2.8) holds for all $x \in \Omega$ and u, therefore, solves (2.1) and satisfies the boundary condition

$$u = 0 \quad \text{on } \partial\Omega.$$

3. Hopf's lemma. In this section, we shall recall Hopf's lemma which is needed to prove some properties of eigenfunctions associated to the first eigenvalue λ_1 of $-\Delta_p$. Let $\phi \in C^1(\overline{\Omega})$ be a solution of

$$\begin{cases} -\Delta_p \phi = \lambda_1 \phi^{p-1} & \text{in } \Omega, \\ \phi > 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

(cf. [**14, 15**]). The following lemma is well-known when $p = 2$ and is a corollary of Lemma A.3 in [**16**].

Lemma 3.1. *For all $x \in \Omega$*

$$\frac{\partial \phi(x)}{\partial \nu} < 0,$$

where ν is the outward unit normal vector to $\partial\Omega$ at x.

Note that the maximum principle of Vázquez [**19**] is not applicable, since it requires $\Delta_p \phi \in L^2_{\text{loc}}(\Omega)$.

The following lemma gives a property of the eigenfunction in Lemma 3.1, which we will need to prove Remark 5.2.

Lemma 3.2. *Let* $\varphi \in C^1(\overline{\Omega})$. *Assume that for all* $x \in \partial\Omega$,

$$\frac{\partial \varphi(x)}{\partial \nu} < 0.$$

Then

$$\int_\Omega \varphi^r \, dx < \infty,$$

if and only if $r > -1$.

Lazer and McKenna [**10**] have proved this lemma for the eigenfunction $\varphi = \phi$ when $p = 2$. The general case may be proved in a similar way. (Note that this result is a general result implied by the behavior of the function at $\partial\Omega$.)

4. The singular elliptic problem. In this section, we shall present the main result of this paper, Theorem 4.1, and its proof. As mentioned in the first section, we shall employ arguments using the sub-supersolution theorem proved above. Thus, the main point here is the construction of a well-ordered pair of sub-supersolutions of (1.1).

Theorem 4.1. *Assume* g *satisfies*:

$$\exists \gamma > 0,\ C > 0 \text{ such that } g(s) \le C s^{-\gamma},\ \forall s \in (0, \infty). \tag{4.1}$$

Then:

(i) *if* $\limsup_{s \to 0^+} h(s)/s^{p-1} < \infty$, *there exists* $\tilde{\lambda} > 0$ *such that for all* $\lambda \in [0, \tilde{\lambda}]$, *problem* (1.1) *has a solution,*

(ii) *if there exists* $\alpha < p - 1$ *such that*

$$0 \le h(s) \le s^\alpha,\ \forall s \in [1, \infty),$$

then for all $\lambda \ge 0$, *problem* (1.1) *has a solution.*

Proof. For each $b > 0$, define the function Ψ_b on Ω as follows

$$\Psi_b = b\phi^t,$$

where $t \in (0,1)$ is such that

$$t(p-1+\gamma) \le p, \quad tp - t + \gamma t - p \le 0. \tag{4.2}$$

Note that equalities in (4.2) can be satisfied when $\gamma > 1$. A direct calculation shows that ϕ is a weak solution of

$$-\Delta_p \left(b\phi^t(x)\right) = (bt)^{p-1}\phi^{tp-t-p}(x)\left[q(t,x) + \frac{\lambda_1 \phi^p(x)}{2}\right],$$

or equivalently,

$$-\Delta_p \Psi_b(x) = (bt)^{p-1}\phi^{tp-t-p}(x)\left[q(t,x) + \frac{\lambda_1 \phi^p(x)}{2}\right], \tag{4.3}$$

where

$$q(t,x) = (1-t)(p-1)|\nabla\phi(x)|^p + \frac{\lambda_1 \phi^p(x)}{2}.$$

It follows from Lemma 3.1 that $\nabla\phi \ne 0$ on $\partial\Omega$. So, there exists $\beta > 0$, depending on t, such that $q(t,x) > \beta$, $x \in \overline{\Omega}$.

Lemma 4.2. *Assume that* $\limsup_{s\to 0^+} h(s)/s^{p-1} < \infty$. *Then there exists* $\tilde{\lambda} > 0$, *such that for all* $\lambda \in [0, \tilde{\lambda}]$, *problem* (1.1) *has a supersolution* $\overline{u} \in L^\infty(\Omega)$.

Proof. When b is large, with the help of (4.1) and (4.2), we conclude that

$$\begin{aligned}
\beta(bt)^{p-1}\phi^{tp-t-p} &- ag(\Psi_b) \\
&\ge \beta(bt)^{p-1}\phi^{tp-t-p} - C\|a\|_{L^\infty(\Omega)}\Psi_b^{-\gamma} \\
&= \beta(bt)^{p-1}\phi^{tp-t-p} - C\|a\|_{L^\infty(\Omega)}(b\phi^t)^{-\gamma} \\
&\ge \phi^{-t\gamma}\left[\beta(bt)^{p-1}\phi^{tp-t-p+t\gamma} - \frac{C\|a\|_{L^\infty(\Omega)}}{b^\gamma}\right] \\
&\ge 0,
\end{aligned}$$

where the constant C in the above calculation is given by (4.1). Thus,

$$(bt)^{p-1}\phi^{tp-t-p}(x)q(t,x) - a(x)g(\Psi_b(x)) \ge 0, \quad x \in \Omega. \tag{4.4}$$

Now, choose $\tilde{\lambda}$ small enough, so that

$$\tilde{\lambda}h(s) < \frac{\lambda_1 t^{p-1}}{2} s^{p-1}, \quad \forall s \in (0, \max_{x\in\overline{\Omega}} \Psi_b^t(x)].$$

For all $\lambda \in [0, \tilde{\lambda}]$,

$$\begin{aligned}\frac{\lambda_1}{2}(bt)^{p-1}\phi^{tp-t} - \lambda h(\Psi_b) &\geq \frac{\lambda_1}{2}(bt)^{p-1}\phi^{t(p-1)} - \frac{\lambda_1 t^{p-1}}{2}\Psi_b^{p-1} \\ &= \frac{\lambda_1}{2}(bt)^{p-1}\phi^{t(p-1)} - \frac{\lambda_1 (bt)^{p-1}}{2}\phi^{t(p-1)} \\ &= 0.\end{aligned}$$

This, (4.3) and (4.4) imply that $\overline{u} = \Psi_b$ is a supersolution of (1.1). □

Lemma 4.3. *Assume that there exists $\alpha < p-1$ such that*

$$0 \leq h(s) \leq s^\alpha, \quad \textit{for all } s \in [1, \infty).$$

Then for all $\lambda \geq 0$, (1.1) has a supersolution $\overline{u} \in L^\infty(\Omega)$.

Proof. We first choose b large, such that

$$\frac{1}{2}\beta b^{p-1+\gamma} t^{p-1} \min_{x\in\overline{\Omega}}\{\phi^{tp-t-p+\gamma t}(x)\} \geq C\|a\|_{L^\infty(\Omega)} \tag{4.5}$$

and

$$\frac{1}{2}\beta (bt)^{p-1} \geq \lambda M \max_{x\in\overline{\Omega}}\{\phi(x)\}^{t+p-tp}, \tag{4.6}$$

where

$$M = \max_{s\in[0,\Lambda]} h(s), \;\; \Lambda = \max\left\{\left(\frac{2\lambda}{\lambda_1 t^{p-1}}\right)^{1/(p-1-\alpha)}, 1\right\}.$$

Define $\overline{u} := \Psi_b$. The choice of b in (4.5) implies

$$\frac{1}{2}\beta b^{p-1} t^{p-1} \phi^{tp-t-p} \geq a g(\overline{u}), \;\text{ in } \Omega. \tag{4.7}$$

Using (4.3), (4.6) and (4.7), we obtain

$$\begin{aligned}-\Delta_p \overline{u} &\geq ag(\overline{u}) + \frac{\beta(bt)^{p-1}\phi^{tp-t-p}}{2} + \frac{\lambda_1 t^{p-1}\overline{u}^{p-1}}{2} \\ &\geq ag(\overline{u}) + \lambda h(\overline{u}) + \frac{\lambda_1 t^{p-1}\overline{u}^{p-1}}{2} \\ &\geq ag(\overline{u}) + \lambda h(\overline{u}),\end{aligned}$$

on the set $\{x \in \Omega : 0 < \overline{u}(x) \leq \Lambda\}$. On the complementary set, $\overline{u} \geq 1$, and

$$\begin{aligned}\frac{\lambda_1 t^{p-1}\overline{u}^{p-1}}{2} &= \frac{\lambda_1 t^{p-1}\overline{u}^{p-1-\alpha}}{2}\overline{u}^{\alpha} \\ &\geq \frac{\lambda_1 t^{p-1}\Lambda^{p-1-\alpha}}{2}h(\overline{u}) \\ &\geq \lambda h(\overline{u}).\end{aligned}$$

Hence, by (4.3), (4.6) and (4.7)

$$\begin{aligned}-\Delta_p \overline{u} &\geq ag(\overline{u}) + \frac{\beta(bt)^{p-1}\phi^{tp-t-p}}{2} + \lambda h(\overline{u}) \\ &\geq ag(\overline{u}) + \lambda h(\overline{u}),\end{aligned}$$

whenever $\overline{u} \geq \Lambda$. So, $\overline{u}$ is a supersolution of (1.1). □

Next, we find a subsolution for (1.1). Since

$$\lim_{\varepsilon \to 0^+} \left(g(\varepsilon\phi(x)) + \lambda h(\varepsilon\phi(x))\right) = \infty$$

and

$$\lim_{\varepsilon \to 0^+} (\varepsilon\phi(x))^{p-1} = 0,$$

uniformly with respect to $x \in \Omega$, we can find $\varepsilon > 0$ and $M > 0$ such that

$$\lambda_1(\varepsilon\phi(x))^{p-1} < M < g(\varepsilon\phi(x)) + \lambda h(\varepsilon\phi(x)), \quad x \in \Omega.$$

Thus, for all $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$

$$\begin{aligned}\int_\Omega |\nabla(\varepsilon\phi)|^{p-2}\nabla(\varepsilon\phi)\cdot\nabla\varphi\,dx &= \int_\Omega \lambda_1(\varepsilon\phi)^{p-1}\varphi\,dx \\ &\leq \int_\Omega \left(g(\varepsilon\phi) + h(\varepsilon\phi)\right)\varphi\,dx.\end{aligned}$$

It follows that $\underline{u} = \varepsilon\phi$ is a subsolution of (1.1).

Since the supersolution $\overline{u}$, obtained in Lemma 4.2 and Lemma 4.3 is of the form

$$\overline{u} = b\phi^t,$$

for some $b > 0$ and $0 < t < 1$, we can find ε small enough that

$$\underline{u}(x) \le \overline{u}(x), \quad x \in \Omega.$$

It follows from Theorem 2.4 and Remark 2.7 that there exists a solution u of (1.1) satisfying

$$\varepsilon\phi(x) \le u(x) \le b\phi^t(x), \quad x \in \Omega$$

for some $0 < \varepsilon \ll 1$, $b \gg 1$, and $0 < t < 1$. □

5. Concluding remarks.

Remark 5.1. It follows from Theorem 4.1 that there exist $b > 0$ and $t \in (0, 1)$ such that

$$0 < u(x) < b\phi^t(x), \quad x \in \Omega,$$

where u is a solution of (1.1) obtained by Theorem 4.1.

When $\gamma \ge (2p - 1)/(p - 1)$, we let $t = p/(p - 1 + \gamma) \in (0, 1)$ so that the inequalities in condition (4.2) hold. In this case, under an additional condition on g, the solution u in Theorem 4.1 is not a weak solution of (1.1). This is shown by the following remark.

Remark 5.2. Assume in addition to (4.1) that g satisfies

$$g(s) \ge C^{-1}s^{-\gamma}, \quad \text{for all } s > 0, \tag{5.1}$$

where C and γ are defined in (4.1). Then if $\gamma \ge (2p - 1)/(p - 1)$, the solution u obtained in Theorem 4.1 is not in $W_0^{1,p}(\Omega)$.

Proof. Let u be the solution obtained from Theorem 4.1. It follows from Remark 5.1 that there exists a $b > 0$ such that

$$u(x) \le b\phi^{p/(p-1+\gamma)}(x), \quad \text{a.e. } x \in \Omega.$$

Thus

$$\int_\Omega a u^{1-\gamma}\,dx \geq \int_\Omega a(b\phi^{p/(p-1+\gamma)})^{1-\gamma}\,dx = \infty, \tag{5.2}$$

which follows from Lemma 3.2.

Suppose, contrary to the assertion of the remark, that $u \in W_0^{1,p}(\Omega)$. Then, there exists a sequence $\{w_n\} \subset C_0^\infty(\Omega)$ such that

$$w_n \longrightarrow u \text{ in } W_0^{1,p}(\Omega).$$

Define

$$w_n^+ := \max\{w_n, 0\} \in W_0^{1,p}(\Omega), \quad n = 1, 2, \dots .$$

Since $u \geq 0$,

$$w_n^+ \longrightarrow u \text{ in } W_0^{1,p}(\Omega).$$

Without loss of generality, assume that w_n^+ converges to u almost everywhere in Ω. Using Fatou's lemma and inequality (5.2), we obtain

$$\lim_{n\to\infty} \int_\Omega w_n^+ a u^{-\gamma}\,dx = \infty.$$

Since $w_n^+ \in W_0^{1,p}(\Omega)$ and condition (5.1) holds, we have

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w_n^+ dx &= \int_\Omega (a w_n^+ g(u) + \lambda h(u) w_n^+)\,dx \\ &\geq \int_\Omega (C^{-1} a w_n^+ u^{-\gamma} + \lambda h(u) w_n^+)\,dx.\end{aligned}$$

Hence

$$\int_\Omega |\nabla u|^p dx = \lim_{n\to\infty} \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w_n^+ dx = \infty,$$

which contradicts the assumption that $u \in W_0^{1,p}(\Omega)$. $\square$

The following example illustrates this remark.

Example 5.3. In the case $N = 1$ and $\Omega = (0,1)$, the function u defined by

$$x \longmapsto \sqrt{2x(1-x)}$$

does not belong to $W_0^{1,2}(0,1)$ and is the unique solution of the boundary value problem

$$-u'' = u^{-3} \text{ in } (0,1), \quad u(0) = u(1) = 0.$$

The uniqueness of the solution can be deduced from the fact that the function

$$s \longmapsto s^{-3}$$

is nonincreasing and its smoothness in $(0,1)$.

When $p = 2$, we may use regularity techniques from [**10**] to show that the solution obtained in Theorem 2.4 is a classical solution, provided that the function f is Lipschitz continuous and $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$. Thus, if a, g and h are Lipschitz continuous, then the solution u obtained in Theorem 4.1 is a classical solution.

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