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SUB-SUPERSOLUTION METHOD IN VARIATIONAL INEQUALITIES WITH MULTIVALUED OPERATORS GIVEN BY INTEGRALS

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In memory of Professor Lloyd Jackson

ABSTRACT. We are concerned in this paper with the existence of solutions of the variational inequality

$$\begin{cases} \int_\Omega A(x, \nabla u)(\nabla v - \nabla u)\, dx \\ + \int_\Omega f(x,u)(v-u)\, dx \geq \langle L, v-u\rangle, \forall v \in K \\ u \in K, \end{cases}$$

where the functions A and f are multivalued. Both coercive and noncoercive cases are considered. In the noncoercive case, we follow a sub-supersolution approach and obtain further properties of solutions and also of sub-supersolutions.

1. Introduction. This paper is the next step in our study of nonsmooth nonconvex problems by sub-supersolution method developed in [**4–6, 12, 13, 15, 16**], etc. We are concerned here with (multivalued) variational inequalities of the form

(1.1)
$$\begin{cases} \int_\Omega A(x, \nabla u)(\nabla v - \nabla u)\, dx + \int_\Omega f(x,u)(v-u)\, dx \geq \langle L, v-u\rangle, \\ \forall v \in K, \quad u \in K, \end{cases}$$

where all involved terms are multivalued, that is, the functions $A(x, \xi)$ in the principal term and $f(x,u)$ in the lower order term are set-valued functions and the right hand side L could vary in a subset of the dual space. The integrals in (1.1) are therefore integrals of set-valued functions. By considering multivalued lower order terms, we include as a particular case hemivariational-variational inequalities where $f(x,u) = \partial j(x,u)$, the Clarke generalized gradient of a locally Lipschitz function j. However, f can be a general set-valued function

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without being a generalized derivative of some nonsmooth potential function. On the other hand, the consideration of the multivalued principal term $A(x, \xi)$ could describe some cases of variational inequalities. The theorems here therefore complement, generalize and unify several results previously established in the works cited above. The consideration of multivalued terms in (1.1) is motivated by the papers [**8, 10**]. Compared to [**8**], we consider here variational inequalities instead of equations and the lower order term is not in general given by Clarke's subdifferentials of locally Lipschitz functions as considered there. The principal operator function $A(x, \xi)$ in (1.1) does not depend on u as assumed in that paper; on the other hand, we obtain the existence and comparison properties of solutions of (1.1), together with the existence of its extremal solutions as well. Moreover, the results here are obtained by a different approach from that in [**8**] where a fixed point theorem in [**7**] was employed. The main difference between [**10**] and this paper is that all the operators here are multivalued. Moreover, the right hand side L could vary in the dual space, which means that in the sub-supersolution approach which we follow here for the noncoercive case, the right hand sides L in (1.1) can be different from the right hand sides in the inequalities that the sub- and supersolutions satisfy. Therefore, the existence/enclosure theorem in the sequel could also be seen as a range theorem. It also reflects in a certain sense the monotone dependence of the solutions on the right hand side. Another property of sub- and supersolutions, well known for equalities with single-valued operators, is that the maximum (respectively, minimum) of subsolutions (respectively, supersolutions) is also a subsolution (respectively, supersolution) (cf., e.g., [**6, 9, 11, 16**]). We extend this property to variational inequalities with multivalued terms.

The plan of this paper is as follows. In Section 2, we present a precise setting of problem (1.1) together with necessary assumptions and preparatory results. The existence of solutions of (1.1) under certain coercivity conditions is next considered. Appropriate concepts of sub- and supersolutions of (1.1) are introduced in Section 3. Next, we present our main results about existence and other properties of solutions of (1.1) such as the directness and compactness of the solution sets and the existence of extremal solutions. The extension of the property on minima of subsolutions to inequalities with multivalued operators is given in Section 4.

2. Setting of the problem–Coercive case. For a normed vector space X, we use the notation $\mathcal{K}(X) = \{P \subset X : P \neq \varnothing, P \text{ is convex and compact}\}$. Let $A : \Omega \times \mathbf{R}^N \to \mathcal{K}(\mathbf{R}^N)$ satisfy the following conditions:

(A1) A is graph measurable.

(A2) For almost every $x \in \Omega$, $A(x, \cdot)$ is strictly monotone, i.e., for all $\xi_1, \xi_2 \in \mathbf{R}^N$, all $\xi_1^* \in A(x, \xi_1), \xi_2^* \in A(x, \xi_2)$, if $\xi_1 \neq \xi_2$ then $(\xi_2^* - \xi_1^*)(\xi_2 - \xi_1) > 0$.

(A3) For almost every $x \in \Omega$, $A(x, \cdot)$ has closed graph.

(A4) There exist $p \in (1, \infty)$, $a_1 \in L^{p'}(\Omega)$ (p' is the Hölder conjugate of p), and $b_1 \in [0, \infty)$ such that for almost every $x \in \Omega$, all $\xi \in \mathbf{R}^N$,

$$\sup\{|\zeta| : \zeta \in A(x, \xi)\} \leq a_1(x) + b_1|\xi|^{p-1}. \tag{2.1}$$

(A5) There exist $a_2 \in L^1(\Omega)$ and $b_2 \in (0, \infty)$ such that for almost every $x \in \Omega$, all $\xi \in \mathbf{R}^N$,

$$\zeta\xi \geq b_2|\xi|^p - a_2(x), \quad \text{for all } \zeta \in A(x, \xi). \tag{2.2}$$

Concerning f, we assume the following.

(F1) $f : \Omega \times \mathbf{R} \to \mathcal{K}(\mathbf{R})$ is graph measurable.

(F2) For almost every $x \in \Omega$, $f(x, \cdot)$ is upper semicontinuous from $\mathbf{R}$ to $\mathcal{K}(\mathbf{R})$.

We also need the following (sub-critical) growth condition:

(F3) There exist $q \in (1, p^*)$ (p^* is the Sobolev conjugate of p) and $a_3 \in L^{q'}(\Omega)$, $b_3 \geq 0$ such that

$$\sup\{|v| : v \in f(x, u)\} \leq a_3(x) + b_3|u|^{q-1}, \tag{2.3}$$

for almost every $x \in \Omega$, all $u \in \mathbf{R}$.

Let $W^{1,p}(\Omega)$ be the first-order Sobolev space with the usual norm

$$\|u\| = \|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |u|^p \, dx + \sum_{j=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_j} \right|^p dx \right)^{1/p}, \quad u \in W^{1,p}(\Omega).$$

Assume K is a (nonempty) closed, convex subset of $W^{1,p}(\Omega)$. For measurable functions $u : \Omega \to \mathbf{R}$ and $w : \Omega \to \mathbf{R}^N$, we denote by $\tilde{f}(u)$ and $\widetilde{A}(u)$ the sets of measurable selections of $f(\cdot, u)$ and $A(\cdot, w)$, that is, $\tilde{f}(u) = \{\eta : \Omega \to \mathbf{R} : \eta \text{ is measurable and } \eta(x) \in f(x, u(x)) \text{ for almost every } x \in \Omega\}$ and $\widetilde{A}(w) = \{\xi : \Omega \to \mathbf{R}^N : \xi \text{ is measurable and } \xi(x) \in A(x, w(x)) \text{ for almost every } x \in \Omega\}$. From our assumptions on A and f, $\tilde{f}(u)$ and $\widetilde{A}(w)$ are nonempty. Moreover, from (A4), if $u \in W^{1,p}(\Omega)$ then $\widetilde{A}(\nabla u) \subset [L^{p'}(\Omega)]^N$. Similarly, (F3) implies that $\tilde{f}(u) \subset L^{q'}(\Omega)$ whenever $u \in L^q(\Omega)$.

Let us define $\mathcal{A} : W^{1,p}(\Omega) \to 2^{[W^{1,p}(\Omega)]^*}$ by $\mathcal{A}(u) = \langle \widetilde{A}(\nabla u), \nabla(\cdot)\rangle$, i.e.,

$$(2.4) \qquad \langle \mathcal{A}(u), v\rangle = \int_\Omega \widetilde{A}(\nabla u)\nabla v \, dx = \left\{ \int_\Omega \xi \nabla v \, dx : \xi \in \widetilde{A}(\nabla u)\right\}.$$

Also, we use for simplicity the notation $\tilde{f}$ for the mapping $u \mapsto \tilde{f}(u)$ from $L^q(\Omega)$ from $L^{q'}(\Omega)$. Let us recall some properties of $\mathcal{A}$ and $\tilde{f}$ that will be needed for our arguments later.

Lemma 2.1 ([**8**, Lemma 1]). *Under assumptions* (A1)–(A5)*, the mapping $\mathcal{A}$ is maximal monotone from $\mathcal{K}([W^{1,p}(\Omega)]^*)$.*

This result was in fact established in [**8**] for $\mathcal{A}$ on $W_0^{1,p}(\Omega)$. The proofs there can however be extended to the case of $W^{1,p}(\Omega)$ with trivial modifications.

Let $i_q : W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $u \mapsto i_q(u) = u$, be the identity embedding of $W^{1,p}(\Omega)$ into $L^q(\Omega)$, which is continuous (and compact) and $i_q^* : L^{q'}(\Omega) = [L^q(\Omega)]^* \to [W^{1,p}(\Omega)]^*$ its adjoint, which is the usual projection mapping for linear functionals on $L^q(\Omega)$, $i_q^*(w) = w|_{W^{1,p}(\Omega)}$ for $w \in [L^q(\Omega)]^*$. A useful property of the lower order term is given by the following result.

Lemma 2.2 ([**10**, Lemma 3.4]). *If f satisfies* (F1)–(F3)*, then the operator $i_q^* \tilde{f} i_q$ is pseudomonotone from $W^{1,p}(\Omega)$ to $\mathcal{K}([W^{1,p}(\Omega)]^*)$.*

With the above notation and definitions, we see that the multivalued variational inequality (1.1) can be formulated in a precise way as:

Find $u \in K$ and $\xi \in \mathcal{A}(u)$, $\eta \in \tilde{f}(u)$ such that

$$\int_\Omega \xi(\nabla v - \nabla u)\,dx + \int_\Omega \eta(v-u)\,dx \geq \langle L, v-u\rangle, \quad \text{for all } v \in K, \tag{2.5}$$

where $\langle\cdot,\cdot\rangle$ now denotes the dual pairing between $W^{1,p}(\Omega)$ and $[W^{1,p}(\Omega)]^*$. This inequality is equivalent to finding $u \in K$ and $\zeta \in (\mathcal{A} + i_q^*\tilde{f}i_q)(u)$ such that

$$\langle \zeta - L, v-u\rangle \geq 0, \quad \text{for all } v \in K, \tag{2.6}$$

i.e.,

$$\langle \zeta - L, v-u\rangle + I_K(v) - I_K(u) \geq 0, \quad \text{for all } v \in W^{1,p}(\Omega).$$

We have the following existence theorems for (2.5) (and (2.6)) under some coercivity conditions.

Theorem 2.3. *Under assumptions* (A1)–(A5) *and* (F1)–(F3)*, if* $L \in [W^{1,p}(\Omega)]^*$ *is such that for some* $u_0 \in K$*,*

$$\lim_{\substack{\|u\|_{W^{1,p}(\Omega)}\to\infty \\ u\in K}} \left\{ \inf_{\substack{\xi\in\mathcal{A}(u)\\ \eta\in\tilde{f}(u)}} \left[\int_\Omega \xi\nabla(u-u_0)\,dx + \int_\Omega \eta(u-u_0)\,dx\right] - \langle L, u-u_0\rangle \right\} = \infty, \tag{2.7}$$

then (2.5) (*or equivalently* (2.6)) *has a solution.*

As a corollary of Theorem 2.3, we have

Corollary 2.4. *Under assumptions* (A1)–(A5) *and* (F1)–(F3)*, if for some* $u_0 \in K$*,*

$$\lim_{\substack{\|u\|_{W^{1,p}(\Omega)}\to\infty \\ u\in K}} \left\{ \inf_{\substack{\xi\in\mathcal{A}(u)\\ \eta\in\tilde{f}(u)}} \left[\frac{\int_\Omega \xi\nabla(u-u_0)\,dx + \int_\Omega \eta(u-u_0)\,dx}{\|u\|_{W^{1,p}(\Omega)}}\right]\right\} = \infty, \tag{2.8}$$

then for any $L \in [W^{1,p}(\Omega)]^*$, (2.5) (*or equivalently* (2.6)) *has a solution.*

We state some preparatory results for the proof of Theorem 2.3 and refer to the corresponding papers for further details.

Lemma 2.5. *Let X be a reflexive Banach space.*

(a) ([**1**, Proposition 8]). *A maximal monotone mapping T from X into 2^{X^*} with effective domain $D(T) = X$ is pseudomonotone.*

(b) ([**1**, Proposition 9]). *If T_1 and T_2 are pseudomonotone mappings from X into 2^{X^*}, then $T_1 + T_2$ is a pseudomonotone mapping from X into 2^{X^*}.*

Lemma 2.6 ([**10**, Theorem 3.1]). *Let X be a reflexive Banach space and $T : X \to 2^X$ be a multivalued mapping such that:*

(T1) *For each $x \in X$, $T(x)$ is nonempty, convex, and closed in X^*.*

(T2) (Pseudomonotone property). *If $\{x_n\} \subset X$, $\{x_n^*\} \subset X^*$ are sequences such that $x_n^* \in T(x_n)$, for all $n \in \mathbf{N}$, $x_n \rightharpoonup x$ (weakly) in X, and*

$$\limsup \langle x_n^*, x_n - x\rangle \le 0,$$

then to each $y \in X$, there exists an $x^*(y) \in T(x)$ such that

$$\liminf \langle x_n^*, x_n - y\rangle \ge \langle x^*(y), x - y\rangle.$$

(T3) *For each $x_0 \in K$, each bounded subset B of X, there exists a constant $N(B, x_0) \in \mathbf{R}$ such that $\langle x^*, x - x_0\rangle \ge N(B, x_0)$ for all $x \in B$, all $x^* \in T(x)$.*

Assume K is a nonempty closed convex subset of X and $\phi : X \to \mathbf{R} \cup \{\infty\}$ is a proper convex, lower semicontinuous functional such that $D(\phi) \cap K \neq \varnothing$. Let $f \in X^*$.

If there exists $a \in D(\phi) \cap K$ such that

$$\lim_{\substack{x \in K \\ \|x\| \to \infty}} \left(\inf_{x^* \in T(x)} [\langle x^* - f, x - a\rangle + \phi(x)] \right) = \infty, \tag{2.9}$$

then there exist $x_0 \in K$ and $x_0^* \in T(x_0)$ such that

$$\langle x_0^* - f, x - x_0\rangle + \phi(x) - \phi(x_0) \ge 0, \quad \textit{for all } x \in K. \tag{2.10}$$

Proof of Theorem 2.3. From the assumptions on A, we have $D(\mathcal{A}) = W^{1,p}(\Omega)$. Hence, from Lemmas 2.1 and 2.5 (a), $\mathcal{A}$ is pseudomonotone from $W^{1,p}(\Omega)$ into $2^{[W^{1,p}(\Omega)]^*}$, which together with Lemmas 2.2 and 2.5 (b), implies that the sum $\mathcal{A} + i_q^* \tilde{f} i_q$ is also a pseudomonotone mapping from $W^{1,p}(\Omega)$ into $2^{[W^{1,p}(\Omega)]^*}$.

Next, we note that because $\mathcal{A}$ and $\tilde{f}$ are bounded operators on their corresponding spaces, $\mathcal{A}+i_q^* \tilde{f} i_q$ is bounded from $W^{1,p}(\Omega)$ to $2^{[W^{1,p}(\Omega)]^*}$. This property, together with the pseudomonotonicity of $\mathcal{A} + i_q^* \tilde{f} i_q$, shows that the assumptions (T1), (T2), and (T3) in Lemma 2.6 are fulfilled. The coercivity there, in the particular case of inequality (2.5), is the same as (2.7). The existence of solutions of (2.5) now follows from Lemma 2.6. □

3. Existence and enclosure properties by sub-supersolution method. If coercivity conditions such as (2.7) or (2.8) do not hold then the existence of solutions of (2.5) (or (2.6)) is in general not guaranteed. However, if sub- and supersolutions of this inequality (in certain appropriate sense) exist, then the growth condition of the lower term could be reduced to a local one. In this case, not only the existence but also some other qualitative properties of solutions of (2.5) can be obtained. First, let us introduce the concepts of sub- and supersolutions for (2.5).

Definition 3.1. Let $\underline{L} \in [W^{1,p}(\Omega)]^*$. We say that $\underline{u} \in W^{1,p}(\Omega)$ is a subsolution of (2.5) with respect to $\underline{L}$ if there exist $q \in [1, p^*)$ and $\underline{\zeta} \in [L^{p'}(\Omega)]^N$, $\underline{\eta} \in L^{q'}(\Omega)$ such that:

$$\underline{\zeta}(x) \in A(x, \nabla \underline{u}(x)), \tag{3.1}$$

$$\underline{\eta}(x) \in f(x, \underline{u}(x)) \text{ for almost every } x \in \Omega, \tag{3.2}$$

(that is, $\underline{\zeta} \in [L^{p'}(\Omega)]^N \cap \widetilde{A}(\nabla \underline{u})$ and $\underline{\eta} \in L^{q'}(\Omega) \cap \tilde{f}(\underline{u})$), and

$$\int_\Omega \underline{\zeta}(\nabla v - \nabla \underline{u})\, dx + \int_\Omega \underline{\eta}(v - \underline{u})\, dx \geq \langle \underline{L}, v - \underline{u} \rangle, \tag{3.3}$$

for all $v \in \underline{u} \wedge K := \{\underline{u} \wedge w := \min\{\underline{u}, w\} : w \in K\}$.

Similarly, $\overline{u} \in W^{1,p}(\Omega)$ is a supersolution of (2.5) with respect to $\overline{L} \in [W^{1,p}(\Omega)]^*$ if there exists a $q \in [1, p^*)$ and $\overline{\zeta} \in [L^{p'}(\Omega)]^N$, $\overline{\eta} \in L^{q'}(\Omega)$ such that:

$$\overline{\zeta}(x) \in A(x, \nabla \overline{u}(x)) (\text{i.e. } \overline{\zeta} \in [L^{p'}(\Omega)]^N \cap \widetilde{A}(\nabla \overline{u})), \tag{3.4}$$

$$\overline{\eta}(x) \in f(x, \overline{u}(x) \text{ for a.e. } x \in \Omega \text{ (i.e. } \overline{\eta} \in L^{q'}(\Omega) \cap \tilde{f}(\overline{u})), \tag{3.5}$$

and

$$\int_\Omega \overline{\zeta}(\nabla v - \nabla \overline{u})\, dx + \int_\Omega \overline{\eta}(v - \overline{u})\, dx \geq \langle \overline{L}, v - \overline{u} \rangle, \tag{3.6}$$

for all $v \in \overline{u} \vee K := \{\overline{u} \vee w := \max\{\overline{u}, w\} : w \in K\}$.

Let $W_+^{1,p} = \{u \in W^{1,p}(\Omega) : u \geq 0$ almost everywhere in $\Omega\}$ be the positive cone of $W^{1,p}(\Omega)$. We induce from $W_+^{1,p}$ the usual partial ordering on $W^{1,p}(\Omega)$ and on $[W^{1,p}(\Omega)]^*$: For $u_1, u_2 \in W^{1,p}(\Omega)$ and $L_1, L_2 \in [W^{1,p}(\Omega)]^*$, we define the partial orderings "$\leq$" by

$$u_1 \leq u_2 \iff u_2 - u_1 \in W_+^{1,p},$$

and

$$L_1 \leq L_2 \iff \langle L_2 - L_1, w \rangle \geq 0, \text{ for all } w \in W_+^{1,p}$$

(for simplicity, we use the same notation for both ordering relations).

We are now ready to state and prove our main theorem of this section.

Theorem 3.2. *Assume A and f satisfy* (A1)–(A5) *and* (F1)–(F3) *and there exist subsolutions $\underline{u}_i$ of* (2.5) *with respect to $\underline{L}_i$* $(i = 1, \dots, k)$ *and supersolutions $\overline{u}_j$ of* (2.5) *with respect to $\overline{L}_j$* $(j = 1, \dots, m)$. *Assume*

$$\underline{u} = \max\{\underline{u}_i : 1 \leq i \leq k\} \leq \overline{u} = \min\{\overline{u}_j : 1 \leq j \leq m\}, \tag{3.7}$$

and $L \in [W^{1,p}(\Omega)]^*$ satisfies

$$\underline{L}_i \leq L \leq \overline{L}_j, \text{ for all } i \in \{1, \dots, k\}, \quad \text{for all } j \in \{1, \dots, m\}. \tag{3.8}$$

Furthermore, f has the following subcritical local growth between $\underline{u}$ and $\overline{u}$: There exist $q \in [1, p^*)$ and $a_4 \in L^{q'}(\Omega)$ such that

$$\sup\{|\xi| : \xi \in f(x, u)\} \leq a_4(x), \tag{3.9}$$

for almost every $x \in \Omega$, *all* $u \in [\underline{u}(x), \overline{u}(x)]$. *Then there exists a solution* u *of* (2.5) *such that*

$$\underline{u} \le u \le \overline{u}. \tag{3.10}$$

Proof. The proof follows the same lines as those in Theorem 3.7 of [**10**] and only an outline is given here together with necessary adaptations and modifications; we refer to [**10**] for more details and complete arguments. First, we note that the exponents q's in the definitions of sub- and supersolutions and the growth condition (3.9) can be assumed, without loss of generality, to be the same.

Let $\underline{u}_i, \underline{\eta}_i, \underline{\zeta}_i$ $(1 \le i \le k)$ and $\overline{u}_j, \overline{\eta}_j, \overline{\zeta}_j$ $(1 \le j \le m)$ satisfy (3.1)–(3.3) and (3.4)–(3.6) as in the definitions of sub- and supersolutions. We define the functions $\underline{\eta}$ and $\overline{\eta}$ as follows. Let $\Omega_1 = \{x \in \Omega : \underline{u}(x) = \underline{u}_1(x)\}$, and

$$\Omega_i = \left\{ x \in \Omega \Big\backslash \bigcap_{l=1}^{i-1} \Omega_l : \underline{u}(x) = \underline{u}_i(x) \right\}$$

for $i = 2, \dots, k$. Similarly, let $\Omega^1 = \{x \in \Omega : \overline{u}(x) = \overline{u}_1(x)\}$, and

$$\Omega^j = \left\{ x \in \Omega \Big\backslash \bigcap_{l=1}^{j-1} \Omega^l : \overline{u}(x) = \overline{u}^j(x) \right\}$$

for $j = 2, \dots, m$. From their definitions, Ω_i $(1 \le i \le k)$ (respectively, Ω^j $(1 \le j \le m)$) are disjoint measurable subsets of Ω and $\Omega = \cup_{i=1}^k \Omega_i = \cup_{j=1}^m \Omega^j$. We define $\underline{\eta} = \sum_{i=1}^k \underline{\eta}_i \chi_{\Omega_i}$ and $\overline{\eta} = \sum_{j=1}^m \overline{\eta}_j \chi_{\Omega^j}$, where χ_A $(A \subset \Omega)$ is the characteristic function of A. It is clear that $\underline{\eta}, \overline{\eta} \in L^{q'}(\Omega)$ and $\underline{\eta}(x) \in f(x, \underline{u}(x)), \overline{\eta}(x) \in f(x, \overline{u}(x))$ for almost every $x \in \Omega$.

Next, we define the truncated function $f_0(x, u)$ of $f(x, u)$ as in [**10**]: f_0 is a function from $\Omega \times \mathbf{R}$ to $2^{\mathbf{R}}$ given by

$$f_0(x, u) = \begin{cases} \{\underline{\eta}(x)\} & \text{if } u < \underline{u}(x) \\ f(x, u) & \text{if } \underline{u}(x) \le u \le \overline{u}(x) \\ \{\overline{\eta}(x)\} & \text{if } u > \overline{u}(x). \end{cases} \tag{3.11}$$

Then, as observed in [**10**], f_0 satisfies (F_1) and (F_2). Moreover, from (3.9), we see that

$$\sup\{|\xi| : \xi \in f_0(x, u(x))\} \le a_4(x) + |\overline{\eta}(x)| + |\underline{\eta}(x)| \text{ for a.e. } x \in \Omega,$$

where $a_4 + |\overline{\eta}| + |\underline{\eta}| \in L^{q'}(\Omega)$. We also need the truncation function $b : \Omega \times \mathbf{R} \to \mathbf{R}$ given by

$$b(x,u) = \begin{cases} [u - \overline{u}(x)]^{p-1} & \text{if } u > \overline{u}(x) \\ 0 & \text{if } \underline{u}(x) \le u \le \overline{u}(x) \\ -[\underline{u}(x) - u]^{p-1} & \text{if } u < \underline{u}(x), \text{ for } x \in \Omega,\ u \in \mathbf{R}, \end{cases}$$

and its corresponding Niemytskii operator $\mathcal{B} : L^p(\Omega) \to L^{p'}(\Omega) = [L^p(\Omega)]^*$ defined by

$$\langle \mathcal{B}(u), v\rangle_{L^{p'}(\Omega), L^p(\Omega)} = \int_\Omega b(x,u) v \, dx, \text{ for all } u, v \in L^p(\Omega).$$

We see that $\mathcal{B}$ is a bounded continuous operator and $i_p^* \mathcal{B} i_p$ is a pseudomonotone operator from $W^{1,p}(\Omega)$ into its dual. Next, we put, for $x \in \Omega$ and $u \in \mathbf{R}$, $T_i(x,u) = |\underline{\eta}_i(x) - \underline{\eta}(x)|\sigma[(u - \underline{u}_i(x))/(\underline{u}(x) - \underline{u}_i(x))]$ $(1 \le i \le k)$, and $T^j(x,u) = |\overline{\eta}_j(x) - \overline{\eta}(x)|[1 - \sigma(u - \overline{u}(x))/(\overline{u}_j(x) - \overline{u}(x))]$ $(1 \le j \le m)$, where

$$\sigma(s) = \begin{cases} 1 & s < 0 \\ 1 - s & 0 \le s \le 1 \\ 0 & s > 1. \end{cases}$$

Let us consider the following auxiliary variational inequality of finding $u \in K$, $\zeta \in [L^p(\Omega)]^N$, $\eta \in L^q(\Omega)$ such that

$$\zeta(x) \in A(x, \nabla u(x)), \tag{3.12}$$

$$\eta(x) \in f_0(x, u(x)), \text{ for a.e. } x \in \Omega, \tag{3.13}$$

and

$$\begin{aligned} \int_\Omega \zeta(\nabla v - \nabla u)\, dx + \int_\Omega \eta(v-u)\, dx + \int_\Omega b(x,u)(v-u)\, dx \\ - \sum_{i=1}^k \int_\Omega T_i(x,u)(v-u)\, dx \\ + \sum_{j=1}^m \int_\Omega T^j(x,u)(v-u)\, dx - \langle L, v-u\rangle \\ \ge 0, \quad \text{for all } v \in K. \end{aligned} \tag{3.14}$$

Note that $T_i(\cdot,u), T^j(\cdot,u) \in L^{q'}(\Omega)$ whenever $u \in L^q(\Omega)$. Therefore $\mathcal{T}_i : u \mapsto T_i(\cdot,u)$ and $\mathcal{T}^j : u \mapsto T^j(\cdot,u)$ $(1 \le i \le k,\ 1 \le j \le m)$ are bounded operators from $L^q(\Omega)$ to $L^{q'}(\Omega)$. Problem (3.12)–(3.14) can be equivalently written as:

Find $u \in K$, $\zeta \in \mathcal{A}(u)$, $\eta \in (i_q^* \tilde{f}_0 i_q)(u)$, such that

$$\begin{aligned}(3.15)\quad \langle \zeta + \eta + (i_p^* \mathcal{B} i_p)(u) - \sum_{i=1}^{k} (i_q^* \mathcal{T}_i i_q)(u) \\ + \sum_{j=1}^{m} (i_q^* \mathcal{T}^j i_q)(u), v - u\rangle \\ \ge \langle L, v - u\rangle, \quad \text{for all } v \in K.\end{aligned}$$

Note that the mappings $\mathcal{A}$ and $i_q^* \tilde{f}_0 i_q$ are multivalued while $i_p^* \mathcal{B} i_p$, $i_q^* \mathcal{T}_i i_q$ and $i_q^* \mathcal{T}^j i_q$ are single-valued. The variational inequality (3.15) is of the form (2.10) with $X = W^{1,p}(\Omega)$, $f = L$, $\phi = 0$ and $T = \mathcal{A} + i_q^* \tilde{f}_0 i_q + i_p^* \mathcal{B} i_p - \sum_{i=1}^{k} i_q^* \mathcal{T}_i i_q + \sum_{j=1}^{m} i_q^* \mathcal{T}^j i_q$. Since all components of T are pseudomonotone, so is T, that is, condition (T2) in Lemma 2.6 is fulfilled. The verification that T satisfies the coercivity condition ((T3) in Lemma 2.6)

$$\lim_{\|u\|_{W^{1,p}(\Omega)} \to \infty} \left[\inf_{u^* \in T(u)} \langle u^* - L, u - u_0\rangle \right] = \infty,$$

(u_0 is any (fixed) element of K) is based on similar estimates as in the case where $\mathcal{A}$ is single-valued, and is thus omitted. It now follows from Lemma 2.6 that (3.15), that is, (3.12)–(3.14), has solutions in K.

Let u be any of such solution. Let us check that

$$(3.16)\qquad \underline{u}_s \le u \text{ almost everywhere in } \Omega,$$

for all $s \in \{1, \dots, k\}$. In fact, suppose u, ζ, η satisfy (3.12)–(3.14) and $\underline{u}_s$, $\underline{\zeta}_s$, $\underline{\eta}_s$ satisfy (3.1)–(3.3) (with $\underline{L}_s$). Letting $v = \max\{\underline{u}_s, u\} \in K$ in

(3.14) yields

(3.17)
$$\int_\Omega \zeta \nabla[(\underline{u}_s - u)^+]\,dx + \int_\Omega \eta(\underline{u}_s - u)^+\,dx + \int_\Omega b(x,u)(\underline{u}_s - u)^+\,dx - \sum_{i=1}^{k}\int_\Omega \underline{T}_i(x,u)(\underline{u}_s - u)^+\,dx + \sum_{j=1}^{m}\int_\Omega \overline{T}_j(x,u)(\underline{u}_s - u)^+\,dx - \langle L, (\underline{u}_s - u)^+\rangle \geq 0.$$

From (3.3) with $\underline{\zeta}_s, \underline{\eta}_s, \underline{u}_s$ instead of $\underline{\zeta}, \underline{\eta}, \underline{u}$ and with $v = \min\{\underline{u}_s, u\} \in \underline{u}_s \wedge K$, we have

$$-\int_\Omega \underline{\zeta}_s \nabla[(\underline{u}_s - u)^+]\,dx - \int_\Omega \underline{\eta}_s(\underline{u}_s - u)^+\,dx + \langle \underline{L}_s, (\underline{u}_s - u)^+\rangle \geq 0. \tag{3.18}$$

Adding (3.17) and (3.18) yields

$$\int_\Omega (\zeta - \underline{\zeta}_s)\nabla[(\underline{u}_s - u)^+]\,dx + \int_\Omega (\eta - \underline{\eta}_s)(\underline{u}_s - u)^+\,dx + \int_\Omega b(x,u)(\underline{u}_s - u)^+\,dx - \sum_{i=1}^{k}\int_\Omega T_i(x,u)(\underline{u}_s - u)^+\,dx + \sum_{j=1}^{m}\int_\Omega T^j(x,u)(\underline{u}_s - u)^+\,dx + \langle \underline{L}_s - L, (\underline{u}_s - u)^+\rangle \geq 0.$$

It follows from (3.8) that $\langle \underline{L}_s - L, (\underline{u}_s - u)^+\rangle \geq 0$ and from (A2), (3.1) and (3.12) that

$$\int_\Omega (\underline{\zeta}_s - \zeta)\nabla[(\underline{u}_s - u)^+]\,dx \geq \int_{\{x\in\Omega : \underline{u}_s(x) > u(x)\}} (\underline{\zeta}_s - \zeta)[\nabla \underline{u}_s - \nabla u]\,dx \geq 0.$$

Using these estimates and the calculations in the proof of Theorem 4.2 of [**10**], we obtain (3.16). Since (3.16) holds for every $s \in \{1, \dots, k\}$, we have $u \geq \underline{u}$. Analogously, we can demonstrate that $u \leq \overline{u}$ almost everywhere in Ω, that is, u satisfies (3.10). As a consequence of (3.10), one can check easily that

$$b(x, u) = \underline{T}_i(x, u) = \overline{T}_j(x, u) = 0,$$

for almost every $x \in \Omega$, all $i \in \{1, \dots, k\}$, $j \in \{1, \dots, m\}$, and thus (3.14) reduces to (2.5). □

Using this main theorem, we can derive other properties of solutions between sub- and supersolutions. Assume there are subsolutions $\underline{u}_i$ $(1 \leq i \leq k)$ and supersolutions $\overline{u}_j$ $(1 \leq j \leq m)$ of (2.5), and let $\mathcal{S}$ be the set of solutions of (2.5) between $\underline{u}$ and $\overline{u}$:

$$\mathcal{S} = \{u \in W^{1,p}(\Omega) : u \text{ is a solution of (2.5) and } \underline{u} \leq u \leq \overline{u}\}.$$

Then $\mathcal{S} \neq \varnothing$ by Theorem 3.2. Some properties of $\mathcal{S}$ are collected in the following theorem, the proof of which is a combination of the above arguments with those in the single-valued case, and is therefore omitted.

Theorem 3.3. (a) *$\mathcal{S}$ is a compact subset of $W^{1,p}(\Omega)$.*

(b) *If*

$$K \wedge K \subset K \quad \text{and} \quad K \vee K \subset K, \tag{3.19}$$

then S is a directed set in the following sense: If $u_1, u_2 \in \mathcal{S}$ then there are u and w in $\mathcal{S}$ such that $u \geq \max\{u_1, u_2\}$ and $w \leq \min\{u_1, u_2\}$.

(c) *Under assumption (3.19), there exist maximal and minimal solutions of (2.5) between $\underline{u}$ and $\overline{u}$, that is, there are $u^*, u_* \in \mathcal{S}$ such that $u_* \leq u \leq u^*$ for all $u \in \mathcal{S}$.*

Remark 3.4. To illustrate the ideas and to keep the notation simple, we consider here the case where the higher order term $A = A(x, \nabla u)$ does not depend on u and the lower order term $f = f(x, u)$ does not depend on ∇u. By imposing conditions on the continuity with respect to u on the higher order term and on the growth with respect to ∇u on the lower order term (cf., e.g., [**6**] and the corresponding references therein), one can adapt and extend the above arguments and results to variational inequalities in which $A = A(x, u, \nabla u)$ and $f = f(x, u, \nabla u)$. Moreover, as in [**16**] (see also [**2, 14**]), by adding a lower order multivalued term on the boundary such as $\int_{\partial\Omega} g(x, u)(v - u)\, d\Gamma$ ($d\Gamma$ is the surface measure on $\partial\Omega$) to (1.1), we can include as particular cases problems with multivalued/nonsmooth nonhomogeneous Neumann, Robin, or no-flux boundary conditions.

The above argument in Section 3, with appropriate adaptations and extensions, can be used to establish theorems similar to Theorems 2.3, 3.2 and 3.3, for inequalities of the following general form:

$$\begin{cases} \int_\Omega A(x, u, \nabla u)(\nabla v - \nabla u)\, dx + \int_\Omega f(x, u, \nabla u)(v - u)\, dx \\ + \int_{\partial\Omega} g(x, u)(v - u)\, d\Gamma \\ \geq \langle L, v - u\rangle, \forall v \in K \\ u \in K, \end{cases}$$

where A, f and g are all set-valued functions.

4. Minimum of subsolutions. We know in smooth equations with single-valued operators that, under certain conditions, minima of subsolutions are also subsolutions (see, e.g., [**3, 6, 9, 11**]). In this section, we extend this property to variational inequalities with multivalued operators. The following arguments are motivated by those in [**11**]. We use the notation

$$\begin{aligned} (K - K)^+ &= \{v \in K - K : v \geq 0 \text{ almost everywhere in } \Omega\} \\ &= \{u - w : u, w \in K, u \geq w \text{ almost everywhere in } \Omega\}. \end{aligned}$$

Theorem 4.1. *Assume* $\underline{u}_1$ *and* $\underline{u}_2$ *are subsolutions of* (2.5) *corresponding to* $\underline{L}_1$ *and* $\underline{L}_2$ *such that* $\underline{u}_i \vee K \subset K$ $(i = 1, 2)$. *Suppose furthermore that there exists a* $K_0 \subset (K - K)^+ \cap L^\infty(\Omega)$ *dense in* $(K - K)^+$ *such that*

$$\underline{u}_i - \Lambda \subset \underline{u}_i \wedge K, \quad i = 1, 2, \tag{4.1}$$

where $\Lambda = \{\theta v : v \in K_0, \theta \in W^{1,p}(\Omega), 0 \le \theta \le 1\}$. *Then,* $\underline{u} = \min\{\underline{u}_1, \underline{u}_2\}$ *is also a subsolution of* (2.5) *corresponding to any* $\underline{L} \in [W^{1,p}(\Omega)]^*$ *such that* $\underline{L}_i \le \underline{L}$ $(i = 1, 2)$.

Remark 4.2. Note that the conditions in Theorem 4.1 are satisfied in the cases where $K = W_0^{1,p}(\Omega)$ (Dirichlet boundary condition) or $K = W^{1,p}(\Omega)$ (Neumann boundary condition). Also, we have a similar result for minima of supersolutions.

Proof of Theorem 4.1. Let $\underline{\zeta}_1, \underline{\eta}_1$ and $\underline{\zeta}_2, \underline{\eta}_2$ be the functions corresponding to $\underline{u}_1$ and $\underline{u}_2$ as in Definition 3.1. Put

$$\Omega_1 = \{x \in \Omega : \underline{u}_1(x) \ge \underline{u}_2(x)\}, \Omega_2 = \Omega \setminus \Omega_1,$$

$\underline{\eta} = \underline{\eta}_1 \chi_{\Omega_1} + \underline{\eta}_2 \chi_{\Omega_2}$, and $\underline{\zeta} = \underline{\zeta}_1 \chi_{\Omega_1} + \underline{\zeta}_2 \chi_{\Omega_2}$. Then $\underline{\eta} \in L^{q'}(\Omega)$, $\underline{\zeta} \in L^{p'}(\Omega)$ satisfy (3.1) and (3.2) for $\underline{u} = \underline{u}_1 \wedge \underline{u}_2$. We only need to check (3.3). Let $w \in K$ and $v = \underline{u} \wedge w$. We have $v - \underline{u} = -(\underline{u} - w)^+ = w - \underline{u} \vee w$. On the other hand, we have $\underline{u} \vee w = \underline{u}_1 \vee (\underline{u}_2 \vee w) \in K$. Hence, $\underline{u} - v = (\underline{u} \vee w) - w \in (K - K)^+$. Let $\{\psi_n\}$ be a sequence in K_0 such that

$$\psi_n \longrightarrow \underline{u} - v \text{ in } W^{1,p}(\Omega). \tag{4.2}$$

Assume γ is a nondecreasing function in $C^\infty(\mathbf{R})$ such that $0 \le \gamma(t) \le 1$ for all $t \in \mathbf{R}$, $\gamma(t) = 1$ if $t \ge 1$ and $\gamma(t) = 0$ if $t \le 0$. For $n \in \mathbf{N}$, we define $\gamma_n(t) = \gamma(nt)$ $(t \in \mathbf{R})$. Let $\phi \in K_0$ and, for each $n \in N$, put

$$\phi_1 = \phi_{1n} = [1 - \gamma_n(\underline{u}_2 - \underline{u}_1)]\phi, \qquad \phi_2 = \phi_{2n} = [\gamma_n(\underline{u}_2 - \underline{u}_1)]\phi.$$

It is clear that $\gamma_n(\underline{u}_2 - \underline{u}_1) \in W^{1,p}(\Omega)$. Since $0 \le \gamma_n(\underline{u}_2 - \underline{u}_1) \le 1$, we have $\phi_1, \phi_2 \in \Lambda$. Hence, from (4.1), there are $v_1, v_2 \in K$ such that $\underline{u}_i - \phi_i = \underline{u}_i \wedge v_i$ $(i = 1, 2)$. From (3.3) with $\underline{u}_i, \underline{\eta}_i, \underline{\zeta}_i$, and $v = \underline{u}_i \wedge v_i$ $(i = 1, 2)$, noting that $\underline{u}_i \wedge v_i - \underline{u}_i = -\phi_i$, we obtain

$$\int_\Omega \underline{\zeta}_i \nabla \phi_i \, dx + \int_\Omega \underline{\eta}_i \phi_i \, dx - \langle \underline{L}_i, \phi_i \rangle \le 0, \quad i = 1, 2,$$

that is, with $\underline{\zeta}_i = (\underline{\zeta}_{i1}, \dots, \underline{\zeta}_{iN})$,

$$\begin{aligned}(4.3)\qquad \int_\Omega \sum_{j=1}^N \underline{\zeta}_{1j} \bigg\{ &-\gamma_n'(\underline{u}_2 - \underline{u}_1)\frac{\partial}{\partial x_j}(\underline{u}_2 - \underline{u}_1)\phi \\ &+ [1-\gamma_n(\underline{u}_2 - \underline{u}_1)]\frac{\partial \phi}{\partial x_j}\bigg\}\,dx \\ &+ \int_\Omega \underline{\eta}_1 [1-\gamma_n(\underline{u}_2 - \underline{u}_1)]\phi\,dx - \langle \underline{L}_1, \phi_1\rangle \\ &\le 0,\end{aligned}$$

and

$$\begin{aligned}(4.4)\qquad \int_\Omega \sum_{j=1}^N \underline{\zeta}_{2j} \bigg\{ &\gamma_n'(\underline{u}_2 - \underline{u}_1)\frac{\partial}{\partial x_j}(\underline{u}_2 - \underline{u}_1)\phi + \gamma_n(\underline{u}_2 - \underline{u}_1)\frac{\partial \phi}{\partial x_j}\bigg\}\,dx \\ &+ \int_\Omega \underline{\eta}_2 \gamma_n(\underline{u}_2 - \underline{u}_1)\phi\,dx - \langle \underline{L}_2, \phi_2\rangle \\ &\le 0.\end{aligned}$$

Adding (4.3) and (4.4) yields

$$\begin{aligned} 0 \ge &\int_\Omega \sum_{j=1}^N (\underline{\zeta}_{2j} - \underline{\zeta}_{1j})\gamma_n'(\underline{u}_2 - \underline{u}_1)\frac{\partial}{\partial x_j}(\underline{u}_2 - \underline{u}_1)\phi\,dx \\ &+ \int_\Omega \sum_{j=1}^N (\underline{\zeta}_{2j} - \underline{\zeta}_{1j})\gamma_n(\underline{u}_2 - \underline{u}_1)\frac{\partial \phi}{\partial x_j}\,dx \\ (4.5)\qquad &+ \int_\Omega \sum_{j=1}^N \underline{\zeta}_{1j}\frac{\partial \phi}{\partial x_j}\,dx + \int_\Omega \underline{\eta}_1 \phi\,dx \\ &+ \int_\Omega (\underline{\eta}_2 - \underline{\eta}_1)\gamma_n(\underline{u}_2 - \underline{u}_1)\phi\,dx \\ &- \langle \underline{L}_1, \phi_1\rangle - \langle \underline{L}_2, \phi_2\rangle.\end{aligned}$$

If $x \in \Omega_1$ then $(\underline{u}_2 - \underline{u}_1)(x) \le 0$ and thus $[\gamma_n(\underline{u}_2 - \underline{u}_1)](x) = 0$ for all $n \in \mathbf{N}$. If $x \in \Omega_2$ then $[\gamma_n(\underline{u}_2 - \underline{u}_1)](x) = 1$ for all n sufficiently large. Hence, by Lebesgue's dominated convergence theorem,

$$\begin{aligned}&\int_\Omega (\underline{\eta}_2 - \underline{u}_1)\gamma_n(\underline{u}_2 - \underline{u}_1)\phi\,dx \\ &\qquad = \int_{\Omega_2} (\underline{\eta}_2 - \underline{u}_1)\gamma_n(\underline{u}_2 - \underline{u}_1)\phi\,dx \longrightarrow \int_{\Omega_2} (\underline{\eta}_2 - \underline{u}_1)\phi\,dx,\end{aligned}$$

and thus

$$\begin{aligned}(4.6)\quad &\int_\Omega \underline{\eta}_1 \phi\,dx + \int_\Omega (\underline{\eta}_2 - \underline{u}_1)\gamma_n(\underline{u}_2 - \underline{u}_1)\phi\,dx \\ &\longrightarrow \int_\Omega \underline{\eta}_1 \phi\,dx + \int_{\Omega_2} (\underline{\eta}_2 - \underline{u}_1)\phi\,dx \\ &= \int_{\Omega_1} \underline{\eta}_1 \phi\,dx + \int_{\Omega_2} \underline{\eta}_2 \phi\,dx \\ &= \int_\Omega \underline{\eta}\phi\,dx.\end{aligned}$$

Similarly,

$$\int_\Omega (\underline{\zeta}_2 - \underline{\zeta}_1)\nabla\phi\gamma_n(\underline{u}_2 - \underline{u}_1)\,dx \longrightarrow \int_{\Omega_2} (\underline{\zeta}_2 - \underline{\zeta}_1)\nabla\phi\,dx,$$

and thus

$$\begin{aligned}(4.7)\quad &\int_\Omega (\underline{\zeta}_2 - \underline{\zeta}_1)\nabla\phi\gamma_n(\underline{u}_2 - \underline{u}_1)\,dx + \int_\Omega \underline{\zeta}_1 \nabla\phi\,dx \\ &\longrightarrow \int_{\Omega_2} (\underline{\zeta}_2 - \underline{\zeta}_1)\nabla\phi\,dx + \int_\Omega \underline{\zeta}_1 \nabla\phi\,dx \\ &= \int_{\Omega_1} \underline{\zeta}_1 \nabla\phi\,dx + \int_{\Omega_2} \underline{\zeta}_2 \nabla\phi\,dx \\ &= \int_\Omega \underline{\zeta}\nabla\phi\,dx.\end{aligned}$$

Lastly, since $\phi(x) \geq 0$ and $\gamma_n'(\underline{u}_2 - \underline{u}_1)(x) \geq 0$ for almost every $x \in \Omega$, from the monotonicity of A, we have

$$(4.8)\quad \begin{gathered}[\underline{\zeta}_2(x) - \underline{\zeta}_1(x)]\nabla(\underline{u}_2 - \underline{u}_1)(x)\gamma_n'(\underline{u}_2 - \underline{u}_1)(x)\phi(x) \geq 0 \\ \text{for almost every } x \in \Omega.\end{gathered}$$

Passing to the limit in (4.5) and taking into account (4.6)–(4.8), we get

$$0 \geq \int_\Omega \underline{\zeta}\nabla\phi\,dx + \int_\Omega \underline{\eta}\phi\,dx - \langle \underline{L}_1, \phi_1\rangle - \langle \underline{L}_2, \phi_2\rangle.$$

Since $\underline{L} - \underline{L}_i \geq 0$ $(i = 1, 2)$, this implies that

$$\begin{aligned}\int_\Omega \underline{\zeta}\nabla\phi\, dx &+ \int_\Omega \underline{\eta}\phi\, dx - \langle \underline{L}, \phi\rangle \\ &= \int_\Omega \underline{\zeta}\nabla\phi\, dx + \int_\Omega \underline{\eta}\phi\, dx - \langle \underline{L}_1, \phi_1\rangle - \langle \underline{L}_2, \phi_2\rangle \\ &\quad + \langle \underline{L}_1 - \underline{L}, \phi_1\rangle + \langle \underline{L}_2 - \underline{L}, \phi_2\rangle \\ &\leq \int_\Omega \underline{\zeta}\nabla\phi\, dx + \int_\Omega \underline{\eta}\phi\, dx - \langle \underline{L}_1, \phi_1\rangle - \langle \underline{L}_2, \phi_2\rangle \\ &\leq 0.\end{aligned}$$

In particular, for the sequence $\{\psi_n\}$ in (4.2), we have

$$0 \geq \int_\Omega \underline{\zeta}\nabla\psi_n\, dx + \int_\Omega \underline{\eta}\psi_n\, dx - \langle \underline{L}, \psi_n\rangle, \quad \text{for all } n \in \mathbf{N}.$$

Letting $n \to \infty$ in this inequality yields

$$0 \geq \int_\Omega \underline{\zeta}\nabla(\underline{u} - v)\, dx + \int_\Omega \underline{\eta}(\underline{u} - v)\, dx - \langle \underline{L}, \underline{u} - v\rangle,$$

i.e., condition (3.3) also holds true for $\underline{u} = \min\{\underline{u}_1, \underline{u}_2\}$. □

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