ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 2, 2011

MULTIDIMENSIONAL GRAPH COMPLETIONS AND CELLINA APPROXIMABLE MULTIFUNCTIONS

ALBERTO BRESSAN AND RUSSELL DEFOREST

ABSTRACT. Relying on the continuous approximate selection method of Cellina, ideas and techniques from Sobolev spaces can be applied to the theory of multifunctions and differential inclusions. The first part of this paper introduces a concept of *graph completion*, which extends the earlier construction in [**12**] to functions of several space variables. The second part introduces the notion of *Cellina $W^{1,p}$-approximable multifunction.* To show its relevance, we consider the Cauchy problem on the plane $\dot{x} \in F(x)$, $x(0) = 0 \in \mathbf{R}^2$. If F is an upper semicontinuous multifunction with compact but possibly non-convex values, this problem may not have any solution, even if F is Cellina-approximable in the usual sense. However, we prove that a solution exists under the assumption that F is Cellina $W^{1,1}$-approximable.

1. Introduction. For a vector-valued function of a scalar variable, the concept of a *graph completion* was introduced in [**12**]. Its main motivation came from control theory. The control of mechanical systems by means of active constraints [**9, 11, 19, 22**] leads to a system of equations of the form

$$\dot{x} = f_0(x) + \sum_{k=1}^{m} f_k(x)\dot{u}_k. \tag{1.1}$$

Here $t \mapsto x(t) \in \mathbf{R}^n$ describes the state of the system, while $t \mapsto u(t) \in \mathbf{R}^m$ is the control function. An upper dot denotes derivative with respect to time. Assume that each f_k is a globally Lipschitz continuous vector field on $\mathbf{R}^n$. Since the right hand side of (1.1) contains the time derivatives $\dot{u}_k$, given an initial data

$$x(0) = \overline{x}, \tag{1.2}$$

to achieve existence and uniqueness of the solution it is natural to consider control functions $t \mapsto u(t) = (u_1, \dots, u_m)(t)$ which are

This research was partially supported by NSF through grant DMS-0807420, "New problems in nonlinear control."
Received by the editors on June 15, 2010.
DOI:10.1216/RMJ-2011-41-2-411

absolutely continuous. As shown in [**12**], a solution to (1.1)–(1.2) can be uniquely determined also in the case where the control function $u(\cdot)$ is a function with bounded variation, provided that we "complete" its graph, bridging the points where a jump occurs.

Definition 1. A *graph-completion* of a BV function $u : [0,T] \mapsto \mathbf{R}^m$ is a Lipschitz continuous path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_m) : [0,S] \mapsto [0,T] \times \mathbf{R}^m$ such that

(i) $\gamma(0) = (0, u(0))$, $\gamma(S) = (T,\ u(T))$,

(ii) $\gamma_0(s_1) \le \gamma_0(s_2)$ whenever $0 \le s_1 < s_2 \le S$,

(iii) for each $t \in [0,T]$ there exists some s such that $\gamma(s) = (t, u(t))$.

Notice that the path γ provides a continuous parametrization of the graph of u in the (t,u) space. At a time τ where u has a jump, the continuous curve γ must include an arc joining the left and right limits $(\tau, u(\tau-))$, $(\tau, u(\tau+))$. As soon as a graph completion of $u(\cdot)$ is assigned, we can uniquely solve the Cauchy problem

$$\frac{d}{ds}\, y(s) = f_0\big(y(s)\big)\, \dot\gamma_0(s) + \sum_{k=1}^{m} f_k\big(y(s)\big)\, \dot\gamma_k(s), \quad y(0) = \overline{x}. \tag{1.3}$$

The (possibly multivalued) function

$$t \longmapsto x(t,\gamma) = \{y(s,\gamma);\ \gamma_0(s) = t\} \tag{1.4}$$

is then called the *generalized trajectory* of (1.1)–(1.2) determined by the graph-completion $\gamma(\cdot)$ of the control function $u(\cdot)$.

This same construction has been later used in [**16**] and then in [**15**], in order to define *nonconservative products*, in connection with hyperbolic systems of PDEs which are not in conservation form.

The first part of the present paper develops an extension of these concepts to functions of several variables. Our basic approach is here very different from [**12**], although in the one-dimensional case it produces essentially the same construction.

Let $\Omega \subset \mathbf{R}^n$ be an open domain with compact closure $\overline{\Omega}$. Denote by Hom$\,(\Omega)$ the family of all homeomorphisms $\phi : \overline{\Omega} \mapsto \overline{\Omega}$ which keep fixed the boundary of Ω, namely,

$$\phi(x) = x \quad \text{for all } x \in \partial\Omega. \tag{1.5}$$

Given two continuous maps $f, g : \overline{\Omega} \mapsto \mathbf{R}^m$, we consider an alternative measure of their distance, defined as

$$(1.6) \qquad d^{\diamondsuit}(f,g) \doteq \inf_{\phi \in \mathrm{Hom}\,(\Omega)} \sup_{x\in\Omega} \Big(|x - \phi(x)| + |f(x) - g(\phi(x))| \Big).$$

In general, the space $\mathcal{C}(\overline{\Omega}; \mathbf{R}^m)$ is not complete with respect to the metric $d^{\diamondsuit}$. As shown by our analysis, to a Cauchy sequence of functions $(f_k)_{k\geq 1}$, one can associate a unique multifunction $F : \overline{\Omega} \hookrightarrow \mathbf{R}^m$. The graph of F can be parameterized by a continuous map $\Phi : \overline{\Omega} \mapsto \overline{\Omega} \times \mathbf{R}^m$. If $f : \overline{\Omega} \mapsto \mathbf{R}^m$ is a function such that $\mathrm{graph}\,(f) \subseteq \mathrm{graph}\,(F)$, and $F(x) = \{f(x)\}$ for almost every $x \in \Omega$, we regard F as a *graph completion* of f.

It is of interest to study Cauchy sequences (always with respect to the metric $d^{\diamondsuit}$) consisting of functions whose Sobolev norm $\|f_k\|_{W^{1,p}(\Omega)}$ is uniformly bounded. This gives rise to the notion of $W^{1,p}$*-graph completions*. Taking $p = 1$, we obtain graph completions of bounded variation. In the one-dimensional case, these are equivalent to the graph completions introduced in [**12**].

In the second part of this paper we explore the connection between graph completions and Cellina-approximable multifunctions [**21, 23**]. We recall that a compact valued multifunction $F : \Omega \hookrightarrow \mathbf{R}^m$ is Cellina-approximable if for every $\varepsilon > 0$ there exists a continuous function $f_\varepsilon : \Omega \mapsto \mathbf{R}^m$ such that

$$(1.7) \qquad \mathrm{graph}\,(f_\varepsilon) \subset B\Big(\mathrm{graph}\,(F), \varepsilon\Big).$$

According to (1.7), the graph of f_ε should thus be contained in an ε-neighborhood of the graph of F. A fundamental theorem proved in [**13**] states that every upper semicontinuous multifunction with compact, convex values is Cellina-approximable.

By definition, it is clear that every multifunction F, arising as a limit of a Cauchy sequence of continuous functions $(f_k)_{k\geq 1}$ with respect to the metric $d^{\diamondsuit}$, is Cellina-approximable. Our analysis leads to the introduction of a more refined concept:

Definition 2. Let $\Omega \subseteq \mathbf{R}^n$, and let $F : \Omega \hookrightarrow \mathbf{R}^m$ be an upper semicontinuous multifunction with compact values. For $1 \leq p < \infty$, we say that F is *Cellina $W^{1,p}$-approximable* if for every $\varepsilon > 0$ there exists a smooth function $f_\varepsilon : \Omega \mapsto \mathbf{R}^m$ such that (1.7) holds, and the Sobolev norm $\|f_\varepsilon\|_{W^{1,p}(\Omega)}$ satisfies a bound independent of ε.

As shown in the sequel, this definition can be relevant in the theory of differential inclusions. Given a bounded, compact valued multifunction $F : \mathbf{R}^n \hookrightarrow \mathbf{R}^n$, it is well known that the Cauchy problem

$$\dot{x} \in F(x), \quad x(0) = 0 \in \mathbf{R}^n \tag{1.8}$$

admits a Carathéodory solution in the following cases.

(i) F is upper semicontinuous, with convex values [**3, 4, 24**].

(ii) F is continuous [**2, 18**], or merely lower semicontinuous [**6, 8**], possibly with non-convex values.

In the case where F is upper semicontinuous but with non-convex values, it is easy to see that the initial value problem (1.8) need not have solutions. The standard one-dimensional example is

$$\dot{x} \in F(x) = \begin{cases} \{-1\} & \text{if } x > 0, \\ \{-1, 1\} & \text{if } x = 0, \\ \{1\} & \text{if } x < 0. \end{cases} \tag{1.9}$$

A two-dimensional example is

$$(\dot{x}_1, \dot{x}_2) \in G(x_1, x_2) = \begin{cases} \left\{1/\sqrt{x_1^2 + x_2^2}(x_2, -x_1)\right\} & \text{if } (x_1, x_2) \neq (0,0), \\ \left\{(y_1, y_2);\ y_1^2 + y_2^2 = 1\right\} & \text{if } (x_1, x_2) = (0,0). \end{cases} \tag{1.10}$$

Clearly, neither of the multifunctions F in (1.9) or G in (1.10) is Cellina-approximable. For some time, this was regarded as the main topological obstruction to the existence of solutions. Indeed, it seemed natural to consider the following

Conjecture. *Let $F : \mathbf{R}^n \hookrightarrow \mathbf{R}^n$ be a bounded upper semicontinuous multifunction with compact values, which is Cellina-approximable. Then the Cauchy problem* (1.8) *has at least one solution.*

As shown in [**7**], the conjecture is false. We recall here a counterexample (see Figure 1).

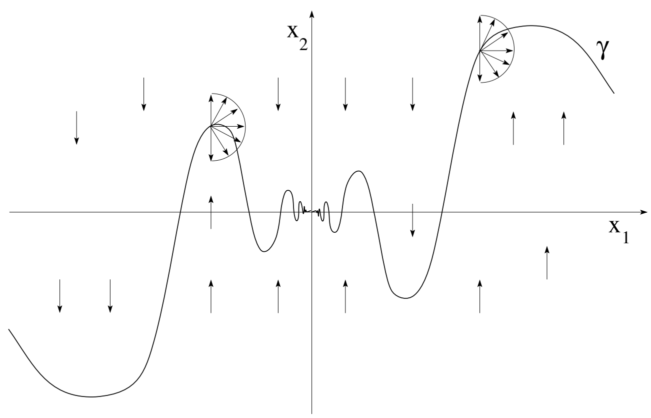

FIGURE 1. The multifunction F defined at (5.35) is Cellina-approximable. However, the corresponding differential inclusion has no solution starting from the origin. Here the curve γ has infinite length.

Example 1. Consider the continuous function

$$\phi(s) \doteq s\,\cos\frac{1}{s}\quad \text{if } s \neq 0, \quad \phi(0) = 0.$$

Define the multifunction $F : \mathbf{R}^2 \mapsto \mathbf{R}^2$ as

$$(1.11)\quad F(x_1, x_2) \doteq \begin{cases} \{(0,-1)\} & \text{if } x_2 > \phi(x_1), \\ \{(0,1)\} & \text{if } x_2 < \phi(x_1), \\ \Big\{(y_1,y_2);\ y_1 \geq 0,\ y_1^2 + y_2^2 = 1\Big\} & \text{if } x_2 = \phi(x_1). \end{cases}$$

Observe that this multifunction can be written as a composition: $F(x) = \psi(G(x))$, where

$$(1.12)\quad \begin{aligned} \psi(\xi) &= (\cos\xi,\ \sin\xi), \\ G(x_1,x_2) &\doteq \begin{cases} \{-\pi/2\} & \text{if } x_2 > \phi(x_1), \\ \{\pi/2\} & \text{if } x_2 < \phi(x_1), \\ [-\pi/2,\ \pi/2] & \text{if } x_2 = \phi(x_1). \end{cases} \end{aligned}$$

Since ψ is Lipschitz continuous and G is an upper semicontinuous multifunction with compact, convex values, it follows that F is Cellina-approximable. However, in this case the Cauchy problem (1.8) has no solution. Indeed, consider the curve $\gamma = \text{graph}\,(\phi) = \{(s, \phi(s))\,;\, s \in \mathbf{R}\}$. By the definition of F, every solution starting on γ cannot move away from γ, at any positive time. On the other hand, a solution starting at the origin cannot move along γ, because every portion of this curve connecting the origin to any other point has infinite length.

Apparently, what goes wrong in this example is that the multifunction F has a jump along a curve of infinite length. This possibility is ruled out if we impose that F is Cellina $W^{1,p}$-approximable, for a suitable exponent p. A precise result in this direction, valid for differential inclusions in the plane, is the following.

Theorem 1. *Let $F : \mathbf{R}^2 \hookrightarrow \mathbf{R}^2$ be a bounded, upper semicontinuous multifunction with compact, possibly non-convex values. Assume that F is Cellina $W^{1,1}$-approximable. Then the Cauchy problem* (1.8) *admits a Caratheodory solution, defined for all times $t \in \mathbf{R}$.*

A key ingredient in the proof is the following lemma, which rules out the existence of arbitrarily long trajectories remaining inside the square $Q \doteq \{(x_1, x_2);\, 0 \le x_1 \le 1,\ 0 \le x_2 \le 1\}$.

Lemma 1. *Let $f : Q \mapsto \mathbf{R}^2$ be a smooth vector field defined on the square Q, such that $|f(x)| = 1$ for every $x \in Q$. Then every trajectory of the ODE $\dot{x} = f(x)$ starting inside Q reaches the boundary of Q within time*

$$T \doteq 4 + \frac{1}{2}\,\|Df\|_{\mathbf{L}^1(Q)}. \tag{1.13}$$

Here $\|Df\|$ is the norm of the 2×2 Jacobian matrix $Df = (\partial f_i/\partial x_j)$. More precisely,

$$\|Df\|_{\mathbf{L}^1(Q)} \doteq \sum_{i,j=1,2} \int_Q \left|\frac{\partial f_i}{\partial x_j}(x)\right| dx.$$

Remark 1. Both Lemma 1 and Theorem 1 rely on topological properties of the plane, related to the Jordan curve theorem and the Poincaré-Bendixon theory, which have no analogue in dimension $n \geq 3$. We observe that, by the classical Sobolev embedding theorems, when $p > n$, every Cellina $W^{1,p}$-approximable multifunction F admits a Hölder continuous selection. In this case, the existence of solutions to the Cauchy problem (1.8) is trivial. The really interesting case arises when $n-1 \leq p \leq n$. Notice that here one must choose p large enough, so that functions $f \in W^{1,p}(\mathbf{R}^n)$ can be discontinuous only on small sets. In particular, one must avoid the possibility of a vector field f having jumps along a one-dimensional curve of infinite length; otherwise, a counterexample such as (5.35) can again be produced. At the present time it is not clear whether Theorem 1 can be extended to Cellina $W^{1,n-1}$-approximable multifunctions defined on $\mathbf{R}^n$, for $n \geq 3$. We leave this as an open problem.

A comprehensive introduction to the theory of differential inclusions and set valued functions can be found in [**3, 4**]. For the basic theory of Sobolev spaces and BV functions in several variables we refer to [**1, 17**]. An elementary introduction to Sobolev spaces can also be found in Chapter 8 of the lecture notes [**10**].

2. Multifunctions with continuously parameterizable graph. Let $\Omega \subset \mathbf{R}^n$ be a compact domain with piecewise smooth boundary $\partial\Omega$. Given two continuous maps $f, g : \Omega \mapsto \mathbf{R}^m$, we define their graph distance according to (1.6). We recall that the Hausdorff distance between two compact sets K_1, K_2 is

$$d_H(K_1, K_2) \doteq \inf\Big\{\varepsilon > 0;\ K_2 \subset \overline{B}(K_1, \varepsilon) \text{ and } K_1 \subset \overline{B}(K_2, \varepsilon)\Big\}.$$

Here $\overline{B}(K_i, \varepsilon)$ denotes the closed ε-neighborhood around the set K_i, i.e., the set of all points having distance $\leq \varepsilon$ from K_i.

Lemma 2. *The function $d^\diamond(\cdot,\cdot)$ defined at* (1.6) *provides a distance within the space of all continuous functions $f : \overline{\Omega} \mapsto \mathbf{R}^m$. Moreover,*

$$d_H\Big(\text{graph}\,(f),\ \text{graph}\,(g)\Big) \leq d^\diamond(f, g), \tag{2.1}$$

where d_H denotes the Hausdorff distance between two compact sets.

Proof. 1. We start by proving (2.1). Let $d(\cdot,\cdot)$ denote Euclidean distance on the product space $\mathbf{R}^n \times \mathbf{R}^m$.

We claim that, for any homeomorphism $\phi \in \mathrm{Hom}\,(\Omega)$, one has

(2.2)
$$d_H\Big(\mathrm{graph}\,(f),\ \mathrm{graph}\,(g)\Big) \le \sup_{x\in\Omega}\Big(|x-\phi(x)| + |f(x) - g(\phi(x))|\Big) \doteq \alpha_\phi.$$

Indeed, the mapping $(x,\ f(x)) \mapsto (\phi(x),\ g(\phi(x))$ is a one-to-one mapping from graph (f) onto graph (g). Moreover,

$$d\Big((x,\ f(x)),\ (\phi(x),\ g(\phi(x))\Big) \le \alpha_\phi$$

for every $x \in \overline{\Omega}$. This implies

$$\mathrm{graph}\,(g) \subseteq \overline{B}(\mathrm{graph}\,(f),\ \alpha_\phi),$$
$$\mathrm{graph}\,(f) \subseteq \overline{B}(\mathrm{graph}\,(g),\ \alpha_\phi).$$

Taking the supremum over all homeomorphisms $\phi \in \mathrm{Hom}\,(\Omega)$, our claim is proved.

2. We now check that $d^\diamondsuit(\cdot,\cdot)$ is a distance on the set of all continuous functions $f : \overline{\Omega} \mapsto \mathbf{R}^m$. From the definition, it is clear that $d^\diamondsuit(f,g) \ge 0$, with equality holding if $f = g$. On the other hand, if $f \ne g$, then by (2.1) $d^\diamondsuit(f,g) \ge d_H\Big(\mathrm{graph}\,(f),\ \mathrm{graph}\,(g)\Big) > 0$.

To show that $d^\diamondsuit(\cdot,\cdot)$ is symmetric, take $\phi \in \mathrm{Hom}\,(\Omega)$ and let $\psi = \phi^{-1}$. Then

$$\sup_{x\in\Omega}\Big(|x-\phi(x)| + |f(x) - g(\phi(x))|\Big) = \sup_{y\in\Omega}\Big(|y-\psi(y)| + |g(y) - f(\psi(y))|\Big).$$

Since ϕ was arbitrary, taking a supremum we conclude that $d^\diamondsuit(f,g) = d^\diamondsuit(g,f)$.

Finally, to prove the triangle inequality, let $f, g, h \in \mathcal{C}(\overline{\Omega}, \mathbf{R}^m)$ and $\varepsilon > 0$ be given. Choose homeomorphisms $\phi, \psi \in \mathrm{Hom}\,(\Omega)$ such that

$$\sup_{x\in\Omega}\Big(|x-\phi(x)| + |f(x) - g(\phi(x))|\Big) \le d^{\diamondsuit}(f,g) + \varepsilon,$$
$$\sup_{y\in\Omega}\Big(|y-\psi(y)| + |g(y) - h(\psi(y))|\Big) \le d^{\diamondsuit}(g,h) + \varepsilon.$$

Observing that the composition satisfies $\psi \circ \phi \in \mathrm{Hom}\,(\Omega)$, we can write

$$\begin{aligned} d^{\diamondsuit}(f,h) &\le \sup_{x\in\Omega}\Big(|x-\psi(\phi(x))| + |f(x) - h(\psi(\phi(x)))|\Big) \\ &\le \sup_{x\in\Omega}\Big(|x-\phi(x)| + |\phi(x) - \psi(\phi(x))| \\ &\qquad + |f(x) - g(\phi(x))| + |g(\phi(x)) - h(\psi(\phi(x)))|\Big) \\ &\le d^{\diamondsuit}(f,g) + d^{\diamondsuit}(g,h) + 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $d^{\diamondsuit}(f,g) \le d^{\diamondsuit}(f,h) + d^{\diamondsuit}(h,g)$, completing the proof. □

We observe that, in general, the space $\mathcal{C}(\Omega; \mathbf{R}^n)$ is not complete with respect to the distance $d^{\diamondsuit}$.

Example 2. Take $\overline{\Omega} = [-1,1]$ and consider the sequence of continuous functions

$$f_k(x) \doteq \begin{cases} 0 & \text{if } x \in [-1,0], \\ kx & \text{if } x \in [0,\ 1/k], \\ 1 & \text{if } x \in [1/k,\ 1]. \end{cases} \tag{2.3}$$

Clearly, this is not a Cauchy sequence with respect to the norm distance in the Banach space $\mathcal{C}([-1,1]; \mathbf{R})$. However, it is Cauchy with respect to the distance $d^{\diamondsuit}$. Indeed, for any given $k, \ell \ge 1$, let $\phi : [-1,1] \mapsto [-1,1]$ be the piecewise affine map such that

$$\phi(-1) = -1, \quad \phi(0) = 0, \quad \phi\left(\frac{1}{k}\right) = \frac{1}{\ell}, \quad \phi(1) = 1.$$

Then

$$d^{\diamondsuit}(f_k, f_\ell) \leq \sup_{x\in[-1,1]} \Big(|x-\phi(x)| + |f_k(x) - f_\ell(\phi(x))|\Big) = \left|\frac{1}{k} - \frac{1}{\ell}\right|.$$

Hence $\limsup_{k,\ell\to\infty} d^{\diamondsuit}(f_k, f_\ell) = 0$.

It is interesting to study the completion of the space $\mathcal{C}(\overline{\Omega}; \mathbf{R}^m)$ with respect to the metric $d^{\diamondsuit}(\cdot,\cdot)$. As will become apparent from the following analysis, elements of this completion can be identified with multifunctions whose graph admits a continuous parametrization: $\overline{\Omega} \mapsto \overline{\Omega} \times \mathbf{R}^m$.

Given any sequence $(f_\ell)_{\ell\geq 1}$ which is Cauchy for the metric (1.6), one can extract a subsequence such that (after relabeling)

$$d^{\diamondsuit}(f_{k+1}, f_k) < 2^{-k}. \tag{2.4}$$

This means that, for each k, there exists a homeomorphism $\phi_k \in \Omega$ such that

$$\sup_{x\in\Omega} \Big(|x-\phi_k(x)| + |f_{k+1}(x) - f_k(\phi(x))|\Big) < 2^{-k}. \tag{2.5}$$

By (2.4), for each $x \in \Omega$ the sequence

$$X_k(x) = \phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_1(x) \tag{2.6}$$

is Cauchy. The same is true of $f_k(X_k(x))$. We thus obtain two continuous maps

$$x \longmapsto X(x) \doteq \lim_{k\to\infty} X_k(x), \qquad x \mapsto f(x) \doteq \lim_{k\to\infty} f_k(X_k(x)). \tag{2.7}$$

Let F be the (possibly multivalued) function whose graph is

$$\text{graph}\,(F) \doteq \Big\{(X(x),\ f(x));\ x\in\Omega\Big\}. \tag{2.8}$$

This graph is a compact subset of $\Omega \times \mathbf{R}^m$, hence the multifunction F is upper semicontinuous. Since the map $x \mapsto X(x)$ need not be one-to-one, F may indeed be multivalued.

Remark 2. Instead of the parameterization (2.8), one can fix an integer ν and perform the same construction starting with f_ν instead of f_1. The definition (2.6) would thus be replaced by

$$X_k(x) = \phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_\nu(x), \quad k \geq \nu. \tag{2.9}$$

In this case, the formulas (2.7)–(2.8) still yield a continuous parameterization of the graph of F, satisfying the sharper estimate

$$\begin{gathered}|x - X(x)| + |f(x) - f_\nu(x)| \leq 2^{1-\nu} \\ \text{for all } x \in \overline{\Omega}.\end{gathered} \tag{2.10}$$

Lemma 3. *The multifunction* $F : \overline{\Omega} \hookrightarrow \mathbf{R}^m$ *defined at* (2.8) *is upper semicontinuous and has nonempty, compact, connected values.*

Proof. 1. Let Y denote the product space $\overline{\Omega} \times \mathbf{R}^m$, and let $\mathcal{K}(Y)$ denote the collection of all nonempty compact subsets of Y. Recall that $\mathcal{K}(Y)$ with the Hausdorff distance d_H is a complete metric space. Consider the sequence of graphs $\{\text{graph}\,(f_\ell)\}_{\ell\geq 1}$ as a sequence of compact sets in $\mathcal{K}(Y)$. By (2.1), this sequence is Cauchy and hence it converges to a unique compact set $S \subset Y$, in the Hausdorff metric.

We claim that $S = \text{graph}\,(F)$. For this purpose, it is sufficient to show that the subsequence $\{\text{graph}\,(f_k)\}_{k\geq 1}$ in (2.4) converges to graph (F), in the Hausdorff metric. Given $\varepsilon > 0$, choose k large enough so that for every $x \in \overline{\Omega}$,

$$\Big|X(x) - X_k(x)\Big| < \frac{\varepsilon}{2} \quad \text{and} \quad \Big|f(x) - f_k(X_k(x))\Big| < \frac{\varepsilon}{2}.$$

Consider the homeomorphism $\phi \doteq X_k^{-1} \in \text{Hom}\,(\Omega)$, i.e., the inverse of the map X_k in (2.6). Observing that $X_k(\phi(x)) = x$ and setting

$y = \phi(x)$, we obtain

$$\begin{aligned} d_H\Big(&\text{graph}\,(f_k),\ \text{graph}\,(F)\Big) \\ &\leq \sup_{x\in\Omega} d\Big((x,\ f_k(x)),\ (X(\phi(x)),\ f(\phi(x)))\Big) \\ &\leq \sup_{x\in\Omega} d\Big((x,\ f_k(x)),\ (X_k(\phi(x)),\ f_k(X_k(\phi(x))))\Big) \\ &\quad + \sup_{x\in\Omega} d\Big((X_k(\phi(x)),\ f_k(X_k(\phi(x)))),\ (X(\phi(x)),\ f(\phi(x)))\Big) \\ &\leq \sup_{y\in\Omega} \Big(|X_k(y) - X(y)| + |f_k(X_k(y)) - f(y)|\Big) < \varepsilon. \end{aligned}$$

This proves our claim. □

We conclude that $\{\text{graph}\,(f_\ell)\}$ converges to graph (F) as $\ell \to \infty$. Notice that the subsequences$\{f_k\}$ and $\{\phi_k\}$ determine the continuous maps (2.7) which provide a parametrization of graph (F). This parameterization depends on the chosen subsequence, while graph (F) does not.

2. Since $S = \text{graph}\,(F)$ is a compact subset of the product space $\overline{\Omega} \times \mathbf{R}^m$, it is clear that the multifunction F has closed graph, hence it is upper semicontinuous.

We claim that F is defined on the whole set $\overline{\Omega}$. Towards this goal, let $\pi : \overline{\Omega} \times \mathbf{R}^m \to \overline{\Omega}$ be the projection operator. Since $\{\text{graph}\,(f_\ell)\}$ is a convergent sequence of compact sets, the continuity of π implies

$$d_H\Big(\pi(\text{graph}\,(f_\ell)),\ \pi(\text{graph}\,(F))\Big) \to 0 \quad \text{as } \ell \to \infty. \tag{2.11}$$

For each ℓ, the domain of f_ℓ is the entire set $\overline{\Omega}$, hence $\pi(\text{graph}\,(f_\ell)) = \overline{\Omega}$. Taking the limit, we conclude that $\pi(\text{graph}\,(F)) = \overline{\Omega}$ as well. Hence the domain of F is the whole set $\overline{\Omega}$. This also implies that the map $X : \overline{\Omega} \mapsto \overline{\Omega}$ is surjective (but possibly not one-to-one).

3. Using the parametrization (2.7), for every $x \in \overline{\Omega}$ one has

$$F(x) = \{f(y);\ X(y) = x\}. \tag{2.12}$$

The surjectivity of $X(\cdot)$ implies $F(x)$ is non-empty. We claim that $F(x)$ is connected. Indeed, consider any two points $a, b \in F(x)$. By assumption, there exist sequences

$$(x_k, f_k(x_k)) \longrightarrow (x, a), \qquad (y_k, f_k(y_k)) \longrightarrow (x, b).$$

For each $k \geq 1$, the segment joining x_k with y_k can be parameterized as

$$\theta \mapsto x_k^\theta \doteq \theta x_k + (1-\theta) y_k, \quad \theta \in [0,1].$$

Consider the compact sets

$$S_k \doteq \{(x_k^\theta,\ f_k(x_k^\theta));\ \theta \in [0,1]\} \subseteq \text{graph}\,(f_k).$$

Taking a subsequence, we can assume the convergence $S_k \to S$ in the Hausdorff metric, for some compact set $S \subset \text{graph}\,(F)$. Since f_k is continuous, each set S_k is path connected, hence S is connected as well. Clearly, $(x,a) \in S$ and $(x,b) \in S$.

We claim that every point $(y,v) \in S$ has the same first component, namely $y = x$. Indeed, given $\varepsilon > 0$, from the definitions it follows that $|x_k^\theta - x| \leq \varepsilon$ for all $\theta \in [0,1]$ and all k large enough. If $(y,v) \in S$, the above implies $|y - x| \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves our claim.

We have thus proved that every two points $a, b \in F(x)$ are contained in a compact connected subset of $F(x)$. Therefore the set $F(x)$ is connected. □

The fact that the graph of F is obtained as a limit of graphs of continuous functions imposes further topological properties on the multifunction F.

Lemma 4. *Let $m = 1$, and assume that the domain $\Omega \subset \mathbf{R}^n$ is convex. Then for any two real numbers $a < b$ there exist finitely many compact connected sets $K_1, \ldots, K_N$ such that*

$$\begin{aligned}\Big\{x \in \Omega;\ F(x) \cap [a,\infty[\ \neq \varnothing\Big\} &\supseteq \bigcup_{i=1}^N K_i \\ &\supseteq \Big\{x \in \Omega;\ F(x) \cap [b,\infty[\ \neq \varnothing\Big\}.\end{aligned} \tag{2.13}$$

Proof. Fix $\varepsilon = (b-a)/3$. Recalling (2.10), we can choose ν large enough so that the corresponding parameterization $y \mapsto (X(y), f(y))$ of the graph of F satisfies

$$\sup_{y \in \overline{\Omega}} \Big(|X(y) - y| + |f(y) - f_\nu(y)|\Big) < \varepsilon. \tag{2.14}$$

Since f_ν is uniformly continuous on the compact set $\overline{\Omega}$, there exists a $\delta > 0$ such that $|y - y'| \le \delta$ implies $|f_\nu(y) - f_\nu(y')| < \varepsilon$. Consider the compact sets

$$V_{b-\varepsilon} \doteq \Big\{ y \in \overline{\Omega};\ f_\nu(y) \ge b - \varepsilon \Big\}$$
$$V_{a+\varepsilon} \doteq \Big\{ y \in \overline{\Omega};\ f_\nu(y) \ge a + \varepsilon \Big\}.$$

Cover V_b with finitely many closed balls $\overline{B}(x_i, \delta)$, $i = 1, \dots N$, centered at points $x_i \in V_{b-\varepsilon}$ with radius δ. Define

$$K_i \doteq X(\overline{\Omega} \cap \overline{B}_i(x_i, \delta)).$$

Being the continuous image of a closed convex set, it is clear that each K_i is a compact connected set. By (2.14) it follows that

$$\begin{aligned}
&\Big\{ x \in \overline{\Omega};\ F(x) \cap [a, \infty[\ \neq \varnothing \Big\} \\
&\qquad = \Big\{ x \in \overline{\Omega};\ x = X(y),\ \ f(y) \ge a \text{ for some } y \in \overline{\Omega} \Big\} \\
&\qquad \supseteq \Big\{ x \in \overline{\Omega};\ x = X(y),\ \ f_\nu(y) \ge a + \varepsilon \text{ for some } y \in \overline{\Omega} \Big\} \\
&\qquad = \Big\{ x \in \overline{\Omega};\ x = X(y),\ \ f_\nu(y) \ge b - 2\varepsilon \text{ for some } y \in \overline{\Omega} \Big\} \\
&\qquad \supseteq \Big\{ x \in \overline{\Omega};\ x = X(y),\ \ y \in \cup_{i=1}^N (\overline{\Omega} \cap \overline{B}_i(x_i, \delta)) \Big\} \\
&\qquad = \bigcup_{i=1}^N K_i \\
&\qquad \supseteq \Big\{ x \in \overline{\Omega};\ x = X(y),\ \ f_\nu(y) \ge b - \varepsilon \text{ for some } y \in \overline{\Omega} \Big\} \\
&\qquad \supseteq \Big\{ x \in \overline{\Omega};\ x = X(y),\ \ f(y) \ge b \text{ for some } y \in \overline{\Omega} \Big\} \\
&\qquad = \Big\{ x \in \Omega;\ F(x) \cap [b, \infty[\ \neq \varnothing \Big\}.
\end{aligned}$$

This proves (2.13). $\square$

3. A multidimensional example. As shown by Example 2, the map F can indeed be multivalued. In the one-dimensional case, where $\overline{\Omega} = [0, T]$ is an interval, the multifunction F must be single-valued at all but countably many points x_j. Indeed, in this case the map

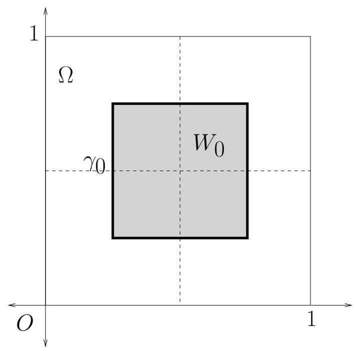

FIGURE 2. The initial curve γ_0, enclosing the square W_0.

$X : [0, T] \mapsto [0, T]$ must be a surjective, nondecreasing map. If $F(x)$ is multivalued at some point $x \in [0, T]$, then there exists an $\alpha < \beta$ such that $X(y) = x$ for all $y \in [\alpha, \beta]$. Since $[0, T]$ can contain at most countably many disjoint intervals $[\alpha_j, \beta_j]$, the map F can be multivalued only at countably many points.

On the other hand, in dimension $n \geq 2$, the map F can be multivalued at all points $x \in \Omega$.

Example 3. Let $\overline{\Omega} \doteq [0, 1] \times [0, 1]$ be the closed unit square. We construct a sequence of functions $f_k : \overline{\Omega} \mapsto \mathbf{R}$ which is Cauchy for the metric $d^\diamondsuit$ in (1.6). The limit as $k \to \infty$ will determine a multifunction F, multivalued at every point $x \in \overline{\Omega}$.

1. We start with a simple, closed curve γ_0 in the interior of Ω enclosing a compact set W_0.

2. We use a sequence of homeomorphisms $\varphi_k : \overline{\Omega} \mapsto \overline{\Omega}$ to continuously deform γ_0 into a Peano curve, filling $\overline{\Omega}$.

3. We consider a continuous function f_0 such that $f_0 \equiv 1$ on W_0 and $f_0 \equiv 0$ outside of some ε-neighborhood of W_0.

4. The sequence $(f_k)_{k \geq 1}$ is defined recursively, by setting $f_{k+1}(\varphi_{k+1}(x)) = f_k(x)$.

5. The resulting multifunction then satisfies $F(x) = [0, 1]$ at every $x \in \Omega$.

These steps are now explained in more detail.

1. As initial curve γ_0 (see Figure 2) we take the perimeter of the square

$$W_0 \doteq \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right].$$

Next consider the curve γ_1 depicted in Figure 3. This curve bounds a region W_1 consisting of 5 squares having sides of length 1/4, connected by narrow necks as shown. These necks have some small width $\delta > 0$, so that γ_1 is again a closed, simple curve. The curve γ_1 can be obtained as the image of γ_0 under a certain homeomorphism $\varphi_1 \colon \mapsto \overline{\Omega}$.

Referring to Figure 2, we divide $\overline{\Omega}$ into 4 equal squares, each having sides of length 1/2. We can choose a homeomorphism φ_1 having the following properties.

- φ_1 keeps fixed the boundary $\partial\Omega$.
- φ_1 maps each of the 4 smaller squares onto itself.
- φ_1 maps the boundary of each smaller square into itself. In fact each side of every smaller square is mapped to itself, although not by the identity map.
- When we further subdivide each square into 4 smaller squares having sides of length 1/4, the curve γ_1 passes through each of the resulting 16 subregions.

The procedure can now be iterated. After k iterations, we achieve the following:

(i) The original square $\overline{\Omega}$ has been divided into 4^{k+1} smaller squares, each having sides of length 2^{-k-1}.

(ii) The curve γ_k is a simple, closed curve, enclosing a compact region W_k.

(iii) W_k consists of $\sum_{i=0}^{k} 4^i$ squares joined by "necks" of width $\delta/2^{k-1}$.

(iv) γ_k is the image of γ_{k-1} under a homeomorphism $\varphi_k : \overline{\Omega} \mapsto \overline{\Omega}$ which keeps fixed the boundary $\partial\Omega$.

(v) γ_k passes through each of the 4^{k+1} smaller square regions in $\overline{\Omega}$.

Figures 2, 3 and 4 show the cases $k = 0, 1, 2$.

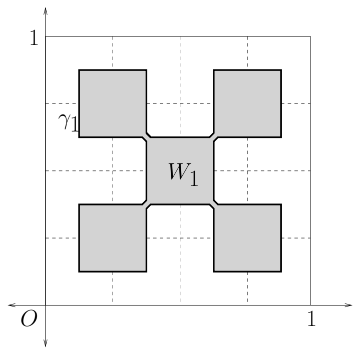

FIGURE 3. The curve γ_1, enclosing the connected region W_1.

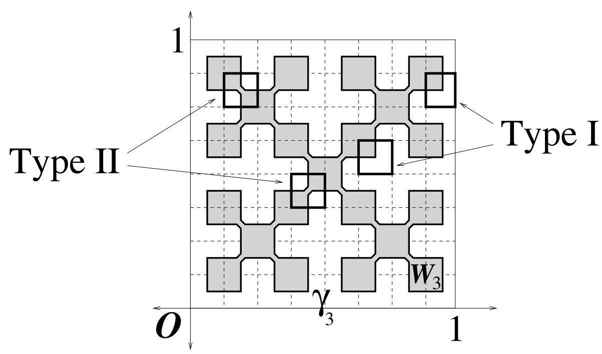

FIGURE 4. The curve γ_2, enclosing the connected region W_2, and the two different types of subregions.

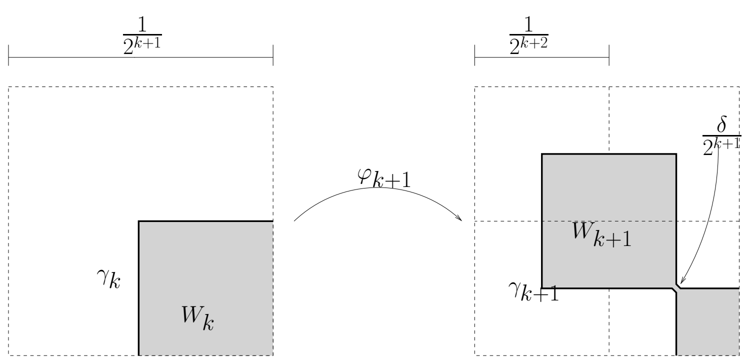

FIGURE 5. The transformation φ_{k+1}, in a type I region.

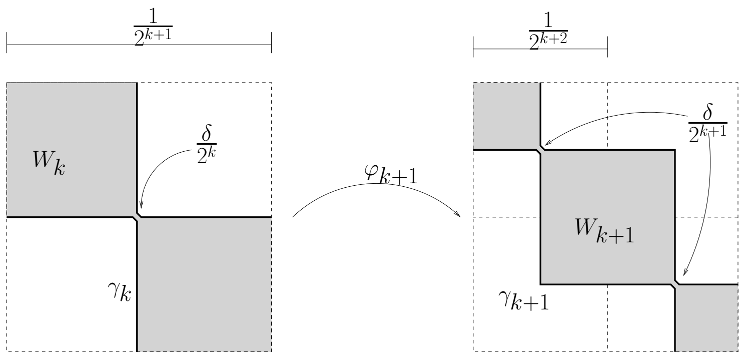

FIGURE 6. The transformation φ_{k+1}, in a type II region.

Notice that, up to rotations, there are two distinct types of regions, which we label type I and type II, as shown in Figure 4. We describe the homeomorphism φ_{k+1} by its action on each type of region, see Figures 5 and 6.

(i) The homeomorphism φ_{k+1} maps each region onto itself, maintaining the orientation of the boundary of each region. Consequently, for each $x \in \overline{\Omega}$ we have the inequality

$$|x - \varphi_k(x)| \leq \frac{\sqrt{2}}{2^k}. \tag{3.15}$$

(ii) The boundary of each region consists of 4 segments. If γ_k does not pass through a given segment, then φ_{k+1} keeps fixed all points on that segment. In particular, φ_{k+1} keeps fixed all points in $\partial\Omega$.

(iii) Let γ_{k+1} be the image of γ_k under φ_{k+1}. Then γ_{k+1} is a simple closed curve, bounding a region W_{k+1} consisting of squares joined by "necks" of width $\delta/2^k$.

As in (2.7), let us define $X_k(x) \doteq \varphi_k \circ \cdots \circ \varphi_1(x)$, and set

$$X(x) \doteq \lim_{k\to\infty} X_k(x).$$

For each $x \in \overline{\Omega}$, the sequence $\{X_k(x)\}$ is Cauchy. Hence the limit $X(x)$ is well-defined for every $x \in \overline{\Omega}$. Furthermore, γ_k is constructed so that each point of $\overline{\Omega}$ is within a distance $\sqrt{2}/2^k$ of some point in γ_k. It follows that the limit curve

$$\gamma \doteq \lim_{k\to\infty} \gamma_k = X(\gamma_0) \tag{3.16}$$

is dense in $\overline{\Omega}$. But since $X(\cdot)$ is continuous and γ_0 is closed, γ must be all of $\overline{\Omega}$.

2. Choose $\varepsilon = 1/4$, and define a continuous function $f_0 : \overline{\Omega} \mapsto [0,1]$ by setting

$$f_0(x) = \begin{cases} 1 & \text{if } x \in W_0, \\ 0 & \text{if } d(x, W_0) \geq 1/4, \\ 1 - 4\,d(x, W_0) & \text{if } 0 < d(x, W_0) < 1/4. \end{cases} \tag{3.17}$$

3. Now consider the sequence of functions $(f_k)_{k\geq 1}$ defined recursively as

$$f_k(\varphi_k(x)) = f_{k-1}(x) \tag{3.18}$$

Since φ_k is a homeomorphism, f_k is well defined on the whole square $\overline{\Omega}$. Furthermore, calling $Y_0 \doteq \{x \in \overline{\Omega},\ d(x, W_0 \geq 1/4\}$, we have

$$\begin{aligned} f_k(X_k(x)) &= f_0(x) \quad \text{for all } x \in \Omega \\ f_k &\equiv 1 \quad \text{on } W_k = X_k(W_0) \\ f_k &\equiv 0 \quad \text{on } Y_k \doteq X_k(Y_0). \end{aligned}$$

Moreover, by construction the sequence $(f_k)_{k\geq 1}$ is Cauchy with respect to the distance $d^{\diamondsuit}$.

4. Consider the limit functions $X(\cdot)$ and $f(\cdot)$, defined at (2.7), and the corresponding multifunction F in (2.8). Let any point $x \in \Omega$ be given.

Since γ is a Peano curve, there exists a point $w \in \gamma_0$ such that $X(w) = x$. Hence

$$f(w) = \lim_k f_k(X_k(w)) = 1, \tag{3.19}$$

Therefore $1 = f(w) \in F(X(w)) = F(x)$.

On the other hand, the set $X(Y_0)$ is also dense and closed in $\overline{\Omega}$. Hence it is equal to $\overline{\Omega}$. We can thus find a point $z \in Y_0$ such that $x = X(z)$ and

$$f(z) = \lim_k f_k(X_k(z)) = 0. \tag{3.20}$$

Therefore $0 = f(z) \in F(X(z)) = F(x)$.

By Lemma 2, $F(x)$ is a connected set. We thus conclude that $F(x) = [0, 1]$ for every $x \in \overline{\Omega}$.

4. Multidimensional graph completions. Given a function $f : \overline{\Omega} \mapsto \mathbf{R}^m$, relying on the previous ideas we would like to construct a "graph completion" of f. Roughly speaking, this should be a multifunction F whose graph is obtained as limit of graphs of functions f_k, converging in the $d^\diamondsuit$ metric. Moreover, $F(x)$ should be single valued and coincide with $\{f(x)\}$ at almost every point x. As shown by Example 3, however, in several space dimensions the multifunction F can be multivalued everywhere. One can easily achieve the inclusion graph $(f) \subseteq$ graph (F), but there may be little relation between F and the original function f.

To make further progress, we need to impose some restrictions on the Cauchy sequences of continuous functions f_k used in (2.4). A natural assumption is that all these functions f_k should have uniformly bounded $W^{1,1}$ norm. This leads to

Definition 3. Given a bounded open set $\Omega \subset \mathbf{R}^n$ and function $f : \overline{\Omega} \mapsto \mathbf{R}^m$ having bounded variation, we say that a multifunction $F : \overline{\Omega} \hookrightarrow \mathbf{R}^m$ is a *graph completion* of f if there exists a sequence of functions f_k which is Cauchy with respect to the distance $d^\diamondsuit$ and satisfies

$$\sup_k \|f_k\|_{W^{1,1}(\Omega)} < \infty, \tag{4.21}$$

$$d_H\Big(\text{graph}\,(f_k),\ \text{graph}\,(F)\Big) \to 0, \qquad \text{graph}\,(f) \subseteq \text{graph}\,(F). \tag{4.22}$$

Notice that in this case there exists a continuous surjective map $\tilde{f} : \overline{\Omega} \mapsto \text{graph}\,(F)$.

In the one-dimensional case, a slight modification of the arguments in [**12**] yields the existence of a graph completion, for every map $f : [0,T] \mapsto \mathbf{R}^m$ having bounded variation.

Proposition 1. *Let $f : [0,T] \mapsto \mathbf{R}^m$ be a right continuous function with bounded variation. Then f admits a graph completion.*

Proof. For $t \in [0,T]$ and $k \geq 1$, define

$$s_k(t) \doteq t + \frac{1}{k}\,\text{Tot. Var.}\,\{f;\,[0,t]\}$$

where the right hand side involves the total variation of f on the subinterval $[0,t]$. The map $t \mapsto s_k(t)$ is strictly increasing, right continuous, from $[0,T]$ into $[0,S_k]$, with

$$S_k = T + \frac{1}{k}\text{Tot. Var.}\,\{f;\,[0,T]\}.$$

Its inverse $s \mapsto t_k(s)$ is Lipschitz continuous.

Define the Lipschitz continuous function $\hat{f}_k : [0,S_k] \mapsto \mathbf{R}^m$ by setting

$$\begin{aligned}
&s_k^-(t) \doteq \inf\{s;\, t = t_k(s)\},\\
&s_k^+(t) \doteq \sup\{s;\, t = t_k(s)\}\\
&\hat{f}_k(s) \doteq \theta f(t) + (1-\theta) f(t-) \text{ if } t = t_k(s),\\
&\qquad s = \theta s_k^+(t) + (1-\theta) s_k^-(t).
\end{aligned}$$

By a reparameterization, we obtain a function $f_k : [0,T] \mapsto \mathbf{R}^m$, defined as

$$f_k(t) \doteq \hat{f}_k\left(\frac{S_k}{T}\,t\right).$$

One now checks that the functions f_k form a Cauchy sequence with respect to the distance $d^\diamond$. Moreover, their limit yields a multifunction F satisfying (4.22). □

On the other hand, when the domain Ω has dimension $n \geq 2$, Lemma 4 puts some topological obstructions to the general existence of a graph completion.

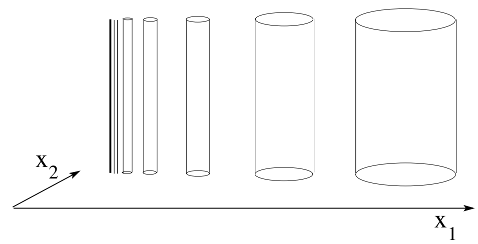

FIGURE 7. A function with bounded variation, which does not admit a graph completion.

Example 4. Let B_j be the open disc centered at the point $p_j = ((1/j),\, 0)$, with radius $r_j = j^{-3}$. Observe that, for $j \geq 4$ the countably many discs B_j are mutually disjoint. Moreover, since the radii r_j converge to zero very fast, the union of their boundaries $\cup_{j\geq 4}\, \partial B_j$ has finite length. Therefore, the function $f : \mathbf{R}^2 \mapsto \mathbf{R}$ defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in \cup_{j\geq 4} B_j, \\ 0 & \text{otherwise}, \end{cases} \tag{4.23}$$

has bounded variation. The multifunction

$$F(x) = \begin{cases} \{1\} & \text{if } x \in \cup_{j\geq 4} B_j, \\ \{0\} & \text{if } x \notin \overline{\cup_{j\geq 4} B_j}, \\ [0,1] & \text{if } x \in \cup_{j\geq 4} \partial B_j \text{ or if } x = (0,0), \end{cases} \tag{4.24}$$

is upper semicontinuous with compact convex values. However, by Lemma 4 the BV function f in (4.23) does not admit a graph completion. Here the obstruction lies in the fact that, although the graph of the multifunction F has locally bounded two-dimensional measure, there does not exist any continuous parameterization $\psi : \mathbf{R}^2 \mapsto \text{graph}\,(F)$.

Proposition 2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set, and let F be a graph completion of a function $f : \overline{\Omega} \mapsto \mathbf{R}^m$. Then*

(i) *The graph of F has bounded n-dimensional measure.*

(ii) *For almost every $x \in \Omega$ one has $F(x) = \{f(x)\}$.*

Proof. By the assumption (4.21), for every $k \geq 1$ the graph of f_k has bounded n-dimensional Hausdorff measure, namely

$$m_n\Big(\mathrm{graph}\,(f_k)\Big) \leq C$$

for some constant C independent of k. We now consider the maps $x \mapsto \Psi_k(x) \doteq \Big(X_k(x),\, f_k(X_k(x))\Big)$ defined at (2.6)–(2.7). These are continuous maps, converging to a continuous map $x \mapsto \Psi(x) \doteq (X(x), \tilde{f}(x))$ uniformly on $\overline{\Omega}$. Define the functional

$$J(\Psi) \doteq m_n(\Psi(\overline{\Omega})).$$

Taking the Hausdorff limit of the graphs of f_k and using the lower semicontinuity result in [**1**], we conclude

$$\begin{aligned} m_n\Big(\mathrm{graph}\,(F)\Big) &= J(\Psi) \leq \liminf_{k\to\infty} J(\Psi_k) \\ &= \liminf_{k\to\infty}\, m_n\Big(\mathrm{graph}\,(f_k)\Big) \leq C. \end{aligned}$$

This already implies that $F(x)$ must be single-valued for almost every $x \in \overline{\Omega}$. Since graph $(f) \subseteq$ graph (F), this implies $F(x) = \{f(x)\}$ for almost every $x \in \overline{\Omega}$. □

5. Upper semicontinuous differential inclusions. The main goal of this section is to prove Theorem 1, on the existence of solutions to differential inclusions with Cellina $W^{1,1}$-approximable right hand side.

We begin with a couple of results, providing interesting classes of multifunctions which are Cellina $W^{1,1}$-approximable. In the following, we denote by $BV(\mathbf{R}^n)$ the set of all maps $g : \mathbf{R}^n \mapsto \mathbf{R}^m$ having bounded variation. See [**17**] for the general theory of BV functions of several variables.

Proposition 3. *Let $G : \mathbf{R}^n \mapsto \mathbf{R}^m$ be an upper semicontinuous multifunction with compact, convex values. Assume that G admits a (possibly discontinuous) selection $g(x) \in G(x)$, with $g \in BV(\Omega)$. Then G is Cellina $W^{1,1}$-approximable.*

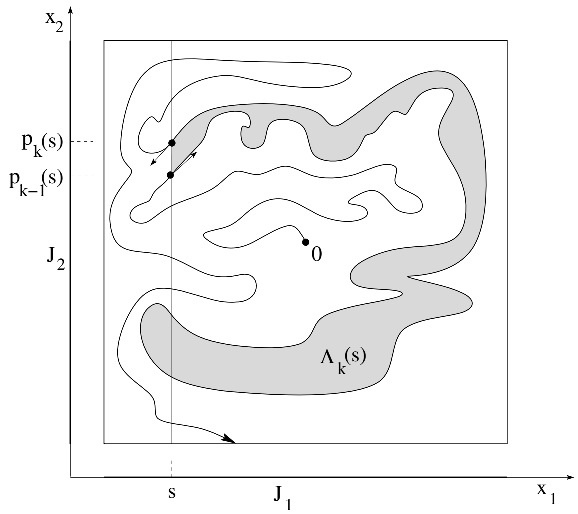

FIGURE 8. If a vector field f with $|f(x)| \equiv 1$ has a very long integral curve remaining inside the unit square, then the total variation of f must be large.

Proof. Indeed, consider the mollifications $g_\varepsilon \doteq J_\varepsilon * g$. Here the function $J_\varepsilon : \mathbf{R}^n \mapsto \mathbf{R}$ is defined by setting

$$J(x) \doteq \begin{cases} C_n \exp\{1/(|x|^2 - 1)\} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where the constant C_n is chosen so that $\int_{\mathbf{R}^n} J(x)\,dx = 1$. For each $\varepsilon > 0$ we then define

$$J_\varepsilon(x) \doteq \frac{1}{\varepsilon^n}\, J\left(\frac{x}{\varepsilon}\right).$$

For every $\varepsilon > 0$, standard properties of mollifications now imply

$$\|g_\varepsilon\|_{W^{1,1}} \leq \|g\|_{BV}$$

Since the graph of g_ε is contained in an ε-neighborhood of the graph of the multifunction G, we conclude that G is $W^{1,1}$-Cellina approximable. □

Proposition 4. *If $G : \Omega \mapsto \mathbf{R}^m$ is $W^{1,p}$-Cellina approximable and $\psi : \mathbf{R}^m \mapsto \mathbf{R}^k$ is a Lipschitz continuous function, then the composition $F(x) \doteq \psi(G(x))$ is $W^{1,p}$-Cellina approximable.*

Proof. If $g_\varepsilon : \Omega \mapsto \mathbf{R}^m$ is a continuous approximate selection of G, then the composition $f_\varepsilon(x) = \psi(g_\varepsilon(x))$ is a continuous approximate selection of F. Moreover, there exists a constant C such that $\|f_\varepsilon\|_{W^{1,p}} \leq C\,\|g_\varepsilon\|_{W^{1,p}}$. □

Proof of Lemma 1. To derive a lower bound on the total variation of f, we proceed as follows. Consider the square $Q = [-1, 1] \times [-1, 1]$. Let $\gamma : [0, T] \mapsto Q$ be a trajectory of the vector field f. Denote its components along the axes as $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Then its length is computed as

$$\|\gamma\| = T = \int_0^T \sqrt{\dot\gamma_1^2 + \dot\gamma_2^2}\, dt.$$

Consider the domains

$$J_1 \doteq \{x_1 \in [-1,\, 1];\ \gamma_1(t) = x_1 \text{ for finitely many times } t\},$$
$$J_2 \doteq \{x_2 \in [-1,\, 1];\ \gamma_2(t) = x_2 \text{ for finitely many times } t\}.$$

Since γ is a smooth curve, we must have

$$\text{meas}\,(J_1) = \text{meas}\,(J_2) = 2.$$

For $s \in J_1$ we shall denote by $p_0(s) < p_1(s) < \cdots < p_N(s)$ the x_2-coordinates of the intersections of the trajectory γ with the vertical line $\{(x_1, x_2);\, x_1 = s\}$. Similarly, for $s \in J_2$ we denote by $q_0(s) < q_1(s) < \cdots < q_M(s)$ the x_1-coordinates of the intersections of the trajectory γ with the horizontal line $\{(x_1, x_2);\, x_2 = s\}$. By $\tau(p_k(s))$ and $\tau'(q_k(s))$ we denote the times where these intersections occur.

Observing that $dt = (\dot\gamma_1^2 + \dot\gamma_2^2)dt = |\dot\gamma_1|\, dx_1 + |\dot\gamma_2| dx_2$, the length of γ can be estimated as

(5.25)
$$\begin{aligned}
\|\gamma\| &= \int_{J_1} \sum_{\gamma_1(t_k)=x_1} |\dot\gamma_1(t_k)|\, dx_1 + \int_{J_2} \sum_{\gamma_2(t_k)=x_2} |\dot\gamma_2(t_k)|\, dx_2 \\
&= \int_{J_1} \sum_{k=0}^{N(s)} \Big|\dot\gamma_1(\tau(p_k(s)))\Big|\, ds + \int_{J_2} \sum_{k=0}^{M(s)} \Big|\dot\gamma_2(\tau'(q_k(s)))\Big|\, ds \\
&= \int_{J_1} \sum_{k=0}^{N(s)} \Big|f_1(s,\, p_k(s))\Big|\, ds + \int_{J_2} \sum_{k=0}^{M(s)} \Big|f_2(q_k(s),\, s)\Big|\, ds.
\end{aligned}$$

To relate this length with the total variation of the vector field $f = (f_1, f_2) : Q \mapsto \mathbf{R}^2$, we observe that

$$\|Df\|_{\mathbf{L}^1(Q)} \geq \int_{J_1} \int_{-1}^{1} \left| \frac{\partial f_1}{\partial x_2}(s, x_2) \right| ds\, dx_2.$$

We claim that the component f_1 must have a zero on each subinterval $[p_{k-1}(s),\, p_k(s)]$. To see this, consider the simple closed curve obtained as the union of the portion of trajectory γ between $P_{k-1} \doteq (s, p_{k-1}(s))$ and $P_k \doteq (s, p_k(s))$, together with the vertical segment S joining the two points P_{k-1}, P_k. By the Jordan curve theorem, this curve encloses a compact region $\Lambda_k(s) \subset Q$. If the component f_1 does not change sign on the segment S, then the set $\Lambda_k(s)$ would be either positively invariant or negatively invariant for the flow of the vector field f. Since f is continuous, by the classical Poincaré-Bendixon theory it must have a zero inside $\Lambda_k(s)$. But this is impossible, because we are assuming $|f(x)| \equiv 1$.

For $k = 0, 1, \ldots N(s) - 1$, let $z_{k-1}(s)$ denote the location of a zero of the component f_1 on the subinterval $[p_{k-1}(s),\, p_k(s)]$. For a fixed $s \in [-1, 1]$, we estimate the total variation of the function $x_2 \mapsto f_1(s, x_2)$ using the partition

$$p_0(s) < z_0(s) < p_1(s) < z_1(s) < \cdots < z_{N-1}(s) < p_N(s).$$

For a fixed $s \in J_1$, this yields

$$\begin{aligned} \int_{-1}^{1} \left| \frac{\partial f_1}{\partial x_2}(s, x_2) \right| dx_2 &\geq |f_1(s, p_0(s))| + |f_1(s, p_N(s))| \\ &\quad + \sum_{k=1}^{N(s)-1} 2 \Big| f_1(s, p_k(s)) \Big| \\ &\geq -2 + 2 \sum_{k=0}^{N(s)} \Big| f_1(x_1, p_k(s)) \Big|. \end{aligned}$$

Similarly, for a given $s \in J_2$, we have

$$\int_{-1}^{1} \left| \frac{\partial f_2}{\partial x_1}(x_1, s) \right| dx_1 \geq -2 + 2 \sum_{k=0}^{M(s)} \Big| f_2(q_k(s), s) \Big|.$$

Using the two above inequalities in (5.25) we obtain

$$\begin{aligned}\|\gamma\| &\leq \int_{J_1}\left\{1+\frac{1}{2}\int_{-1}^{1}\left|\frac{\partial f_1}{\partial x_2}(x_1,x_2)\right|dx_2\right\}dx_1 \\ &\quad+\int_{J_2}\left\{1+\frac{1}{2}\int_{-1}^{1}\left|\frac{\partial f_2}{\partial x_1}(x_1,x_2)\right|dx_1\right\}dx_2 \\ &\leq 4+\frac{1}{2}||Df||_{L^1(Q)}. \qquad \square\end{aligned}$$

Proof of Theorem 1. In the trivial case $0 \in F(0)$, the function $x(t) \equiv 0$ is a global solution. Throughout the following, we thus assume $0 \notin F(0)$. Since F is bounded, to prove the theorem it suffices to show the existence of a local solution, defined for $t \in [0, \delta]$, with $\delta > 0$ small. The proof will be worked out in various steps.

1. By upper semicontinuity one has $0 \notin F(x)$ for all x in a neighborhood of the origin. By a rescaling of coordinates, we can assume $B(0,\rho) \cap F(x) = \varnothing$ for all $x \in Q \doteq [-1,1] \times [-1,1]$.

Next, we observe that it is not restrictive to further assume

$$|v| = 1 \quad \text{for all } v \in F(x), \quad x \in \mathbf{R}^2. \tag{5.26}$$

Indeed, consider the normalized multifunction

$$\widetilde{F}(x) = \left\{\frac{v}{|v|};\ v \in F(x)\right\}.$$

Observe that $\widetilde{F}$ is also Cellina $W^{1,1}$-approximable. Indeed, if f_ε is an ε-approximate selection of F, then the map $x \mapsto \tilde{f}_\varepsilon(x) \doteq f_\varepsilon(x)/|f_\varepsilon(x)|$ is an ε-approximate selection of $\widetilde{F}$. Moreover $\|\tilde{f}_\varepsilon\|_{W^{1,1}(Q)} \leq C\,\|f_\varepsilon\|_{W^{1,1}(Q)}$, for some constant C depending only on ρ.

If $t \mapsto y(t)$ is a solution to the auxiliary problem

$$\dot{y} \in \widetilde{F}(y), \quad y(0) = 0 \in \mathbf{R}^2, \tag{5.27}$$

then there exists a bounded measurable function $\lambda(t) \geq \rho$ such that

$$\lambda(t)\,\dot{y}(t) \in F(y(t)) \quad \text{for a.e. } t.$$

Define the rescaled time

$$\tau(t) \doteq \int_0^t \frac{ds}{\lambda(s)},$$

and call $\tau \mapsto t(\tau)$ the inverse transformation. Setting $x(\tau) \doteq y(t(\tau))$, we find

$$\frac{dx}{d\tau} = \frac{dy}{dt}\,\frac{dt}{d\tau} = \dot{y}(t(\tau))\,\lambda(t(\tau)) \in F(x(\tau)).$$

2. From now on, we thus assume that (5.26) holds. Given a sequence $\varepsilon_k \downarrow 0$, consider a corresponding sequence of smooth approximate selections $\{f_k\}$, whose norms$\{\|f_k\|_{W^{1,1}}\}$ remain uniformly bounded.

For each k, the Cauchy problem

$$\dot{x}(t) = f_k(x), \quad x(0) = 0 \in \mathbf{R}^2 \tag{5.28}$$

has a unique solution, which we denote as $t \mapsto x_k(t)$. By Lemma 1, each trajectory $x_k(\cdot)$ reaches the boundary ∂Q of the unit square Q at a finite time t_k, with $T \doteq \sup_k\{t_k\} < \infty$.

By possibly taking a subsequence, using the Ascoli-Arzelà compactness theorem we obtain the existence of $T > 0$ and a Lipschitz continuous map $x(\cdot)$ such that $t_k \to T > 0$ and $x_k(t) \to x(t)$, uniformly for $t \in [0, T]$.

3. We now parameterize this limit trajectory $x(\cdot)$ by arc-length. For this purpose, we first define

$$s(t) \doteq \int_0^t |\dot{x}(\tau)|\,d\tau. \tag{5.29}$$

Since $\dot{x}(t) \in \mathrm{co}\,F(x(t))$, one has

$$0 \le \frac{ds}{dt} = |\dot{x}(t)| \le 1.$$

Furthermore,

$$S \doteq s(T) \ge 1, \qquad \mathrm{meas}\left(\left\{s(t);\ \frac{ds(t)}{dt} = 0\right\}\right) = 0. \tag{5.30}$$

Consider the inverse transformation $s \mapsto t(s)$, so that

$$t(s) \doteq \min \left\{ t;\ \int_0^t |\dot{x}(\tau)|\, d\tau \geq s \right\}, \tag{5.31}$$

and reparameterize the trajectory $x(\cdot)$ by arc-length, setting

$$x(s) = x(t(s)), \quad s \in [0, S]. \tag{5.32}$$

4. We claim that, with the parameterization (5.32), one has

$$\frac{d}{ds}\, x(s) \in F(x(s)) \quad \text{for a.e. } s \in [0, S]. \tag{5.33}$$

Indeed, if this is not the case, we could find a time $0 < \tau < T$ such that

(i) τ is a Lebesgue point of the bounded measurable map $t \mapsto \dot{x}(t)$,

(ii) $|\dot{x}(\tau)| > 0$, and

(iii) the inclusion (5.33) fails at $s = s(\tau)$.

Without loss of generality, we can assume that the conditions are chosen so that $\dot{x}(\tau) = (\lambda, 0) \in \mathbf{R}^2$, for some $\lambda > 0$.

If (5.33) fails, by upper semicontinuity there exists a cone $\Gamma_\delta \doteq \{(x_1, x_2);\ |x_2| \leq \delta x_1\}$ such that

$$F(x) \cap \Gamma_\delta = \varnothing \quad \text{for all } x \in \mathbf{N}. \tag{5.34}$$

where $\mathcal{N}$ is a neighborhood of $x(\tau)$. With reference to Figure 9, we now consider a rectangle R with vertices A, B, C, D, contained in $\mathcal{N}$ and with sides parallel to the coordinate axes, containing the point $x(\tau)$ in its interior. By choosing $h > 0$ small enough, we achieve

$$x(\tau + h) \in R \cap \operatorname{int} \Big(x(\tau) + \Gamma_\delta \Big). \tag{5.35}$$

By the uniform convergence $x_k \to x$, we can assume that

$$x_k(\tau) \longrightarrow x(\tau), \qquad x_k(\tau + h) \longrightarrow x(\tau + h) \quad \text{as } k \to \infty,$$

while $x_k(t) \in R$, for all $t \in [\tau,\ \tau + h]$ and all k sufficiently large.

However, we now show that this leads to a contradiction.

Call A_k, B_k the intersections of the boundary $x_k(\tau) + \partial\Gamma_\delta$ with the segments AD and BC, as shown in Figure 9. By shrinking the height of the rectangle R, these points are well defined, for every k large enough. Since $x_k(\tau + h) - x_k(\tau) \in \operatorname{int}\Gamma_\delta$, there will be a smallest time

$$\tau' \doteq \inf\Big\{t \geq \tau;\ x_k(t) - x_k(\tau) \in \operatorname{int}\Gamma_\delta\Big\}.$$

Call $P = x(\tau') \in x_k(\tau) + \partial\Gamma_\delta$. Without loss of generality, assume $P = (p_1, p_2)$ with $p_2 \geq x_{k,2}(\tau)$. Notice that, by (5.34), the second coordinate of the trajectory $x_k = (x_{k,1}, x_{k,2})$ cannot attain its maximum at the crossing time $t = \tau'$, because this would imply

$$\dot{x}_{k,1} \geq 0, \quad 0 \leq \dot{x}_{k,2} \leq \delta\dot{x}_{k,1},$$

in contradiction with (5.34). We thus have

$$m_2 \doteq \max\Big\{x_{k,2}(t);\ t \in [\tau, \tau']\Big\} > p_2.$$

Therefore, there exists an intermediate time $\tau < t_M < \tau'$ where this maximum is attained. Set $M = (m_1, m_2) \doteq x(t_M)$. Consider the domain $\mathcal{D}$ (the shaded domain in Figure 9), bounded by:

– the curve $\{x(t);\ \tau \leq t \leq \tau_M\}$,

– the horizontal line $\{(x_1, x_2);\ x_2 = m_2\}$,

– the vertical segment AB,

– the horizontal segment BB_k,

– the segment joining $x_k(\tau)$ with B_k.

The previous construction implies that, for $t \in [\tau, \tau + h]$, the trajectory $x_k(\cdot)$ cannot leave the domain $\mathcal{D}$. This yields a contradiction, because, for k large, $x_k(\tau + h)$ is arbitrarily close to $x(\tau + h)$.

This contradiction shows that (5.33) holds, thus proving the theorem. □

Example 5. We now show that in dimension $n = 3$ the assumption that the multifunction F is $W^{1,1}$-Cellina approximable does not guarantee the existence of solutions to the Cauchy problem (1.8). This is obtained as a modification of the multifunction in Example 1.

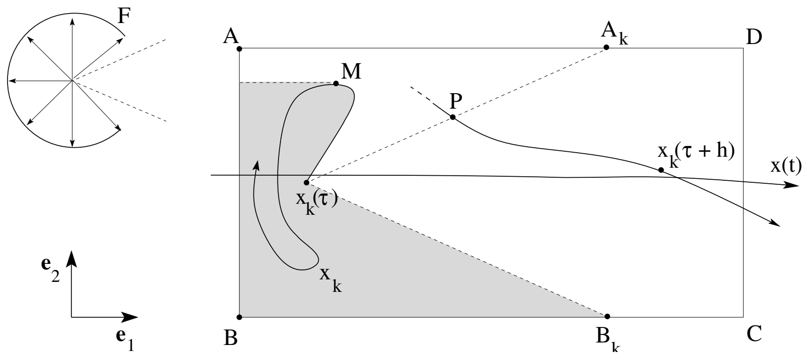

FIGURE 9. If $\dot{x}_k = f_k(x_k)$, where f_k is an approximate selection from F, the sequence of functions $x_k(\cdot)$ cannot converge to a function $x(\cdot)$ with $\dot{x} \approx \theta \mathbf{e}_1$, for any $\theta > 0$.

Consider the continuous function

$$\phi(x_1, x_3) \doteq \begin{cases} x_1 \cos(1/x_1) & \text{if } 0 < |x_3| \leq |x_1|, \\ x_1 \cos(1/x_3) & \text{if } |x_1| < |x_3|, \\ 0 & \text{if } x_1 = x_3 = 0. \end{cases}$$

Define the multifunction $F : \mathbf{R}^2 \mapsto \mathbf{R}^2$ as

$$F(x_1, x_2) \doteq \begin{cases} \{(0, -1, 0)\} & \text{if } x_2 > \phi(x_1, x_3), \\ \{(0, 1, 0)\} & \text{if } x_2 < \phi(x_1, x_3), \\ \Big\{(y_1, y_2, 0);\ y_1 \geq 0,\ y_1^2 + y_2^2 = 1\Big\} & \text{if } x_2 = \phi(x_1, x_3). \end{cases} \tag{5.36}$$

Since the third component of every velocity vector $v \in F(x)$ vanishes, the third component of any solution of the Cauchy problem (1.8) must be identically zero. However, restricted to the plane $\{x_3 = 0\}$, the multifunction F coincides with the multifunction in (1.11). Hence no solution exists.

We observe that the multifunction F is Cellina $W^{1,1}$-approximable. Indeed, we can write $F(x) = \psi(G(x))$ where

$$\psi(\xi) \doteq (\cos \xi,\ \sin \xi,\ 0),$$

and $G : \mathbf{R}^3 \mapsto \mathbf{R}$ is the upper semicontinuous, convex valued multifunction defined as

$$G(x_1, x_2, x_3) \doteq \begin{cases} \{-\pi/2\} & \text{if } x_2 > \phi(x_1, x_3), \\ \{\pi/2\} & \text{if } x_2 < \phi(x_1, x_3), \\ [-\pi/2,\ \pi/2] & \text{if } x_2 = \phi(x_1, x_3). \end{cases} \tag{5.37}$$

We observe that G is almost everywhere single valued. Indeed, $G(x) = \{g(x)\}$ for some function g and all $x = (x_1, x_2, x_3)$ with $x_2 \neq \phi(x_1, x_3)$. We claim that this function g, defined by (5.37), has bounded variation restricted to bounded sets $\Omega \subset \mathbf{R}^3$. Indeed, the partial derivatives of ϕ are computed by

$$\frac{\partial \phi}{\partial x_1}(x_1, x_3) = \begin{cases} \cos(1/x_1) - (1/x_1)\ \sin(1/x_1) & \text{if } 0 < |x_3| < |x_1|, \\ \cos(1/x_3) & \text{if } 0 < |x_1| < |x_3|, \end{cases}$$
$$\frac{\partial \phi}{\partial x_3}(x_1, x_3) = \begin{cases} 0 & \text{if } 0 < |x_3| < |x_1|, \\ -(x_1/x_3^2)\ \sin(1/x_3) & \text{if } 0 < |x_1| < |x_3|. \end{cases}$$

Hence, both partial derivatives are locally integrable. As a consequence, the two-dimensional measure of the graph of ϕ (i.e., the set where g is discontinuous), restricted to any bounded domain, is bounded. Hence the restriction of g to any bounded domain $\Omega \subset \mathbf{R}^3$ is in BV. By Propositions 3 and 4, we conclude that F is Cellina $W^{1,1}$-approximable on any bounded domain $\Omega \subset \mathbf{R}^3$.

This example shows that an n-dimensional extension of Theorem 1 cannot be valid if $n \geq 3$ and we only assume F to be $W^{1,1}$-Cellina approximable. As remarked earlier, to achieve an existence result one apparently should assume that F is Cellina $W^{1,p}$-approximable, for some $p \geq n - 1$.

REFERENCES

1. L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford University Press, Oxford, 2000.

2. H.A. Antosiewicz and A. Cellina, *Continuous selections and differential relations*, J. Differential Equations **19** (1975), 386–398.

3. J.P. Aubin and A. Cellina, *Differential inclusions*, Springer-Verlag, Berlin, 1984.

4. J.P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston, 1990.

5. M. Bardi and I. Capuzzo Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, Boston, 1997.

6. A. Bressan, *On differential relations with lower continuous right-hand side. An existence theorem*, J. Differential Equations **37** (1980), 89–97.

7. ———, *Nonexistence of solutions for differential inclusions with upper semicontinuous non-convex right-hand side*, Rend. Sem. Mat. Univ. Padova **79** (1988), 297–299.

8. ———, *Upper and lower semicontinuous differential inclusions*: *A unified approach*, in *Nonlinear controllability and optimal control*, H. Sussmann, ed., Marcel Dekker, New York, 1990.

9. ———, *Impulsive control of Lagrangian systems and locomotion in fluids*, Discrete Continu. Dynam. Syst. **20** (2008), 1–35.

10. ———, *Lecture notes on functional analysis and linear PDEs*, available at http://www.math.psu.edu/bressan/

11. Aldo Bressan, *Hyper-impulsive motions and controllizable coordinates for Lagrangean systems*, Atti Accad. Naz. Lincei, Memorie, Serie VIII, **19** (1990), 197–246.

12. A. Bressan and F. Rampazzo, *On differential systems with vector-valued impulsive controls*, Boll. Un. Matematica Italiana **2-B**, (1988), 641–656.

13. A. Cellina, *Approximation of set valued functions and fixed point theorems*, Annal. Mat. Pura Appl. **82** (1969), 17–24.

14. V.V. Chistyakov, *On mappings of bounded variation*, J. Dynam. Control Systems **3** (1997), 261–289.

15. G. Dal Maso, P. LeFloch and F. Murat, *Definition and weak stability of nonconservative products*, J. Math. Pures Appl. **74** (1995), 483–548.

16. G. Dal Maso and F. Rampazzo, *On systems of ordinary differential equations with measures as controls*, Differential Integral Equat. **4** (1991), 739–765.

17. L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992.

18. A.F. Filippov, *The existence of solutions of multivalued differential equations*, Mat. Z. **10** (1971), 307–313.

19. C. Marle, *Géométrie des systèmes mécaniques à liaisons actives*, in *Symplectic geometry and mathematical physics*, P. Donato, C. Duval, J. Elhadad and G.M. Tuynman, eds., Birkhäuser, Boston, 1991.

20. J.R. Munkres, *Topology*: *A first course*, Prentice Hall, Englewood Cliffs, NJ, 1975.

21. B. Piccoli and E. Girejko, *On some concepts of generalized differentials*, Set Valued Anal. **15** (2007), 163–183.

22. F. Rampazzo, *On the Riemannian structure of a Lagrangean system and the problem of adding time-dependent coordinates as controls*, European J. Mechanics A/Solids **10** (1991), 405–431.

23. H.J. Sussmann, *Generalized differentials, variational generators, and the maximum principle with state constraints*, in *Nonlinear and optimal control theory*, Lecture Notes Math. **1932**, Springer, Berlin, 2008.

24. S. Zaremba, *Sur les équations au paratingent*, Bull Sci. Math. **60** (1936), 139–160.

Department of Mathematics, Penn State University University Park, PA 16802
Email address: bressan@math.psu.edu

Department of Mathematics, Penn State University University Park, PA 16802
Email address: deforest@math.psu.edu