

SOME FIXED POINT THEOREMS OF LEGGETT-WILLIAMS TYPE

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ABSTRACT. This paper presents a fixed point theorem of compression and expansion of functional type through which straightforward extensions lead to double and triple fixed point theorems in the spirit of the original fixed point work of Leggett-Williams. Only portions of the boundaries are required to be mapped outward or inward.

1. Introduction. The Leggett-Williams triple fixed point theorem [8] has been a staple in the existence of triple positive solutions to boundary value problems for two decades. The five functionals fixed point theorem [2] generalized the Leggett-Williams fixed point theorem by providing flexibility in choosing a convex functional instead of using the norm. The key to both of these results is that only a subset of the elements in the cone in which $\alpha(x) = a$ are mapped outwards in the sense that $\alpha(Tx) \geq a$, where α is a concave positive functional defined on the cone. The technique that characterizes Leggett-Williams type fixed point theorems hinges on the boundary of sets and the interweaving role of fixed point index theory and the properties of concave and convex functionals which will be illustrated in Lemmas 8 and 9. In the original work of Leggett-Williams the subset can be thought of as the set of all elements of the cone in which $\|x\| \leq b$ and $\alpha(x) = a$. There were no outward conditions on the operator T in the Leggett-Williams fixed point theorem nor the five functionals fixed point theorem concerning those elements with $\|x\| > b$ and $\alpha(x) = a$, and hence these results avoided any invariance-like conditions with respect to one of the three boundaries that play a significant role applying index theory. The entire upper and lower boundaries were mapped inward in both the Leggett-Williams triple fixed point theorem and the five functionals fixed point theorem. That is, all of the elements in the cone for which $\|x\| = c$ (and d for the lower

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boundary) were mapped inward in the sense that $\|Tx\| \leq c$ (and d for the lower boundary). In this paper we use techniques similar to those of Leggett-Williams that will require only subsets of all three boundaries to be mapped inward and outward, respectively. We thus provide more general results than those obtained in the Leggett-Williams triple fixed point theorem, the five functionals fixed point theorem and the topological generalizations of fixed point theorems introduced by Kwong [7] which all require certain boundaries to be mapped inward or outward (invariance-like conditions).

2. Preliminaries. In this section we will state the definitions that are used in the remainder of the paper.

Definition 1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if for all $x \in P$ and $\lambda \geq 0$, $\lambda x \in P$ and if $x, -x \in P$ then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 3. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \rightarrow [0, \infty)$ is continuous and

$$\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \rightarrow [0, \infty)$ is continuous and

$$\beta(tx + (1 - t)y) \leq t\beta(x) + (1 - t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let ψ and δ be nonnegative continuous functionals on P ; then, for positive real numbers a and b , we define the following sets:

$$P(\psi, b) = \{x \in P : \psi(x) < b\}$$

and

$$P(\psi, \delta, a, b) = \{x \in P : a < \psi(x) \text{ and } \delta(x) < b\}.$$

Definition 4. Let D be a subset of a real Banach space E . If $r : E \rightarrow D$ is continuous with $r(x) = x$ for all $x \in D$, then D is a *retract* of E , and the map r is a *retraction*. The *convex hull* of a subset D of a real Banach space X is given by

$$\text{conv}(D) = \left\{ \sum_{i=1}^n \lambda_i x_i : x_i \in D, \lambda_i \in [0, 1], \sum_{i=1}^n \lambda_i = 1, \text{ and } n \in \mathbf{N} \right\}.$$

The following theorem is due to Dugundji and its proof can be found in [4, page 44].

Theorem 5. *For Banach spaces X and Y , let $D \subset X$ be closed and let $F : D \rightarrow Y$ be continuous. Then F has a continuous extension $\tilde{F} : X \rightarrow Y$ such that $\tilde{F}(X) \subset \overline{\text{conv}(F(D))}$.*

Corollary 6. *Every closed convex set of a Banach space is a retract of the Banach space.*

The following theorem, which establishes the existence and uniqueness of the fixed point index, is from [5, pages 82–86]; an elementary proof can be found in [4, pages 58, 238]. The proof of our main result in the next section will invoke the properties of the fixed point index.

Theorem 7. *Let X be a retract of a real Banach space E . Then, for every bounded relatively open subset U of X and every completely continuous operator $A : \overline{U} \rightarrow X$ which has no fixed points on ∂U (relative*

to X), there exists an integer $i(A, U, X)$ satisfying the following conditions:

(G1) *Normality:* $i(A, U, X) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in \bar{U}$;

(G2) *Additivity:* $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ whenever U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\bar{U} - (U_1 \cup U_2)$;

(G3) *Homotopy invariance:* $i(H(t, \cdot), U, X)$ is independent of $t \in [0, 1]$ whenever $H : [0, 1] \times \bar{U} \rightarrow X$ is completely continuous and $H(t, x) \neq x$ for any $(t, x) \in [0, 1] \times \partial U$;

(G4) *Solution:* If $i(A, U, X) \neq 0$, then A has at least one fixed point in U .

Moreover, $i(A, U, X)$ is uniquely defined.

3. Main results. Anderson, Avery and Henderson [1] proved an expansion-compression fixed point theorem of Leggett-Williams type; embedded in the proof were the following two lemmas which are the primary means of proving the multiple fixed point theorems of Leggett-Williams type which follow.

Lemma 8. Suppose P is a cone in a real Banach space E , α is a nonnegative continuous concave functional on P , β is a nonnegative continuous convex functional on P and $T : P \rightarrow P$ is a completely continuous operator. If there exist nonnegative numbers a and b such that

(A1) $\{x \in P : a < \alpha(x) \text{ and } \beta(x) < b\} \neq \emptyset$;

(A2) if $x \in P$ with $\beta(x) = b$ and $\alpha(x) \geq a$, then $\beta(Tx) < b$;

(A3) if $x \in P$ with $\beta(x) = b$ and $\alpha(Tx) < a$, then $\beta(Tx) < b$;

and if $\overline{P(\beta, b)}$ is bounded then $i(T, P(\beta, b), P) = 1$.

Proof. By Corollary 6, P is a retract of the Banach space E since it is convex and closed.

Claim 1. $Tx \neq x$ for all $x \in \partial P(\beta, b)$.

Let $z_0 \in \partial P(\beta, b)$; then $\beta(z_0) = b$. We want to show that z_0 is not a fixed point of T ; so suppose to the contrary that $T(z_0) = z_0$. If $\alpha(Tz_0) < a$ then $\beta(Tz_0) < b$ by condition (A3), and if $\alpha(z_0) = \alpha(Tz_0) \geq a$ then $\beta(Tz_0) < b$ by condition (A2). Hence in either case we have that $Tz_0 \neq z_0$, thus T does not have any fixed points on $\partial P(\beta, b)$.

Let $z_1 \in \{x \in P : a < \alpha(x) \text{ and } \beta(x) < b\}$ (see condition (A1)), and let $H_1 : [0, 1] \times \overline{P(\beta, b)} \rightarrow P$ be defined by $H_1(t, x) = (1 - t)Tx + tz_1$. Clearly, H_1 is continuous and $H_1([0, 1] \times \overline{P(\beta, b)})$ is relatively compact.

Claim 2. $H_1(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial P(\beta, b)$.

Suppose not; that is, there exists $(t_1, x_1) \in [0, 1] \times \partial U$ such that $H_1(t_1, x_1) = x_1$. Since $x_1 \in \partial P(\beta, b)$ we have that $\beta(x_1) = b$. Either $\alpha(Tx_1) < a$ or $\alpha(Tx_1) \geq a$.

Case 1: $\alpha(Tx_1) < a$. By condition (A3) we have $\beta(Tx_1) < b$, which is a contradiction since

$$b = \beta(x_1) = \beta((1 - t_1)Tx_1 + t_1z_1) \leq (1 - t_1)\beta(Tx_1) + t_1\beta(z_1) < b.$$

Case 2: $\alpha(Tx_1) \geq a$. We have that $\alpha(x_1) \geq a$ since

$$\alpha(x_1) = \alpha((1 - t_1)Tx_1 + t_1z_1) \geq (1 - t_1)\alpha(Tx_1) + t_1\alpha(z_1) \geq a,$$

and thus by condition (A2) we have $\beta(Tx_1) < b$, which is the same contradiction we arrived at in the previous case.

Therefore, we have shown that $H_1(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial P(\beta, b)$, and thus by the homotopy invariance property (G3) of the fixed point index $i(T, P(\beta, b), P) = i(z_1, P(\beta, b), P)$, and by the normality property (G1) of the fixed point index $i(T, P(\beta, b), P) = i(z_1, P(\beta, b), P) = 1$. \square

Lemma 9. *Suppose P is a cone in a real Banach space E , α is a nonnegative continuous concave functional on P , β is a nonnegative continuous convex functional on P and $T : P \rightarrow P$ is a completely continuous operator. If there exist nonnegative numbers a and b such that*

(A4) $\{x \in P : a < \alpha(x) \text{ and } \beta(x) < b\} \neq \emptyset$;

(A5) if $x \in P$ with $\alpha(x) = a$ and $\beta(x) \leq b$, then $\alpha(Tx) > a$;

(A6) if $x \in P$ with $\alpha(x) = a$ and $\beta(Tx) > b$, then $\alpha(Tx) > a$;

and if $\overline{P(\alpha, a)}$ is bounded then $i(T, P(\alpha, a), P) = 0$.

Proof. By Corollary 6, P is a retract of the Banach space E since it is convex and closed.

Claim 1: $Tx \neq x$ for all $x \in \partial P(\alpha, a)$.

Let $w_0 \in \partial P(\alpha, a)$; then $\alpha(w_0) = a$. We want to show that w_0 is not a fixed point of T ; so suppose to the contrary that $T(w_0) = w_0$. If $\beta(Tw_0) > b$ then $\alpha(Tw_0) > a$ by condition (A6), and if $\beta(w_0) = \beta(Tw_0) \leq b$ then $\alpha(Tw_0) > a$ by condition (A5). Hence in either case we have that $Tw_0 \neq w_0$, thus T does not have any fixed points on $\partial P(\alpha, a)$.

Let $w_1 \in \{x \in P : a < \alpha(x) \text{ and } \beta(x) < b\}$ (see condition (A4)) and let $H_2 : [0, 1] \times \overline{P(\alpha, a)} \rightarrow P$ be defined by $H_2(t, x) = (1 - t)Tx + tw_1$. Clearly, H_2 is continuous and $H_2([0, 1] \times \overline{P(\alpha, a)})$ is relatively compact.

Claim 2: $H_2(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial P(\alpha, a)$.

Suppose not; that is, there exists $(t_2, x_2) \in [0, 1] \times \partial P(\alpha, a)$ such that $H_2(t_2, x_2) = x_2$. Since $x_2 \in \partial P(\alpha, a)$ we have that $\alpha(x_2) = a$. Either $\beta(Tx_2) \leq b$ or $\beta(Tx_2) > b$.

Case 1: $\beta(Tx_2) > b$. By condition (A6) we have $\alpha(Tx_2) > a$, which is a contradiction since

$$\begin{aligned} a = \alpha(x_2) &= \alpha((1 - t_2)Tx_2 + t_2w_1) \\ &\geq (1 - t_2)\alpha(Tx_2) + t_2\alpha(w_1) > a. \end{aligned}$$

Case 2: $\beta(Tx_2) \leq b$. We have that $\beta(x_2) \leq b$ since

$$\beta(x_2) = \beta((1 - t_2)Tx_2 + t_2w_1) \leq (1 - t_2)\beta(Tx_2) + t_2\beta(w_1) \leq b,$$

and thus by condition (A5) we have $\alpha(Tx_2) > a$, which is the same contradiction we arrived at in the previous case.

Therefore, we have shown that $H_2(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial P(\alpha, a)$ and thus by the homotopy invariance property (G3) of

the fixed point index $i(T, P(\alpha, a), P) = i(w_1, P(\alpha, a), P)$, and by the solution property (G4) of the fixed point index (since $w_1 \notin P(\alpha, a)$ the index cannot be nonzero) we have $i(T, P(\alpha, a), P) = i(w_1, P(\alpha, a), P) = 0$. \square

To simplify the statements of our results we will define what it means to say that an operator T is LW-inward or LW-outward with respect to functionals and constants in the next definition.

Definition 10. If P is a cone in a real Banach space E , α is a nonnegative continuous concave functional on P , β is a nonnegative continuous convex functional on P , a and b are positive constants and $T : P \rightarrow P$ is a completely continuous operator then we say that:

- (1) T is LW-inward with respect to $I(\alpha, \beta, a, b)$ if the conditions of Lemma 8 are satisfied; and,
- (2) T is LW-outward with respect to $O(\alpha, \beta, a, b)$ if the conditions of Lemma 9 are satisfied.

In the next theorem we present a generalization of the expansion-compression fixed point theorem of Leggett-Williams type as derived in [1] in the sense that, the concave functionals α and ψ are not required to be equal, and the convex functionals δ and β are not required to be equal.

Theorem 11. Suppose P is a cone in a real Banach space E , α and ψ are nonnegative continuous concave functionals on P , δ and β are nonnegative continuous convex functionals on P , and $T : P \rightarrow P$ is a completely continuous operator. If there exist nonnegative numbers a, b, c and d such that

- (D1) T is LW-inward with respect to $I(\alpha, \delta, a, b)$;
- (D2) T is LW-outward with respect to $O(\psi, \beta, c, d)$;

and if

- (H1) $\overline{P(\delta, b)} \subset P(\psi, c)$, then T has a fixed point x^* in $P(\delta, \psi, b, c)$;
- (H2) $\overline{P(\psi, c)} \subset P(\delta, b)$ then T has a fixed point x^* in $P(\psi, \delta, c, b)$.

Proof. We will prove the expansion result (H1), as the proof of the compression result (H2) is nearly identical. To prove the existence of a fixed point for our operator T in $P(\delta, \psi, b, c)$, it is enough for us to show that $i(T, P(\delta, \psi, b, c), P) \neq 0$.

Since T is LW-inward with respect to $I(\alpha, \delta, a, b)$, we have $i(T, P(\delta, b), P) = 1$, and since T is LW-outward with respect to $O(\psi, \beta, c, d)$, we have $i(T, P(\alpha, c), P) = 0$.

T has no fixed points in $\overline{P(\psi, c)} - (P(\delta, b) \cup P(\delta, \psi, b, c))$, and the sets $P(\delta, b)$ and $P(\delta, \psi, b, c)$ are nonempty, disjoint, open subsets of $\overline{P(\psi, c)}$; since $\overline{P(\delta, b)} \subset P(\psi, c)$ we have that $\{x \in P : b < \delta(x) \text{ and } \psi(x) < c\} \neq \emptyset$. Therefore, by the additivity property (G2) of the fixed point index

$$i(T, P(\psi, c), P) = i(T, P(\delta, b), P) + i(T, P(\delta, \psi, b, c), P).$$

Consequently, we have $i(T, P(\delta, \psi, b, c), P) = -1$, and thus by the solution property (G4) of the fixed point index, the operator T has a fixed point $x^* \in P(\delta, \psi, b, c)$. \square

The following theorem is a double fixed point theorem of Leggett-Williams type.

Theorem 12. *Suppose P is a cone in a real Banach space E , α, ψ and ϕ are nonnegative continuous concave functionals on P , δ, β and θ are nonnegative continuous convex functionals on P , and $T : P \rightarrow P$ is a completely continuous operator. If there exist nonnegative numbers a_1, a_2, a_3, b_1, b_2 , and b_3 such that*

(T1) T is LW-outward with respect to $O(\alpha, \delta, a_1, b_1)$;

(T2) T is LW-inward with respect to $I(\psi, \beta, a_2, b_2)$;

(T3) T is LW-outward with respect to $O(\phi, \theta, a_3, b_3)$;

with $\overline{P(\alpha, a_1)} \subset P(\beta, b_2)$ and $\overline{P(\beta, b_2)} \subset P(\phi, a_3)$, then T has at least two fixed points x^* and x^{**} such that $x^* \in P(\alpha, \beta, a_1, b_2)$ and $x^{**} \in P(\beta, \phi, b_2, a_3)$.

Proof. As in the proof of Theorem 11, we have that $i(T, P(\alpha, a_1), P) = 0$, $i(T, P(\beta, b_2), P) = 1$ and $i(T, P(\phi, a_3), P) = 0$. Thus using the addi-

tivity property (G2) of the fixed point index we have

$$i(T, P(\alpha, \beta, a_1, b_2), P) = 1 \text{ and } i(T, P(\beta, \phi, b_2, a_3), P) = -1.$$

Hence by the solution property (G4) of the fixed point index, T has at least two fixed points x^* and x^{**} such that $x^* \in P(\alpha, \beta, a_1, b_2)$ and $x^{**} \in P(\beta, \phi, b_2, a_3)$. \square

And now, we present a triple fixed point theorem of Leggett-Williams type.

Theorem 13. *Suppose P is a cone in a real Banach space E , α, ψ and ϕ are nonnegative continuous concave functionals on P , δ, β and θ are nonnegative continuous convex functionals on P , and $T : P \rightarrow P$ is a completely continuous operator. If there exist nonnegative numbers a_1, a_2, a_3, b_1, b_2 , and b_3 such that*

- (T1) T is LW-inward with respect to $I(\alpha, \delta, a_1, b_1)$;
- (T2) T is LW-outward with respect to $O(\psi, \beta, a_2, b_2)$;
- (T3) T is LW-inward with respect to $I(\phi, \theta, a_3, b_3)$;

with $\overline{P(\delta, b_1)} \subset P(\psi, a_2)$ and $\overline{P(\psi, a_2)} \subset P(\theta, b_3)$, then T has at least three fixed points x^, x^{**} and x^{***} such that $x^* \in P(\delta, b_1)$, $x^{**} \in P(\delta, \psi, b_1, a_2)$ and $x^{***} \in P(\psi, \theta, a_2, b_3)$.*

Proof. Nearly identical to the proof of Theorem 12, we have that

$$i(T, P(\delta, b_1), P) = 1, \quad i(T, P(\psi, a_2), P) = 0, \text{ and } i(T, P(\theta, b_3), P) = 1.$$

Thus using the additivity property (G2) of the fixed point index we have

$$\begin{aligned} i(T, P(\delta, b_1), P) &= 1, \\ i(T, P(\delta, \psi, b_1, a_2), P) &= -1 \end{aligned}$$

and

$$i(T, P(\psi, \theta, a_2, b_3), P) = 1.$$

And so by the solution property (G4) of the fixed point index, T has at least three fixed points x^* , x^{**} and x^{***} such that $x^* \in P(\delta, b_1)$, $x^{**} \in P(\delta, \psi, b_1, a_2)$ and $x^{***} \in P(\psi, \theta, a_2, b_3)$. \square

4. Application. In this section we will illustrate the key techniques for verifying the existence of a positive solution for a right focal boundary value problem using our main result. Right focal boundary value problems have received substantial study for many years. For an early paper, we mention the classical paper by Jackson [6], and for more recent studies, we cite [1, 3], to name just a couple. For our purposes in this paper, under the expansion condition (H1) we apply the properties of a Green's function, bound the nonlinearity by constants over some intervals, and use concavity to deal with a singularity. To proceed, consider the second-order nonlinear right focal boundary value problem

$$(1) \quad x''(t) + f(x(t)) = 0, \quad t \in (0, 1),$$

$$(2) \quad x(0) = 0 = x'(1),$$

where $f : \mathbf{R} \rightarrow [0, \infty)$ is continuous. If x is a fixed point of the operator T defined by

$$Tx(t) := \int_0^1 G(t, s) f(x(s)) ds,$$

where

$$G(t, s) = \min\{t, s\}, \quad (t, s) \in [0, 1] \times [0, 1]$$

is the Green's function for the operator L defined by

$$Lx(t) := -x'',$$

with right-focal boundary conditions

$$x(0) = 0 = x'(1),$$

then it is well known that x is a solution of the boundary value problem (1), (2). Throughout this section of the paper we will use the facts that $G(t, s)$ is nonnegative, and for each fixed $s \in [0, 1]$, the Green's

function is nondecreasing in t , as well as a concavity property of the Green's function is given by

$$(3) \quad \min_{s \in [0,1]} \frac{G(y,s)}{G(w,s)} \geq \frac{y}{w}.$$

Define the cone $P \subset E = C[0,1]$ by

$$P := \{x \in E : x \text{ is nonnegative, nondecreasing, concave, and } x(0) = 0\},$$

and for $x \in P$ and $\nu \in (0,1)$, define the concave functional α_ν on P by

$$\alpha_\nu(x) := \min_{t \in [\nu,1]} x(t) = x(\nu)$$

and the convex functional β on P by

$$\beta(x) := \max_{t \in [0,1]} x(t) = x(1).$$

Thus if $x \in P$ and $\nu \in (0,1)$, then by the concavity of x we have $x(\nu) \geq \nu x(1)$ since

$$\frac{x(\nu) - x(0)}{\nu - 0} \geq \frac{x(1) - x(0)}{1 - 0}.$$

Therefore, for all $x \in P$, we have

$$(4) \quad \nu\beta(x) \leq \alpha_\nu(x).$$

In the following theorem, we demonstrate how to apply the expansive condition of Theorem 11 to prove the existence of at least one positive solution to (1), (2).

Theorem 14. *If $\tau, \mu \in (0,1)$ are fixed, b and d are positive real numbers with $b < d\mu$, and $f : (0, d] \rightarrow [0, \infty)$ is a continuous function such that*

(a) *$f(w)$ is decreasing for $w \in (0, b\tau]$ with $f(b\tau) \geq f(w)$ for $w \in [b\tau, b]$,*

(b) *$\int_0^\tau s f(bs) ds < [2b - f(b\tau)(1 - \tau^2)]/2$, and*

(c) $d/(1 - \mu) \leq f(d\mu) \leq f(w)$ for $w \in [d\mu, d]$,

then the focal problem (1), (2) has at least one positive solution $x^* \in P(\beta, \alpha_\mu, b, d\mu)$.

Proof. If we let $a = b\tau$ and $c = d\mu$, then we have that $a = b\tau < b < d\mu = c < d$ since $b < \mu d$. For $x \in P(\beta, \alpha_\mu, b, c)$, if $t \in (0, 1)$, then by the properties of the Green's function $(Tx)''(t) = -f(x(t))$ and $Tx(0) = 0 = (Tx)'(1)$. By the Arzela-Ascoli theorem it is a standard exercise to show that T is a completely continuous operator using the properties of G and f and for each $x \in \overline{P(\beta, \alpha_\mu, b, c)}$, $Tx \in P$. Therefore we have that $T : \overline{P(\beta, \alpha_\mu, b, c)} \rightarrow P$ is continuous. Thus by Dugundji's theorem, there is a continuous extension, which we will again denote by T , such that $T : P \rightarrow P$; a proof can be found in [4]. We will now verify that properties (a), (b) and (c) imply that T is LW-inward with respect to $I(\alpha_\tau, \beta, a, b)$ and T is LW-outward with respect to $O(\alpha_\mu, \beta, c, d)$.

Claim 1: T is LW-inward with respect to $I(\alpha_\tau, \beta, a, b)$.

For any $L \in (2b/(2 - \tau), 2b)$ the function x_L defined by

$$\begin{aligned} x_L(t) &\equiv \int_0^1 LG(t, s) ds = \frac{Lt(2-t)}{2} \\ &\in \{x \in P : a < \alpha_\tau(x) \text{ and } \beta(x) < b\}, \end{aligned}$$

since

$$\alpha_\tau(x_L) = x_L(\tau) = \frac{L\tau(2-\tau)}{2} > b\tau = a$$

and

$$\beta(x_L) = x_L(1) = \frac{L}{2} < b.$$

Therefore, we have that

$$\{x \in P : a < \alpha_\tau(x) \text{ and } \beta(x) < b\} \neq \emptyset,$$

and if $x \in \overline{P(\beta, b)}$, then

$$\|x\| = \beta(x) \leq b.$$

Thus $\overline{P(\beta, b)}$ is bounded as well.

Subclaim 1.1: $\beta(Tx) < b$ for all $x \in P$ with $\beta(x) = b$ and $\alpha_\tau(x) \geq a$. Let $x \in P$ with $\beta(x) = b$ and $\alpha_\tau(x) \geq a$. By the concavity of x , for $s \in [0, \tau]$ we have

$$x(s) \geq \left(\frac{x(\tau)}{\tau}\right)s \geq bs,$$

and for all $s \in [\tau, 1]$, we have

$$b\tau \leq x(s) \leq b.$$

Hence by properties (a) and (b), it follows that

$$\begin{aligned} \beta(Tx) &= \int_0^1 G(1, s) f(x(s)) ds = \int_0^1 s f(x(s)) ds \\ &= \int_0^\tau s f(x(s)) ds + \int_\tau^1 s f(x(s)) ds \\ &\leq \int_0^\tau s f(bs) ds + f(b\tau) \int_\tau^1 s ds \\ &\leq \frac{2b - f(b\tau)(1 - \tau^2)}{2} + \frac{f(b\tau)(1 - \tau^2)}{2} = b. \end{aligned}$$

Subclaim 1.2: If $x \in P$ and $\alpha_\tau(Tx) < a$, then $\beta(Tx) < b$. Let $x \in P$ with $\alpha_\tau(Tx) < a$. Thus by the properties of $G(t, s)$ given in (3),

$$\begin{aligned} \beta(Tx) &= \int_0^1 G(1, s) f(x(s)) ds \\ &\leq \left(\frac{1}{\tau}\right) \int_0^1 G(\tau, s) f(x(s)) ds \\ &= \left(\frac{1}{\tau}\right) \alpha_\tau(Tx) < \left(\frac{a}{\tau}\right) = b. \end{aligned}$$

Therefore, we have verified that T is LW-inward with respect to $I(\alpha_\tau, \beta, a, b)$.

Claim 2: T is LW-outward with respect to $O(\alpha_\mu, \beta, c, d)$.

For any $J \in (2d/(2 - \mu), 2d)$, the function x_J defined by

$$\begin{aligned} x_J(t) &\equiv \int_0^1 JG(t, s) ds = \frac{Jt(2-t)}{2} \\ &\in \{x \in P : c < \alpha_\mu(x) \text{ and } \beta(x) < d\}, \end{aligned}$$

since

$$\alpha_\mu(x_J) = x_J(\mu) = \frac{J\mu(2-\mu)}{2} > d\mu = c$$

and

$$\beta(x_J) = x_J(1) = \frac{J}{2} < d.$$

Therefore we have that

$$\{x \in P : c < \alpha_\mu(x) \text{ and } \beta(x) < d\} \neq \emptyset,$$

and $P(\alpha_\mu, c)$ is a bounded subset of the cone P , since if $x \in P(\alpha_\mu, c)$, then by (4) we have that

$$\mu\beta(x) \leq \alpha_\mu(x) \leq c,$$

and so

$$\|x\| = \beta(x) \leq \frac{\alpha_\mu(x)}{\mu} \leq \frac{c}{\mu} = d.$$

Subclaim 2.1: $\alpha_\mu(Tx) > c$ for all $x \in P$ with $\alpha_\mu(x) = c$ and $\beta(x) \leq d$. Let $x \in P$ with $\alpha_\mu(x) = c$ and $\beta(x) \leq d$. Then for $s \in [\mu, 1]$ we have

$$d\mu = c \leq x(s) \leq d.$$

Hence by property (c),

$$\begin{aligned} \alpha_\mu(Tx) &= \int_0^1 G(\mu, s) f(x(s)) ds \\ &\geq \int_\mu^1 G(\mu, s) f(x(s)) ds \\ &= \int_\mu^1 \mu f(c) ds = f(c)\mu(1-\mu) \\ &> \left(\frac{d}{1-\mu}\right)\mu(1-\mu) = c. \end{aligned}$$

Subclaim 2.2: $\alpha_\mu(Tx) > c$ for all $x \in P$ with $\alpha_\mu(x) = c$ and $\beta(Tx) > d$. Let $x \in P$ with $\beta(Tx) > d$. Thus by (4) we have

$$\alpha_\mu(Tx) \geq \mu\beta(Tx) > \mu d = c.$$

Therefore we have verified that T is LW-outward with respect to $O(\alpha_\mu, \beta, c, d)$.

For any $M \in (2b, 2c)$ the function x_M defined by

$$x_M(t) \equiv \int_0^1 MG(t, s)ds = \frac{Mt(2-t)}{2} \in P(\beta, \alpha_\mu, b, c),$$

since

$$\alpha_\mu(x_M) = x_M(\mu) = \frac{M\mu(2-\mu)}{2} < \frac{2c\mu(2-\mu)}{2} \leq c$$

and

$$\beta(x_M) = x_M(1) = \frac{M}{2} > b.$$

Consequently we have that $\{x \in P : b < \beta(x) \text{ and } \alpha_\mu(x) < c\} \neq \emptyset$, and if $x \in \overline{P(\beta, b)}$, then

$$\alpha_\mu(x) \leq \beta(x) \leq b < c.$$

Thus $\overline{P(\beta, b)} \subset P(\alpha_\mu, c)$. Therefore, (H1) of Theorem 11 has been satisfied; thus the operator T has at least one fixed point $x^* \in P(\beta, \alpha_\mu, b, c)$, which is a desired solution of (1), (2). \square

As noted above, because of the concavity of solutions, the proof of Theorem 14 remains valid for certain singularities in the nonlinearity. That was also the case involving only one singularity in the nonlinearity in the papers [1, 3]; yet in this setting, Theorem 14 actually allows for multiple singularities as presented in this example.

Example. Let

$$\tau = \frac{1}{2}, \quad \mu = 0.9999, b = 1 \text{ and } d = 9.9999.$$

Then the boundary value problem

$$x'' + \frac{1}{\sqrt{x}(x-10)^2} = 0,$$

with right focal boundary conditions

$$x(0) = 0 = x'(1),$$

has at least one positive solution x^* , which can be verified by Theorem 14, with

$$1 \leq x^*(1) \quad \text{and} \quad x^*(0.9999) \leq (0.9999)(9.9999).$$

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