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A SINGULARITY ANALYSIS OF POSITIVE SOLUTIONS TO AN EULER-LAGRANGE INTEGRAL SYSTEM

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ABSTRACT. In this paper we study the asymptotic behavior of the positive solutions of the following system of Euler-Lagrange equations of Hardy-Littlewood-Sobolev type in R^n

$$u(x) = \frac{1}{|x|^\alpha} \int_{R^n} \frac{v(y)^q}{|y|^\beta |x-y|^\lambda}\, dy,$$
$$v(x) = \frac{1}{|x|^\beta} \int_{R^n} \frac{u(y)^p}{|y|^\alpha |x-y|^\lambda}\, dy.$$

We obtain the growth rate of the solutions around the origin and the decay rate near infinity. Some new cases beyond the work of Li and Lim [**17**] are studied here. In [**15**], the authors obtained the asymptotic estimates of solutions for the case $\alpha, \beta \geq 0$. In this paper, we extend the case $\alpha, \beta \geq 0$ to $\alpha + \beta \geq 0$ with some restriction, and we obtain asymptotic estimates for the solutions.

1. Introduction.

Let $1 < r,\ s < \infty$, $0 < \lambda < n$, $\alpha + \beta \geq 0$, $\lambda + \alpha > 0$, $\lambda + \beta > 0$ and $\alpha + \beta + \lambda \leq n$. Let $\|f\|_p$ be the $L^p(R^n)$ norm of the function f. The weighted Hardy-Littlewood-Sobolev (WHLS) inequality states that (cf. [**23**])

$$\left| \int_{R^n} \int_{R^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta}\, dx\, dy \right| \leq C_{\alpha,\beta,s,\lambda,n} \|f\|_r \|g\|_s \tag{1.1}$$

where

$$1 - \frac{1}{r} - \frac{\lambda}{n} < \frac{\alpha}{n} < 1 - \frac{1}{r} \text{ and } \frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{n} = 2. \tag{1.2}$$

In order to obtain the best constant in the WHLS inequality (1.1), we maximize the functional

$$J(f,g) = \int_{R^n} \int_{R^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta}\, dx\, dy$$

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under the constraints $\|f\|_r = \|g\|_s = 1$. The corresponding Euler-Lagrange equations are given by the following integral system:

$$(1.3)\qquad \begin{cases} \lambda_1 r f(x)^{r-1} = 1/|x|^\alpha \int_{R^n} (g(y))/|y|^\beta |x-y|^\lambda \, dy \\ \lambda_2 s g(x)^{s-1} = 1/|x|^\beta \int_{R^n} f(y)/(|y|^\alpha |x-y|^\lambda) \, dy \end{cases}$$

where $f, g \geq 0$, $x \in R^n$ and $\lambda_1 r = \lambda_2 s = J(f,g)$. Set $u = c_1 f^{r-1}$, $v = c_2 g^{s-1}$, $1/(p+1) = 1 - (1/r)$, $1/(q+1) = 1 - (1/s)$ with $pq \neq 1$. By a proper choice of constants c_1 and c_2, (1.3) becomes

$$(1.4)\qquad \begin{cases} u(x) = 1/|x|^\alpha \int_{R^n} (v(y)^q/|y|^\beta |x-y|^\lambda) \, dy \\ v(x) = 1/|x|^\beta \int_{R^n} (u(y)^p/|y|^\alpha |x-y|^\lambda) \, dy \end{cases}$$

where $\alpha + \beta + \lambda \leq n$, and

(1.5)
$$\begin{cases} u, v \geq 0, \ 0 < p, q < \infty, \ 0 < \lambda < n, \ \alpha + \beta \geq 0, \\ \alpha/n < 1/(p+1) < (\lambda + \alpha)/n, \ 1/(p+1) + 1/(q+1) = (\lambda + \alpha + \beta)/n. \end{cases}$$

This paper is concerned with the properties of positive solutions of the integral equations (1.4). Jin and Li [**13**] studied the symmetry of the solutions to the more general system of integral equations (1.4) for $\alpha, \beta \geq 0$. They proved that u and v are radially symmetric and decreasing about the origin 0. The regularity of the solutions to (1.4) is discussed in the joint paper of Chen, Jin, Li and Lim [**3**]. In [**3**], the integrability intervals of the solutions for $\alpha, \beta \geq 0$ are obtained. In a subsequent paper, Jin and Li [**14**] also thoroughly analyzed the regularity of the solutions to (1.4) and determined the optimal integrability intervals of the solutions for $\alpha, \beta \geq 0$. The optimal integrability and the radial symmetry are needed for the classification of the regular solutions of the system (1.4) associated with WHLS inequality [**7**].

The following integrability result proved in [**14**] plays an important role in this paper.

Proposition 1.1. *Let $(u,v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)$ be a pair of positive solutions of* (1.4). *Assume that $p, q > 1$, $\alpha + \beta \geq 0$ with $\beta < 0$, and $\bar{\lambda} = \lambda + \alpha + \beta$, then for $1 \leq r, s < \infty$,*

$$u \in L^r(R^n) \quad \text{and} \quad v \in L^s(R^n),$$

provided

(i) $1/r \in ((\alpha/n), (\bar{\lambda}/n))$ *and* $1/s \in (0, (\min\{(p+1)\bar{\lambda} - n,\ \lambda + \beta\})/n)$ *when* $(q+1)(\lambda+\beta) \geq 2n$.

(ii) $1/r \in ((\alpha/n),\ (\min\{\bar{\lambda},\ q(\lambda+\beta) + \bar{\lambda} - n\})/n)$ *and* $1/s \in ((\max\{0,\ p\alpha + \bar{\lambda} - n\})/n,\ (\lambda+\beta)/n)$ *when* $(q+1)(\lambda+\beta) < 2n$.

We wish now to study the asymptotic behavior of solutions of (1.4) in a neighborhood of the origin and at infinity.

Definition 1. A function u is asymptotic to $A/|x|^s$ at $x = 0$, written

$$u(x) \simeq \frac{A}{|x|^s} \text{ at } |x| \simeq 0, \tag{1.6}$$

if $\lim_{|x|\to 0} |x|^s u(x) = A$ for a positive number s and $0 < A < \infty$.

Definition 2. A function u is asymptotic to $B/|x|^t$ near $x = \infty$, written

$$u(x) \simeq \frac{B}{|x|^t} \text{ at } |x| \simeq \infty, \tag{1.7}$$

if $\lim_{|x|\to\infty} |x|^t u(x) = B$ for a positive number t and $0 < B < \infty$.

Remark 1.1. If

$$p \geq 1, \qquad q \geq 1, \quad pq \neq 1, \tag{1.8}$$

then either

$$\lambda + \alpha(p+1) < n \text{ or } \lambda + \beta(q+1) < n \tag{1.9}$$

always holds. For if it fails, then $\lambda + \beta(q+1) \geq n$ and $\lambda + \alpha(p+1) \geq n$. This implies $\alpha + \beta \geq (n-\lambda)(1/(p+1) + 1/(q+1))$. Applying (1.5), $\lambda = n(1/(p+1) + 1/(q+1)) - (\alpha+\beta) \leq \lambda(1/(p+1) + 1/(q+1))$. Thus, $1/(p+1) + 1/(q+1) \geq 1$, which is a contradiction to (1.8).

Hence, the condition $\lambda + \beta(q+1) < n$ may be assumed for small $|x|$ without loss of generality in Theorem 1.2.

Theorem 1.2. *Let $(u,v) \in L^{p+1}(R^n)\times L^{q+1}(R^n)$ be a pair of positive solutions of the system (1.4)–(1.5). Suppose that $p \geq 1$, $q \geq 1$, $pq \neq 1$, $\alpha+\beta \geq 0$ and $\lambda+(q+1)\beta < n$. If either one of the following conditions holds*

(H1) $\lambda + (p+1)\alpha < n + \alpha$,

(H2) $\lambda + (p+1)\alpha \geq n + \alpha$ *and* $\alpha + \beta + (\beta/q(p+1)) \geq 0$,

then for small $|x|$,

$$u(x) \simeq \frac{A_0}{|x|^\alpha}, \tag{1.10}$$

and

$$v(x) \simeq \begin{cases} A_1/|x|^\beta & \text{if } \lambda + \alpha(p+1) < n \\ (A_2|\ln|x||)/|x|^\beta & \text{if } \lambda + \alpha(p+1) = n \\ A_3/|x|^{\alpha(p+1)+\beta+\lambda-n} & \text{if } \lambda + \alpha(p+1) > n \end{cases} \tag{1.11}$$

where $A_0 = \int_{R^n} v^q(y)/|y|^{\lambda+\beta}\, dy$, $A_1 = \int_{R^n} u^p(y)/|y|^{\lambda+\alpha}\, dy$, $A_2 = |S^{n-1}|(\int_{R^n}(v^q(y)/|y|^{\lambda+\beta})\, dy)^p$,

$$A_3 = \left(\int_{R^n} \frac{v^q(y)}{|y|^{\lambda+\beta}}\, dy\right)^p \int_{R^n} \frac{dz}{|z|^{\alpha(p+1)}|e-z|^\lambda},$$

e is a unit vector in R^n, and $|S^{n-1}|$ is the surface area of the unit sphere.

Remark 1.2. For $|x|$ large, we utilize a Kelvin-type transform to change the problem from the origin to one at infinity. With this transform, we derive a new system of integral equations with indices $\overline{\alpha}$, $\overline{\beta}$, p, q, satisfying (1.5). From the definition of $\overline{\alpha} = (2n/p+1) - \alpha - \lambda$, $\overline{\beta} = (2n/q+1) - \beta - \lambda$, we can observe directly that the condition $\lambda + \overline{\alpha}(p+1) < n$ or $\lambda + \overline{\beta}(q+1) < n$ is equivalent to

$$\lambda p + \alpha(p+1) > n \text{ or } \lambda q + \beta(q+1) > n, \tag{1.12}$$

respectively. We claim that statement (1.2) always holds, if (1.8) is assumed. For, if not, then $\lambda q + \beta(q+1) \leq n$ and $\lambda p + \alpha(p+1) \leq n$, implying $(\lambda+\beta)(q+1) \leq n+\lambda$ and $(\lambda+\alpha)(p+1) \leq n+\lambda$.

Then, by (1.5), we have $(\lambda+\alpha+\beta)/n = 1/(p+1) + 1/(q+1) \geq ((\lambda+\alpha+\beta)+\lambda)/(n+\lambda)$. This then leads to a contradiction, since we infer from $1/(p+1) + 1/(q+1) = (\lambda+\alpha+\beta/n) < 1$ that $(\lambda+\alpha+\beta)/n < ((\lambda+\alpha+\beta)+\lambda)/(n+\lambda)$. Consequently, without loss of generality, we may assume $\lambda q+\beta(q+1) > n$ for large $|x|$ in the following theorem.

Theorem 1.3. *Let $(u,v) \in L^{p+1}(R^n)\times L^{q+1}(R^n)$ be a pair of positive solutions of system (1.4) with (1.5). Suppose that $p \geq 1$, $q \geq 1$, $pq \neq 1$, $\alpha+\beta \geq 0$ and $\lambda q+\beta(q+1) > n$. If one of the following conditions holds*

(H3) $1/(p+1) \geq (\lambda+\alpha)/2n$ *and* $1/(q+1) \geq (\lambda+\beta)/2n$,

(H4) $1/(q+1) < (\lambda+\beta)/2n$ *and* $p\lambda + (p+1)\alpha > \lambda+\alpha+n(p-1)/(p+1)$,

(H5) $1/(q+1) < (\lambda+\beta)/2n$, $p\lambda+(p+1)\alpha \leq \lambda+\alpha+n(p-1)/(p+1)$ *and* $\alpha+\beta+(1/(q(p+1)))((2n/q+1)-\lambda-\beta) \geq 0$,

then for large $|x|$,

$$u(x) \simeq \frac{B_0}{|x|^{\lambda+\alpha}}, \tag{1.13}$$

and

$$v(x) \simeq \begin{cases} B_1/|x|^{\lambda+\beta} & \text{if } \lambda p+\alpha(p+1) > n \\ B_2|\ln|x||/|x|^{\lambda+\beta} & \text{if } \lambda p+\alpha(p+1) = n \\ B_3/|x|^{(\alpha+\lambda)(p+1)+\beta-n} & \text{if } \lambda p+\alpha(p+1) < n \end{cases} \tag{1.14}$$

where $B_0 = \int_{R^n}(v^q(y)/|y|^\beta)\,dy$, $B_1 = \int_{R^n}(u^p(y)/|y|^\alpha)\,dy$, $B_2 = |S^{n-1}|(\int_{R^n}(v^q(y)/|y|^\beta)\,dy)^p$ *and*

$$B_3 = \left(\int_{R^n}\frac{v^q(y)}{|y|^\beta}\,dy\right)^p \int_{R^n}\frac{dz}{|z|^{2n-(\alpha+\lambda)(p+1)}|e-z|^\lambda}.$$

Remark 1.3. By the Kelvin transform, (H3) is equivalent to $\overline{\alpha}, \overline{\beta} \geq 0$, (H4) is equivalent to $\overline{\beta} < 0$ and (H1), and (H5) is equivalent to $\overline{\beta} < 0$ and (H2).

Remark 1.4. (i) For $\alpha, \beta \geq 0$, analogous asymptotic results were obtained in [**17**], using the regularity result from [**3**]. In paper [**15**], the authors removed the condition $1/s \in ((\beta/n), (\lambda+\beta)/n)$ imposed by Li and Lim in [**17**]. Using the radial symmetry and the integrability intervals of the solutions, they calculated directly the decay rates of $u(x)$ and $v(x)$ as $|x| \to \infty$ instead of using the Kelvin transform as in [**17**]. Consequently, [**15**] completes the study of the asymptotic behavior of positive solutions for $\alpha, \beta \geq 0$.

(ii) When $\beta < 0$ and $\alpha + \beta \geq 0$, the asymptotic behavior of u and v with $\beta < 0$ is more complicated than that for $\beta \geq 0$. In fact, when $\alpha + \beta \geq 0$ and $\beta < 0$, whether the solutions are radially symmetric and decreasing remains an open question. The idea in [**15**] cannot be used here to obtain the asymptotic estimates of the solutions. In addition, (1.11) shows that v decays to zero near the origin when $\lambda + (p+1)\alpha < n - \beta$. On the contrary, v goes to infinity near the origin when $\lambda + (p+1)\alpha > n - \beta$.

(iii) When $\beta < 0$ and $\alpha + \beta \geq 0$, the paper [**17**] applied the regularity results given in [**14**] to obtain the behavior (1.10) and (1.11) with $\lambda + \alpha(p+1) \leq n - \beta$. We improve the asymptotic results from $\lambda + \alpha(p+1) \leq n - \beta$ to $\lambda + \alpha(p+1) < n + \alpha$ using more elaborate estimates. In the case of $\lambda + \alpha(p+1) \geq n + \alpha$, (1.10) and (1.11) still hold as long as $\alpha + \beta + (\beta/(q(p+1))) \geq 0$. It is unknown whether (1.10) and (1.11) remain true when $\alpha + \beta + (\beta/(q(p+1))) < 0$.

By virtue of Remark 1.4 (i), we only need to consider the case of $\beta < 0$ and $\alpha + \beta \geq 0$. According to Proposition 1.1, v has two different integrability intervals. We will study the asymptotic behavior as $|x| \to 0$ in three cases:

Case I: $\lambda + (p+1)\alpha \leq n - \beta$;

Case II: $n - \beta < \lambda + (p+1)\alpha < n + \alpha$;

Case III: $\lambda + (p+1)\alpha \geq n + \alpha$ and $\alpha + \beta + \beta/(q(p+1)) \geq 0$

in Sections 2, 3 and 4, respectively. By applying the Kelvin transform and Theorem 1.2, we will prove Theorem 1.3 in Section 5.

2. Singularity analysis in Case I.

In this section, assume that $(u, v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)$ is a pair of positive solutions of the system (1.4) satisfying (1.5), $p \geq 1$, $q \geq 1$, $pq \neq 1$, $\beta < 0$ and $\alpha + \beta \geq 0$.

First, we prove two propositions which are true in all the three cases.

Proposition 2.1.

$$\int_{R^n\setminus B_1} \frac{v^q(y)dy}{|y|^{\lambda+\beta}} < \infty; \qquad \int_{R^n\setminus B_1} \frac{u^p(y)dy}{|y|^{\lambda+\alpha}} < \infty.$$

Proof. If $x \in B_{1/2} \setminus B_{1/4}$ and $y \in R^n \setminus B_1$, $|x-y| \le C|y|$. Then,

$$\int_{R^n\setminus B_1} \frac{v^q(y)dy}{|y|^{\lambda+\beta}} \le C \int_{R^n\setminus B_1} \frac{v^q(y)dy}{|y|^{\beta}|x-y|^{\lambda}},$$

as long as $x \in B_{1/2} \setminus B_{1/4}$. Integrating on $B_{1/2} \setminus B_{1/4}$ yields

$$\begin{aligned} |B_{1/2} \setminus B_{1/4}| &\int_{R^n\setminus B_1} \frac{v^q(y)dy}{|y|^{\lambda+\beta}} \\ &\le C \int_{B_{1/2}\setminus B_{1/4}} \left(\int_{R^n\setminus B_1} \frac{v^q(y)dy}{|y|^{\beta}|x-y|^{\lambda}} \right) dx \\ &\le C \int_{B_{1/2}\setminus B_{1/4}} |x|^{\alpha} u(x)\, dx \\ &\le C \left(\int_{B_{1/2}\setminus B_{1/4}} u^{p+1}\, dx \right)^{1/(p+1)} |B_{1/2} \setminus B_{1/4}|^{p/p+1} < \infty, \end{aligned}$$

proving $\int_{R^n\setminus B_1} (v^q(y)\, dy)/|y|^{\lambda+\beta}$ is finite. Similarly, $\int_{R^n\setminus B_1} (u^p(y)\, dy)/|y|^{\lambda+\alpha}$ is also finite.

Proposition 2.2. $A_0 := \int_{R^n} (v^q(y)\, dy)/|y|^{\beta+\lambda} < \infty$.

Proof. Applying Hölder's inequality with $1/s + 1/s' = 1$,

$$\int_{B_1} \frac{v^q(y)\, dy}{|y|^{\beta+\lambda}} \le \left(\int_{B_1} v^{qs}(y)\, dy \right)^{1/s} \left(\int_{B_1} |y|^{-(\lambda+\beta)s'}\, dy \right)^{1/s'}.$$

In Case I, by Proposition 1.1, we can choose $1/s$ so small that $(\lambda+\beta)s' < n$. Thus, we have

$$\int_{B_1} \frac{v^q(y)\, dy}{|y|^{\beta+\lambda}} < \infty. \tag{2.1}$$

In Cases II and III, using Proposition 1.1, we only need to choose $1/s$ sufficiently close to $(q(p\alpha+\bar{\lambda}-n))/n$. To verify $(\lambda+\beta)s' < n$, it is sufficient to prove

$$\lambda+\beta < n - q(p\alpha+\bar{\lambda}-n) \tag{2.2}$$

or $(pq-1)\alpha < (q+1)(n-\bar{\lambda})$. Multiplying with $1/(p+1)(q+1)$ and noting $(1/p+1)+(1/q+1)=(\bar{\lambda})/n$, we know that this inequality is true if and only if $(1/p+1) > (\alpha/n)$. Therefore, (2.1) follows. Combined with Proposition 2.1, the proof of Proposition 2.2 is completed.

In the remainder of this section, we will prove (1.10) and (1.11) in Case I.

Proposition 2.3. *$u(x) \simeq |x|^{-\alpha}A_0$ as $|x| \to 0$.*

Proof. By Definition 1, we only need to prove that as $|x| \to 0$,

$$\left| \int_{R^n} \frac{v^q(y)}{|y|^\beta |x-y|^\lambda}\,dy - \int_{R^n} \frac{v^q(y)}{|y|^{\lambda+\beta}}\,dy \right| \longrightarrow 0. \tag{2.3}$$

For $x \in B_\delta := B_\delta(0) \subset R^n$, δ sufficiently small,

$$\begin{aligned}\left| \int_{R^n} \frac{v^q(y)}{|y|^\beta |x-y|^\lambda}\,dy - \int_{R^n} \frac{v^q(y)}{|y|^{\lambda+\beta}}\,dy \right| &\\ \le \int_{B_\delta} \left(\frac{v^q(y)}{|y|^\beta|x-y|^\lambda} + \frac{v^q(y)}{|y|^{\lambda+\beta}} \right) dy &\\ + \int_{R^n\setminus B_\delta} \left| \frac{v^q(y)}{|y|^\beta|x-y|^\lambda} - \frac{v^q(y)}{|y|^{\lambda+\beta}} \right| dy &\\ =: J_1 + J_2. &\end{aligned}$$

By Hölder's inequality,

$$\int_{B_\delta} \frac{v^q(y)dy}{|x-y|^\lambda} \le \left(\int_{B_\delta} v^{qs}(y)\,dy \right)^{1/s} \left(\int_{B_\delta} |x-y|^{-s'\lambda}\,dy \right)^{1/s'}$$

for $(1/s)+(1/s')=1$. In Case I, we can use Proposition 1.1 by choosing $1/s$ so small that $s'\lambda < n$. Then, for any given $\delta > 0$,

$$\int_{B_\delta} \frac{v^q(y)dy}{|x-y|^\lambda} < \infty, \tag{2.4}$$

when $|x|$ is small enough. By virtue of $\beta < 0$, $|y|^{-\beta} \leq \delta^{-\beta}$ if $y \in B_\delta$. By Proposition 2.2 and (2.4), we have

$$J_1 \leq C(\delta) \longrightarrow 0 \quad \text{as} \quad \delta \to 0.$$

Using Lebesgue's bounded convergence theorem, we prove that for each fixed B_δ,

$$J_2 \longrightarrow 0 \quad \text{as} \quad |x| \to 0.$$

In fact, one can easily see that $v^q(y)/(|y|^\beta|x-y|^\lambda) \leq C(v^q(y)/|y|^{\lambda+\beta})$ for $|y| \geq \delta$ and recall that A_0 is finite. This proves (2.3) and Proposition 2.3 follows.

Next we prove (1.11) in Case I. Proposition 2.3 implies that, for fixed small δ,

$$u(x) = \frac{A_0 + o(1)}{|x|^\alpha}, \quad \text{for } |x| \leq \delta. \tag{2.5}$$

We can use (2.5) to estimate the asymptotic properties of v near the origin.

Clearly, Case I divides into three subcases:

(1) $\lambda + \alpha(p+1) < n$;

(2) $\lambda + \alpha(p+1) = n$;

(3) $n < \lambda + \alpha(p+1) \leq n - \beta$.

Proposition 2.4. *If $\lambda + \alpha(p+1) < n$, then $A_1 < \infty$, and $v(x) \simeq |x|^{-\beta}A_1$ as $|x| \to 0$.*

Proof. By Hölder's inequality,

$$\int_{B_1} \frac{u^p(y)\,dy}{|y|^{\alpha+\lambda}} \leq \left(\int_{B_1} u^{pr}(y)\,dy\right)^{1/r} \left(\int_{B_1} |y|^{-(\lambda+\alpha)r'}\,dy\right)^{1/r'}$$

with $(1/r) + (1/r') = 1$. Using Proposition 1.1, we can choose $1/r$ approaching $p\alpha/n$ such that $(\lambda+\alpha)r' < n$. This is easily verified, since $\lambda + (p+1)\alpha < n$. Therefore, the integral above is finite. Combined with Proposition 2.1 we see that $A_1 < \infty$.

Using (2.5), we obtain

$$|x|^{\beta}v(x) = \int_{R^n} \frac{u^p(y)}{|y|^{\alpha}|x-y|^{\lambda}}\,dy = \int_{B_\delta} \frac{(A_0+o(1))^p}{|y|^{\alpha(p+1)}|x-y|^{\lambda}}\,dy + \int_{R^n\setminus B_\delta} \frac{u^p(y)}{|y|^{\alpha}|x-y|^{\lambda}}\,dy.$$

By means of Young's inequality and the condition $\lambda+\alpha(p+1)<n$, we have, when $|x|\to 0$ and $\delta\to 0$,

$$\int_{B_\delta} \frac{(A_0+o(1))^p}{|y|^{\alpha(p+1)}|x-y|^{\lambda}}\,dy \le C(A_0+o(1))^p\delta^{n-\lambda-\alpha(p+1)} \longrightarrow 0.$$

By Lebesgue's bounded convergence theorem as $|x|\to 0$ and then $\delta\to 0$,

$$\int_{R^n\setminus B_\delta} \frac{u^p(y)}{|y|^{\alpha}|x-y|^{\lambda}}\,dy \longrightarrow A_1.$$

Combining these results we see that $|x|^{\beta}v(x)\simeq A_1$ as $|x|\to 0$. This completes the proof of Proposition 2.4.

Proposition 2.5. *If $\lambda+\alpha(p+1)=n$, then $v(x)\simeq |x|^{-\beta}A_2|\ln|x||$ when $|x|\to 0$.*

Proof. If $\lambda+(p+1)\alpha=n$, A_2 is a constant since it depends only on A_0. Using (2.5), a change of variables and Proposition 2.1, we deduce that

$$\begin{aligned}\frac{|x|^{\beta}}{-\ln|x|}\,v(x) &= \frac{1}{-\ln|x|}\int_{B_\delta} \frac{(A_0+o(1))^p}{|y|^{n-\lambda}|x-y|^{\lambda}}\,dy \\ &\quad + \frac{1}{-\ln|x|}\int_{R^n\setminus B_\delta} \frac{u^p(y)}{|y|^{\alpha}|x-y|^{\lambda}}\,dy \\ &\le \frac{(A_0+o(1))^p}{-\ln|x|}\int_{B_{\delta/|x|}} \frac{dz}{|z|^{n-\lambda}|e-z|^{\lambda}} + \frac{C}{-\ln|x|}.\end{aligned} \tag{2.6}$$

Now, fix any point $w \in \partial B_1$. Employing polar coordinates, we arrive at

$$\begin{aligned}
&\frac{1}{-\ln|x|}\int_{B_{\delta/|x|}}\frac{dz}{|z|^{n-\lambda}|e-z|^{\lambda}}\\
&\quad=\frac{1}{-\ln|x|}\int_{S^{n-1}}d\omega\int_0^{\delta/|x|}\frac{r^{n-1}}{r^{n-\lambda}|e-rw|^{\lambda}}\,dr\\
&\quad=\frac{1}{-\ln|x|}\int_{S^{n-1}}d\omega\Big(\int_0^R\frac{r^{\lambda-1}}{|e-rw|^{\lambda}}\,dr+\int_R^{\delta/|x|}\frac{dr^{\lambda}}{\lambda|e-rw|^{\lambda}}\Big)\\
&\quad=\frac{(1+\theta/R)^{\lambda}}{-\ln|x|}\int_{S^{n-1}}d\omega\Big(C(R)+\int_R^{\delta/|x|}\frac{dr^{\lambda}}{\lambda r^{\lambda}}\Big)\\
&\quad\longrightarrow(1+\theta/R)^{\lambda}|S^{n-1}|,
\end{aligned}$$

as $|x|\to 0$. The last equality holds since $r\geq R$ implies $1-(1/R)\leq |e-rw|/r\leq 1+(1/R)$. Here $\theta\in(-1,1)$, and $|S^{n-1}|$ denotes the surface area of the unit sphere. Consequently (2.6) yields

$$\begin{aligned}
\frac{|x|^{\beta}}{-\ln|x|}\,v(x)&\longrightarrow(A_0+o(1))^p(1+\theta/R)^{\lambda}|S^{n-1}|\quad\text{as }|x|\to 0\\
&\longrightarrow(A_0)^p|S^{n-1}|\quad\text{as }\delta\to 0, R\to\infty,
\end{aligned}$$

proving Proposition 2.5.

Proposition 2.6. *If $n<\lambda+\alpha(p+1)\leq n-\beta$, then $A_3<\infty$. Moreover, when $|x|\to 0$, $v(x)\simeq|x|^{n-\alpha(p+1)-\beta\lambda}A_3$.*

Proof. Since $\alpha(p+1)<n$ near the origin; $\lambda<n$ near e and $\lambda+\alpha(p+1)>n$ near infinity,

$$A_3:=\Big(\int_{R^n}\frac{v^q(y)}{|y|^{\lambda+\beta}}\,dy\Big)^p\int_{R^n}\frac{dz}{|z|^{\alpha(p+1)}|e-z|^{\lambda}}<\infty.$$

Let $s=\alpha(p+1)+\lambda-n$. Then, for a fixed B_δ,

$$\begin{aligned}
&|x|^s\int_{R^n}\frac{u^p(y)dy}{|y|^{\alpha}|x-y|^{\lambda}}\\
&\qquad=|x|^s\int_{B_\delta}\frac{(A_0+o(1))^p dy}{|y|^{\alpha(p+1)}|x-y|^{\lambda}}+|x|^s\int_{R^n\setminus B_\delta}\frac{u^p(y)dy}{|y|^{\alpha}|x-y|^{\lambda}}\\
&\qquad=:K_1+K_2.
\end{aligned}$$

By Proposition 2.1 and $s > 0$,

$$K_2 \le |x|^s \int_{R^n \setminus B_\delta} \frac{u^p(y)}{|y|^{\lambda+\alpha}}\, dy \le |x|^s C \longrightarrow 0, \quad \text{as } |x| \to 0.$$

To deal with K_1, we change the variables $y = |x|z$ on the first term to deduce that

$$\begin{aligned} K_1 &= (A_0 + o(1))^p \int_{B_{\delta/|x|}} \frac{dz}{|z|^{\alpha(p+1)}|(x/|x|) - z|^\lambda} \\ &\longrightarrow (A_0)^p \int_{R^n} \frac{dz}{|z|^{\alpha(p+1)}|e - z|^\lambda} \\ &= A_3 \quad \text{as } |x| \to 0, \quad \delta \to 0. \end{aligned}$$

Here, e is defined as a unit vector in R^n. Consequently,

$$|x|^{s+\beta} v(x) = |x|^{\alpha(p+1)+\beta+\lambda-n} v(x) \longrightarrow A_3 \quad \text{as } |x| \to 0.$$

The proof of Proposition 2.6 is complete.

3. Singularity analysis in Case II. In this section, assume that $(u, v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)$ is a pair of positive solutions of the system (1.4) satisfying (1.5) with $p \ge 1$, $q \ge 1$, $pq \ne 1$ and $\beta < 0$, $\alpha + \beta \ge 0$.

In Case II, it seems very difficult to prove (2.4) by means of Hölder's inequality as in the proof of Proposition 2.3, since $\lambda + (p+1)\alpha > n - \beta$ means $p\alpha + \bar{\lambda} - n > 0$ and by Proposition 1.1 $1/s$ cannot be close to 0.

However, if $n - \beta < \lambda + \alpha(p+1) < n + \alpha$, then Hölder's inequality can estimate the growth rate of $v(x)$ roughly when $|x| \to 0$. Then we can use this growth rate to deduce the limit of $u(x)|x|^{-\alpha}$.

Proposition 3.1. *If $n - \beta < \lambda + \alpha(p+1) < n + \alpha$, then u and v are given by* (1.10) *and* (1.11).

Proof. First, we claim that (1.10) holds.

When $y \in R^n \setminus B_1$, we have $1/(|y|^\alpha |x - y|^\lambda) \le C/|y|^{\alpha+\lambda}$. Using Proposition 1.1, we obtain

$$\int_{R^n \setminus B_1} \frac{u^p(y)\, dy}{|y|^\alpha |x - y|^\lambda} \le C \int_{R^n \setminus B_1} \frac{u^p(y) dy}{|y|^{\alpha+\lambda}} < \infty. \tag{3.1}$$

On the other hand, Hölder's inequality shows that

$$\int_{B_1} \frac{u^p(y)\,dy}{|y|^\alpha |x-y|^\lambda} \le \left(\int_{B_1} u^{pr}(y)\,dy\right)^{1/r} \left(\int_{B_1} \frac{dy}{|y|^{r'\alpha}|x-y|^{r'\lambda}}\right)^{1/r'},$$

where $1/r + 1/r' = 1$. In view of $\lambda + p\alpha < n$, we can choose $1/r = (p\alpha + \varepsilon)/n$ such that $r'\alpha < n$ and

$$r'\lambda < n \tag{3.2}$$

when ε is sufficiently small. Hence,

$$\begin{aligned} \int_{B_1} \frac{u^p(y)\,dy}{|y|^\alpha |x-y|^\lambda} &\le C\left(\int_{B_1} \frac{dy}{|y|^{r'\alpha}|x-y|^{r'\lambda}}\right)^{1/r'} \\ &\le \frac{C}{|x|^{\alpha+\lambda-n/r'}} \\ &= \frac{C}{|x|^{p\alpha+\lambda+\alpha-n+\varepsilon}}. \end{aligned} \tag{3.3}$$

Combining (3.1) and (3.3) yields $v(x) \le C/|x|^{p\alpha+\bar\lambda-n+\varepsilon}$. Therefore, noting $|x|/2 \le |y| \le 3|x|/2$ for $y \in B(x, |x|/2)$ and (2.2), we can see that as $|x| \to 0$,

$$\begin{aligned} \int_{B(x,|x|/2)} \frac{v^q(y)\,dy}{|y|^\beta |x-y|^\lambda} &\le C\int_{B(x,|x|/2)} \frac{dy}{|y|^{\beta+q(p\alpha+\bar\lambda-n+\varepsilon)}|x-y|^\lambda} \\ &\le \frac{C}{|x|^{\beta+q(p\alpha+\bar\lambda-n+\varepsilon)}} \int_{B(x,|x|/2)} \frac{dy}{|x-y|^\lambda} \\ &\le C|x|^{n-q(p\alpha+\bar\lambda-n+\varepsilon)-(\lambda+\beta)} \to 0. \end{aligned} \tag{3.4}$$

In addition, when $\delta \to 0$,

$$\begin{aligned} \int_{B_\delta \setminus B(x,|x|/2)} \frac{v^q(y)dy}{|y|^\beta |x-y|^\lambda} &\le C\int_{B_\delta\setminus B(x,|x|/2)} \frac{v^q(y)dy}{|y|^{\beta+\lambda}} \\ &\le C(\delta) \to 0. \end{aligned} \tag{3.5}$$

Similar to the argument given in the proof of Proposition 2.3, we have

$$\left|\int_{R^n\setminus B_\delta} \left[\frac{v^q(y)}{|y|^\beta|x-y|^\lambda} - \frac{v^q(y)}{|y|^{\beta+\lambda}}\right] dy\right| \longrightarrow 0, \quad \text{as } |x| \to 0.$$

Combining this consequence with (3.4) and (3.5), we get (1.10). Then in view of $n-\beta < \lambda+\alpha(p+1) < n+\alpha$, (1.11) can also be deduced from the same proof as for Proposition 2.6. This proves Proposition 3.1.

Remark 3.1. The proof of Proposition 3.1 depends essentially on the condition $(p+1)\alpha+\lambda < n+\alpha$. Otherwise, (3.2) is not true. This implies that the integral $\int_{B_1} dy/(|y|^{r'\alpha}|x-y|^{r'\lambda})$ is not finite near the point x. Therefore, it seems difficult to derive the conclusion of Proposition 3.1 when $(p+1)\alpha+\lambda \geq n+\alpha$, if we try to apply the idea in the proof of Proposition 3.1.

4. Singularity analysis in Case III. According to Proposition 3.1, we only need to consider the case of $\lambda + p\alpha \geq n$.

The proof of Proposition 3.1 shows that (2.3) can still be derived without (2.4), since (2.4) is not sufficiently accurate. To obtain (2.3) for $\beta < 0$, we sharpen the estimation (2.4) to

$$|x|^\alpha u(x) = \int_{R^n} \frac{v^q(y)dy}{|x-y|^\lambda |y|^\beta} \leq C, \quad \text{for all } x \in B_1. \tag{4.1}$$

Proposition 4.1. *If* $\alpha + \beta(1+(1/q)) \geq 0$, *then* (4.1) *is true.*

Proof. Define $v_1(x) = 0$ for $x \in R^n \setminus B_2$ and $v_1(x) = |x|^{-\beta/q}v(x)$ for $x \in B_2$. In view of $\lambda + p\alpha \geq n$ and $1/(p+1) > \alpha/n$, it follows that $\lambda > \alpha$. So,

$$\lambda + \beta + \frac{\beta}{q} > \alpha + \beta + \frac{\beta}{q} \geq 0. \tag{4.2}$$

Hence, we can apply the WHLS inequality to

$$v_1(x) = \frac{1}{|x|^{\beta(1+(1/q))}} \int_{R^n} \frac{u^p(y)\,dy}{|x-y|^\lambda |y|^\alpha}, \quad x \in B_2$$

to obtain

$$\|v_1\|_{L^s(B_2)} \leq C\|u^p\|_{ns/(n+(n-\bar{\lambda}_0)s)},$$

where $\bar{\lambda}_0 = \bar{\lambda} + (\beta/q)$ and $1/s \in ((p\alpha + \bar{\lambda}_0 - n)/n, (\bar{\lambda}_0)/n)$. Then,

$$\frac{n + (n - \bar{\lambda}_0)s}{nps} = \frac{1}{p}\left(\frac{1}{s} + \frac{n - \bar{\lambda}_0}{n}\right) > \frac{\alpha}{n}.$$

By Proposition 1.1, $\|u^p\|_{ns/(n+(n-\bar{\lambda}_0)s)} < \infty$. This implies

$$v_1 \in L^s(B_2) \quad \text{for } \frac{1}{s} > \frac{p\alpha + \bar{\lambda}_0 - n}{n}. \tag{4.3}$$

We next apply (4.3) to deduce (4.1). If $x \in B_1$, $y \in R^n \setminus B_2$,

$$\int_{R^n \setminus B_2} \frac{v^q(y)\, dy}{|y|^\beta |x - y|^\lambda} \leq C \int_{R^n \setminus B_2} \frac{v^q(y)\, dy}{|y|^{\beta+\lambda}} \leq C$$

in view of Proposition 2.1. Therefore, by Hölder's inequality,

$$\begin{aligned} |x|^\alpha u(x) &= \int_{R^n \setminus B_2} \frac{v^q(y)\, dy}{|y|^\beta |x - y|^\lambda} + \int_{B_2} \frac{v_1^q(y)\, dy}{|x - y|^\lambda} \\ &\leq C + \left(\int_{B_2} v_1^{qt}(y)\, dy\right)^{1/t} \left(\int_{B_2} \frac{dy}{|x - y|^{\lambda t'}}\right)^{1/t'} \end{aligned}$$

for $(1/t) + (1/t') = 1$. As a consequence of (4.3), we can choose t such that $1/(qt) > (p\alpha + \bar{\lambda}_0 - n)/n$. Once we verify $\lambda t' < n$, then the integrals above are finite, and hence (4.1) is true. In fact, if we choose $1/t = (q(p\alpha + \bar{\lambda}_0 - n) + \varepsilon)/n$ for ε sufficiently small, then $\lambda t' < n$ can be deduced by (2.2). Thus, the proof of Proposition 4.1 is complete.

Remark 4.1. Using the idea of the proof of Proposition 4.1, we can improve the condition $\alpha + \beta(1 + (1/q)) \geq 0$ to the weaker one: $\alpha + \beta(1 + (1/q(p + 1))) \geq 0$.

Proposition 4.2. *If* $\alpha + \beta(1 + (1/q(p + 1))) \geq 0$, *then* (4.1) *remains valid.*

Proof. Define $u_1(x) = 0$ for $x \in R^n \setminus B_2$, and

$$u_1(x) = |x|^{\alpha+\beta} u(x) = \frac{1}{|x|^{-\beta}} \int_{R^n} \frac{v^q(y) dy}{|x - y|^\lambda |y|^\beta}, \quad x \in B_2.$$

Applying the WHLS inequality, we have that for $1/r_1 \in ((q(p\alpha+\bar{\lambda}-n) + \lambda - n)/n, (\lambda/n))$,

$$\|u_1\|_{L^{r_1}(B_2)} \le C\|v^q\|_{nr_1/(n+(n-\lambda)r_1)}.$$

Take r_1 satisfying $(n+(n-\lambda)r_1)/nr_1 = (q(p\alpha+\bar{\lambda}-n)+\varepsilon)/n$ for $\varepsilon > 0$ sufficiently small. By Proposition 1.1,

$$u_1 \in L^{r_1}(B_2) \quad \text{with} \quad \frac{1}{r_1} = \frac{q(p\alpha+\bar{\lambda}-n)+\lambda-n+\varepsilon}{n}. \tag{4.4}$$

Case A: $\bar{\lambda}_1 := \bar{\lambda} + (\beta/q) + p(\alpha+\beta) \le n$.

Recalling the definition of v_1, when $x \in B_{3/2}$ we have

$$\begin{aligned} v_1(x) &= \frac{1}{|x|^{\beta(1+(1/q))}} \int_{R^n \setminus B_2} \frac{u^p(y)\,dy}{|x-y|^\lambda |y|^\alpha} \\ &\quad + \frac{1}{|x|^{\beta(1+(1/q))}} \int_{B_2} \frac{u_1^p(y)\,dy}{|x-y|^\lambda |y|^{\alpha+p(\alpha+\beta)}} \\ &:= v_{11} + v_{12}. \end{aligned}$$

By Proposition 2.1, for $\beta < 0$,

$$\|v_{11}\|_{L^\infty(B_{3/2})} \le C. \tag{4.5}$$

We now prove the integrability of v_{12} using the WHLS inequality and (4.4). To use the WHLS inequality, we need to verify the following condition

$$\lambda + \beta + \frac{\beta}{q} > 0. \tag{4.6}$$

In fact, (4.2) is not valid since $\alpha+\beta+(\beta/q) \ge 0$ does not hold. However, from $1/(q+1) < *\lambda+\beta)/n \le (\lambda+\beta)/\lambda$, $q+1 > (\lambda/\lambda+\beta)$, proving (4.6).

Moreover, $\bar{\lambda}_1 \le n$, and $\alpha+\beta+(\beta/q)+p(\alpha+\beta) \ge 0$ is implied by $\alpha+\beta(1+(1/(q(p+1))) \ge 0$. Applying the WHLS inequality, we have for $(1/s_1) = (p[q(p\alpha+\bar{\lambda}-n)+\lambda-n+\varepsilon]+\bar{\lambda}_1-n)/n$,

$$\|v_{12}\|_{L^{s_1}(B_{3/2})} \le C\|u_1^p\|_{ns_1/(n+(n-\bar{\lambda}_1)s_1)}.$$

Since $(n+(n-\bar{\lambda}_1)s_1)/(nps_1)=(1/r_1)$, (4.4) implies $v_{12}\in L^{s_1}(B_{3/2})$. Combined with the estimate for v_{11}, we have

(4.7)
$$v_1 \in L^{s_1}(B_{3/2}) \quad \text{with} \quad \frac{1}{s_1}=\frac{p[q(p\alpha+\bar{\lambda}-n)+\lambda-n+\varepsilon]+\bar{\lambda}_1-n}{n}.$$

Then, by Hölder's inequality,

$$\begin{aligned} |x|^\alpha u(x) &= \int_{B_{3/2}} \frac{v_1^q(y)\,dy}{|x-y|^\lambda} + \int_{R^n\setminus B_{3/2}} \frac{v^q(y)\,dy}{|x-y|^\lambda |y|^\beta} \\ &\le \|v_1^q\|_{L^{t_1}(B_{3/2})}\left(\int_{B_{3/2}} \frac{dy}{|x-y|^{\lambda t_1'}}\right)^{1/t_1'} + C, \end{aligned} \tag{4.8}$$

where $1/(t_1)+1/(t_1')=1$, $x\in B_1$. To obtain (4.1), we need to prove the integrals in (4.8) are finite. Noticing (4.7), we can take t_1 such that $qt_1=s_1$. One only needs to verify $\lambda t_1'<n$. In fact, it is sufficient to derive

$$\lambda < n - pq[q(p\alpha+\bar{\lambda}-n)+\lambda-n]-q(\bar{\lambda}_1-n), \tag{4.9}$$

since ε can be chosen sufficiently small. In fact, (4.9) is true if and only if

$$\begin{aligned} n-\lambda > pq[q(p\alpha+\bar{\lambda}-n)+\lambda-n]+q(\bar{\lambda}+p(\alpha+\beta)-n)+\beta \\ = pq[(q+1)(\lambda-n)+(p+1)(q+1)\alpha \\ -(p+1)\alpha+(q+1)\beta-\beta] \\ +q(\lambda-n)+(p+1)(q+1)(\alpha+\beta) \\ -(p+1)(\alpha+\beta)+\beta. \end{aligned}$$

Multiplying both sides by $1/[(p+1)(q+1)]$, we see that

$$\begin{aligned} \frac{n-\lambda}{p+1} > pq\left[\frac{\lambda-n}{p+1}+\alpha-\frac{\alpha}{q+1}+\frac{\beta}{p+1}-\frac{\beta}{(p+1)(q+1)}\right] \\ +(\alpha+\beta)-\frac{\alpha+\beta}{q+1}+\frac{\beta}{(p+1)(q+1)}. \end{aligned}$$

In view of $1/(p+1)+1/(q+1)=\bar{\lambda}/n$ and $\alpha/n<1/(p+1)$, one has $(\lambda-n)/(p+1)+\alpha-[\alpha/(q+1)]+[\beta/(p+1)]=((\alpha/n)-(1/p+1))$

$(n-\bar{\lambda})<0$. Therefore, (4.9) holds as long as $(n-\lambda)/(p+1) \geq [(1-pq)\beta]/[(p+1)(q+1)]+(\alpha+\beta)-(\alpha+\beta)/(q+1)$. This is equivalent to $(n-\bar{\lambda})/(p+1)+(\alpha+\beta)/(p+1)+(\alpha+\beta)/(q+1) \geq [(\bar{\lambda})/n]\beta+\alpha$. Noting $1/(p+1)+1/(q+1)=(\bar{\lambda})/n$ and $1/(p+1)>(\alpha/n)$, we can see easily that the inequality above holds.

Case B: $\bar{\lambda}_1>n$. In this case the WHLS inequality cannot be applied to v_{12} to derive (4.1). We define another function v_2 instead of v_1. For $\sigma>0$, define $v_2(x)=0$ for $x \in R^n \setminus B_{3/2}$, and $v_2(x)=|x|^{-(\beta/q)+\sigma}v(x)$ for $x \in B_{3/2}$. Clearly, when $x \in B_{3/2}$,

$$\begin{aligned} v_2(x) &= \frac{1}{|x|^{\beta+(\beta/q)-\sigma}} \int_{B_2} \frac{u_1^p(y)\,dy}{|x-y|^{\lambda}|y|^{\alpha+p(\alpha+\beta)}} \\ &\quad + \frac{1}{|x|^{\beta+(\beta/q)-\sigma}} \int_{R^n \setminus B_2} \frac{u^p(y)\,dy}{|x-y|^{\lambda}|y|^{\alpha}} \\ &:= v_{21}+v_{22}. \end{aligned}$$

Similar to (4.5), $\|v_{21}\|_{L^\infty(B_{3/2})}<\infty$. To use the WHLS inequality to estimate v_{22}, we take

$$\sigma := \min\left\{\alpha+\beta+\frac{\beta}{q}+p(\alpha+\beta), \lambda+\beta+\frac{\beta}{q}\right\}.$$

Using $\bar{\lambda}_1>n$ and (4.6), we see that $\sigma>0$. And, we claim that

$$\bar{\lambda}_2 := \bar{\lambda}_1-\sigma \leq n.$$

For when $\alpha+\beta+(\beta/q)+p(\alpha+\beta) \leq \lambda+\beta+(\beta/q)$, $\bar{\lambda}_2=\lambda \leq n$; and when $\alpha+\beta+(\beta/q)+p(\alpha+\beta) > \lambda+\beta+(\beta/q)$, $\bar{\lambda}_2=(p+1)\alpha+p\beta \leq (p+1)\alpha<n$.

Thus, applying the WHLS inequality, for $1/s_2=(p[q(p\alpha+\bar{\lambda}-n)+\lambda-n+\varepsilon]+\bar{\lambda}_2-n)/n$,

$$\|v_2\|_{L^{s_2}(B_{3/2})} \leq C\|u_1^p\|_{ns_2/(n+(n-\bar{\lambda}_2)s_2)}.$$

Since $(n+(n-\bar{\lambda}_1)s_2)/ns_2=(p/r_1)$, (4.4) implies $v_{22} \in L^{s_2}(B_{3/2})$. Thus,

$$v_2 \in L^{s_2}(B_{3/2}) \quad \text{with} \quad \frac{1}{s_2}=\frac{p[q(p\alpha+\bar{\lambda}-n)+\lambda-n+\varepsilon]+\bar{\lambda}_2-n}{n}.$$

Using Hölder's inequality, we get

$$\begin{aligned}|x|^\alpha u(x) &= \int_{B_{3/2}} \frac{v_2^q(y)\,dy}{|x-y|^\lambda |y|^{q\sigma}} + \int_{R^n\setminus B_{3/2}} \frac{v^q(y)\,dy}{|x-y|^\lambda |y|^\beta} \\ &\le \|v_2^q\|_{L^{t_2}(B_{3/2})}\left(\int_{B_{3/2}} \frac{dy}{|y|^{q\sigma t_2'}|x-y|^{\lambda t_2'}}\right)^{1/t_2'} + C,\end{aligned}$$

where $(1/t_2)+(1/t_2')=1$, $x \in B_1$. Take t_2 such that $qt_2 = s_2$. Then $\|v_2^q\|_{t_2} < \infty$. To show $\int_{R^n}(dy/|y|^{q\sigma t_2'}|x-y|^{\lambda t_2'}) < \infty$, it is sufficient to verify

$$\lambda + q\sigma < n - pq[q(p\alpha + \bar{\lambda} - n) + \lambda - n] - q(\bar{\lambda}_2 - n), \tag{4.10}$$

since (4.10), together with $qt_2 = s_2$, leads to $(\lambda + q\sigma)t_2' < n$.

To obtain (4.10), we consider two subcases:

$$\text{(i) } \sigma = \alpha + \beta + \frac{\beta}{q} + p(\alpha+\beta); \quad \text{(ii) } \sigma = \lambda + \beta + \frac{\beta}{q}.$$

Subcase (i). Equation (4.10) implies $(p+1)(q+1)(\alpha+\beta) + \beta < (q+1)(n-\lambda) + (p+1)(\alpha+\beta) - pq[(pq-1)\alpha + (q+1)(\bar{\lambda} - n) - \beta]$. This holds if and only if $(\alpha+\beta) + [\beta/(p+1)(q+1)] < [(n-\lambda)/(p+1)] + [(\alpha+\beta)/(q+1)] - pq[((\alpha/n) - (1/p+1))(n-\bar{\lambda}) - [\beta/(p+1)(q+1)]]$. Since $\alpha/n < 1/(p+1)$, this can be deduced from $(\alpha+\beta) + [\beta/(p+1)(q+1)] \le [(n-\bar{\lambda})/(p+1)] + [(\alpha+\beta)/(p+1)] + [(\alpha+\beta)/(q+1)] + [pq\beta/(p+1)(q+1)]$. Noting $(1/p+1)+(1/q+1) = \bar{\lambda}/n$, we see that the inequality above is equivalent to $[(n-\bar{\lambda})/(p+1)] - [(n-\bar{\lambda})/n](\alpha+\beta) + [(pq-1)\beta/(p+1)(q+1)] \ge 0$. Because $[(pq-1)/(p+1)(q+1)] = (n-\bar{\lambda})/n$, the above inequality is easy to obtain knowing $(1/p+1) > (\alpha/n)$.

Subcase (ii). Equation (4.10) implies $pq[pq\alpha + (q+1)(\bar{\lambda} - n) - (\alpha + \beta)] + (p+1)(q+1)\alpha + pq\beta < (q+1)(n-\lambda-\beta) + (p+1)\alpha$. This is true if and only if $pq[(pq-1)\alpha - (q+1)(n-\bar{\lambda})] < (q+1)(n-\bar{\lambda}) + [(p+1)+(q+1)]\alpha - (p+1)(q+1)\alpha$. Multiplying by $[1/(p+1)(q+1)]$, we see that the above inequality is equivalent to $(pq-1)[(\alpha/n) - (1/p+1)](n-\bar{\lambda}) < 0$.

Putting the two cases together completes the proof of Proposition 4.2.

Once (4.1) is verified, it can be used to deduce (2.3). Thus, we have

Proposition 4.3. *In Case* III, (1.10) *holds as* $|x| \to 0$.

Proof. For any given $\delta > 0$, for all sufficiently small $|x|$,

$$\begin{aligned}(4.11)\quad \left|\int_{R^n} \frac{v^q(y)}{|y|^\beta |x-y|^\lambda}\,dy - \int_{R^n} \frac{v^q(y)}{|y|^{\lambda+\beta}}\,dy\right| &\le \int_{B_\delta} \frac{v^q(y)\,dy}{|y|^\beta|x-y|^\lambda} + \int_{B_\delta} \frac{v^q(y)\,dy}{|y|^{\lambda+\beta}} \\ &\quad + \int_{R^n\setminus B_\delta} \left|\frac{v^q(y)}{|y|^\beta|x-y|^\lambda} - \frac{v^q(y)}{|y|^{\lambda+\beta}}\right| dy \\ &=: J_1 + J_2 + J_3.\end{aligned}$$

Equation (4.1) implies $u(x) \le C|x|^{-\alpha}$ in B_1. From this we have

$$\begin{aligned} v(x) &= \frac{1}{|x|^\beta} \int_{R^n} \frac{u^p(y)\,dy}{|x-y|^\lambda |y|^\alpha} \\ &\le \frac{C}{|x|^\beta} \int_{B_1} \frac{dy}{|x-y|^\lambda |y|^{(p+1)\alpha}} + \frac{1}{|x|^\beta} \int_{R^n\setminus B_1} \frac{u^p(y)\,dy}{|x-y|^\lambda|y|^\alpha} \\ &\le \frac{C}{|x|^{p\alpha+\bar{\lambda}-n}}, \end{aligned}$$

for $x \in B_\delta$. Thus, $J_1 = \int_{B_\delta} [v^q(y)dy/|x-y|^\lambda |y|^\beta] \le C\int_{B_\delta}[dy/|x-y|^\lambda |y|^{q(p\alpha+\bar{\lambda}-n)+\beta}]$. In view of (2.2), we see that $J_1 \to 0$ as $\delta \to 0$. By an argument analogous to that for Proposition 2.2, we can again use Hölder's inequality to prove that $J_2 \to 0$ as $\delta \to 0$. Applying Lebesgue's bounded convergence theorem, we obtain that for each $\delta > 0$, $J_3 \to 0$ as $|x| \to 0$. Inserting the estimates of J_1, J_2 and J_3 into (4.11), we can see easily that (1.10) holds as $|x| \to 0$. This completes the proof of Proposition 4.3.

Proposition 4.4. *In Case* III*,* v *satisfies* (1.11)*.*

Proof. By Proposition 4.3, we have (2.5). Then by Proposition 2.6, (1.11) follows when $|x| \to 0$, proving Proposition 4.4.

5. Proof of Theorem 1.3. In this section, assume that $(u, v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)$ is a pair of positive solutions of the system (1.4)–(1.5), $p \ge 1$, $q \ge 1$, $pq \ne 1$, $\alpha + \beta \ge 0$.

(i) Suppose (H3) holds: $1/(p+1) \geq (\lambda+\alpha)/2n$, $1/(q+1) \geq (\lambda+\beta)/2n$ and $\lambda q + \beta(q+1) > n$.

Step 1. By the Kelvin transform, we change the problem from at the center to at infinity.

Define the Kelvin type transform by

$$\begin{cases} \overline{u}(x) = 1/|x|^{2n/(p+1)}\, u(x/|x|^2) \\ \overline{v}(x) = 1/|x|^{2n/(q+1)}\, v(x/|x|^2), \end{cases} \tag{5.1}$$

which means

$$\begin{cases} u(x) = 1/|x|^{2n/(p+1)}\, \overline{u}(x/|x|^2) \\ v(x) = 1/|x|^{2n/(q+1)}\, \overline{v}(x/|x|^2). \end{cases} \tag{5.2}$$

This implies

$$\|\overline{u}\|_{L^{p+1}(R^n)} = \|u\|_{L^{p+1}(R^n)}, \quad \|\overline{v}\|_{L^{q+1}(R^n)} = \|v\|_{L^{q+1}(R^n)}.$$

Inserting (5.1) into (1.4) gives a new system $(\overline{u}, \overline{v})$ from the system for (u, v) as follows:

$$\begin{cases} \overline{u}(x) = 1/|x|^{\overline{\alpha}} \int_{R^n} [\overline{v}^q(y)/|y|^{\overline{\beta}}|x-y|^\lambda]\, dy \\ \overline{v}(x) = 1/|x|^{\overline{\beta}} \int_{R^n} [\overline{u}^p(y)/|y|^{\overline{\alpha}|x-y|^\lambda}]\, dy, \end{cases} \tag{5.3}$$

where $\overline{\alpha} = 2n/(p+1) - \alpha - \lambda$, $\overline{\beta} = (2n/q+1) - \beta - \lambda$. It is easy to see that $\overline{\alpha} + \overline{\beta} = \alpha + \beta \geq 0$, $1/(p+1) - (\lambda/n) < (\overline{\alpha}/n) < 1/(p+1)$, $1/(q+1) + 1/(p+1) = (\lambda + \overline{\alpha} + \overline{\beta})/n$. In addition, $1/(p+1) \geq (\lambda+\alpha)/2n$ and $1/(q+1) \geq (\lambda+\beta)/2n$ imply $\overline{\alpha} \geq 0$ and $\overline{\beta} \geq 0$, respectively.

Step 2. We prove $u(x) \simeq B_0/|x|^{\lambda+\alpha}$ as $|x| \to \infty$.

Since $\lambda q + \beta(q+1) > n$ implies $\lambda + \overline{\beta}(q+1) < n$, using Theorem 1.4 from [**15**],

$$\overline{u}(x) \simeq \frac{\overline{A}_0}{|x|^{\overline{\alpha}}}, \quad \text{for small } |x| \tag{5.4}$$

where $\overline{A}_0 = \int_{R^n} [\overline{v}^q(y)/|y|^{\lambda+\overline{\beta}}]\, dy$. By virtue of (5.2) and (5.4), for large $|x|$,

$$u(x) = \frac{1}{|x|^{2n/(p+1)}}\, \overline{u}\left(\frac{x}{|x|^2}\right) \simeq \frac{B_0}{|x|^{2n/(p+1) - \overline{\alpha}}} = \frac{B_0}{|x|^{\lambda+\alpha}}.$$

Here, B_0 is defined from $\overline{A}_0$ by a change of variable and is computed as follows.

$$B_0 = \int_{R^n} \frac{\overline{v}^q(z/|z|^2)}{|z|^{-\lambda-\overline{\beta}}} \frac{1}{|z|^{2n}}\, dz = \int_{R^n} \frac{v^q(z)}{|z|^{\beta}}\, dz.$$

This establishes (1.13).

Step 3. Note that

$$\text{(5.5)} \qquad \text{(a) } \lambda p + \alpha(p+1) > n \iff \lambda + \overline{\alpha}(p+1) < n;$$

$$\text{(5.6)} \qquad \text{(b) } \lambda p + \alpha(p+1) = n \iff \lambda + \overline{\alpha}(p+1) = n;$$

$$\text{(5.7)} \qquad \text{(c) } \lambda p + \alpha(p+1) < n \iff \lambda + \bar{\alpha}(p+1) > n.$$

Case (a): Using (5.5), applying Theorem 1.4 from [**15**], as $|x| \to 0$, $\overline{v}(x) \simeq \overline{A}_1/|x|^{\overline{\beta}}$, where $\overline{A}_1 = \int_{R^n} (\overline{u}^p(y))/(|y|^{\lambda+\overline{\alpha}})\, dy$. Thus for large $|x|$, we get

$$\text{(5.8)} \qquad v(x) = \frac{1}{|x|^{2n/(q+1)}}\, \overline{v}\left(\frac{x}{|x|^2}\right) \simeq \frac{B_1}{|x|^{2n/(q+1)-\overline{\beta}}} = \frac{B_1}{|x|^{\lambda+\beta}},$$

where $B_1 := \int_{R^n} (u^p(y))/|y|^{\alpha}\, dy$ can be calculated from $\overline{A}_1$.

Case (b): In view of Theorem 1.4 in [**15**] with (5.6), for small $|x|$, $\overline{v}(x) \simeq (\overline{A}_2 |\ln|x||)/|x|^{\overline{\beta}}$, where $\overline{A}_2 = |S^{n-1}|\, (\overline{A}_0)^p$. For large $|x|$,

$$\text{(5.9)} \qquad v(x) \simeq \frac{B_2 |\ln|x||}{|x|^{\lambda+\beta}}$$

where $B_2 := |S^{n-1}|(B_0)^p = |S^{n-1}|(\int_{R^n} (v^q(y))/|y|^{\beta}\, dy)^p$.

Case (c): From (5.7), by [**15**, Theorem 1.4], $\overline{v}(x) \simeq \overline{A}_3/|x|^{\overline{\alpha}(p+1)+\overline{\beta}+\lambda-n}$ for small $|x|$, where $\overline{A}_3 = (\overline{A}_0)^p \int_{R^n} (dz/|z|^{\overline{\alpha}(p+1)}|e-z|^{\lambda})$. Consequently, by (5.2), for large $|x|$,

$$\text{(5.10)} \qquad v(x) \simeq \frac{1}{|x|^{2n/(q+1)}} \frac{B_3}{|x|^{-\overline{\alpha}(p+1)-\overline{\beta}-\lambda+n}} = \frac{B_3}{|x|^{(\alpha+\lambda)(p+1)+\beta-n}},$$

where $B_3 = (B_0)^p \int_{R^n} dz/(|z|^{2n-(\alpha+\lambda)(p+1)}|e-z|^{\lambda}) = (\int_{R^n} (v^q(y)/|y|^{\beta}) dy)^p \int_{R^n} (dz/|z|^{2n-(\alpha+\lambda)(p+1)}|e-z|^{\lambda})$.

Combining (5.8)–(5.10), (1.14) follows.

(ii) Consider the cases (H4) and (H5).

It is essentially the same as the proof of (i). We still apply the Kelvin transform (5.1) to derive the integral system (5.3). Since $1/(q+1) < (\lambda+\beta)/2n$ implies $\overline{\beta} < 0$, from $\overline{\alpha} + \overline{\beta} \geq 0$ it follows that $\overline{\alpha} > 0$. Noting Remark 1.3, we use Theorem 1.2 to see that (1.10) and (1.11) hold for $(\overline{u}, \overline{v})$. Finally, using (5.2), we obtain (1.13) and (1.14).

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