ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 2, 2011

LINEAR SYSTEMS OF FRACTIONAL NABLA DIFFERENCE EQUATIONS

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ABSTRACT. In this paper we shall consider a linear system of fractional nabla difference equations with constant coefficients. We shall construct the fundamental matrix for the homogeneous system and the causal Green's function for the nonhomogeneous system. We employ transform methods and series methods and we illustrate analogies with classical first order differential or difference equations. We shall close the paper with an asymptotic result that follows from the analysis of a half-order nabla difference equation.

1. Introduction. In this article we shall provide an introductory study to a system of fractional difference equations of the form

$$\nabla_0^\nu y(t) = Ay(t) + f(t), \quad t = 1, 2, \dots, \tag{1.1}$$

where A denotes an $n \times n$ matrix with constant entries, y and f denote $n-$vector valued functions and $0 < \nu < 1$. The operator ∇_a^ν, a Riemann-Liouville fractional difference, is defined as follows. If $\mu > 0$, define the μth fractional sum by

$$\nabla_a^{-\mu} y(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)} y(s)$$

where $\rho(s) = s - 1$ and the raising factorial power function is defined by $t^{\overline{\alpha}} = \Gamma(t+\alpha)/\Gamma(t)$. Then, if $0 \le n-1 < \nu \le n$, define the νth fractional difference (a Riemann-Liouville fractional difference) by $\nabla^\nu y(t) = \nabla^n \nabla^{\nu-n} y(t)$ where ∇^n denotes the standard nth order backward difference. So, in this article, with $0 < \nu < 1$, $\nabla_0^\nu y(t) = \nabla \nabla_0^{\nu-1} y(t)$. Anastassiou [**3**] has introduced the study of nabla fractional calculus in the case of the Caputo fractional difference.

2010 AMS *Mathematics subject classification.* Primary 39A12, 34A25, 26A33.
Keywords and phrases. Discrete fractional calculus, discrete Mittag-Leffler function.
Received by the editors on May 24, 2010.
DOI:10.1216/RMJ-2011-41-2-353

We shall study (1.1) with a transform method [**5**] and with the Neumann series method. We shall be interested in analogies to the study of $y' = Ay + f(t)$. A related study of the fractional differential operator is found in [**10**].

The Mittag-Leffler function is named for Gösta Mittag-Leffler who defined and studied the special function in 1903 [**13**]. The function is a direct generalization of the exponential function e^x, and it plays a major role in fractional calculus. The one and two-parameter representations of the Mittag-Leffler function can be defined in terms of a power series as

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \tag{1.2}$$

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \tag{1.3}$$

where α and β are positive real numbers. The Mittag-Leffler function with two-parameters was first defined by Agarwal in 1953 [**2**].

We shall define the discrete Mittag-Leffler function with one and two-parameters in the following way. Related definitions are given by Nagai in [**14**].

$$F_\alpha(at) = \sum_{k=0}^{\infty} \frac{a^k t^{\overline{k}}}{\Gamma(\alpha k + 1)}, \tag{1.4}$$

$$F_{\alpha,\beta}(at) = \sum_{k=0}^{\infty} \frac{a^k t^{\overline{k}}}{\Gamma(\alpha k + \beta)}, \tag{1.5}$$

where α and β are positive real numbers and $|a| < 1$. For any real number ν, we shall define the discrete Mittag-Leffler function as

$$F_{\alpha,\beta}(at^{\overline{\nu}}) = \sum_{k=0}^{\infty} \frac{a^k t^{\overline{k\nu}}}{\Gamma(\alpha k + \beta)}.$$

In Section 2, so that the article is self-contained, we shall give the definition of the discrete Laplace transform ($\mathcal{N}$-transform) and provide the basic properties employed in this work. We shall also provide some basic algebra properties related to the function $t^{\overline{\alpha}}$. In Section 3, we shall define the convolution product of two functions and obtain a Convolution theorem. We shall also formalize the ∇^ν-fractional derivative of the convolution product of two functions. In Section 4, we shall argue existence and uniqueness of solutions of initial value problems related to (1.1), provide two constructions of the fundamental matrix and solve the nonhomogeneous equation (1.1). In Section 5, we shall focus on the discrete half-order equation and obtain an asymptotic result for the general solution of the fractional equation $\nabla^\nu y(t) = ay(t)$ where $0 < a < 1$ and $1/2 < \nu \le 1$.

For further reading in this area, we refer the reader to the books on the fractional differential equations [**12, 15, 16**] and to the articles on the fractional difference equations [**6**–**9**].

2. The discrete transform and some algebra properties. For the sake of exposition, we first introduce algebra and calculus properties related to the raising factorial power function.

Lemma 2.1. (i) $\nabla t^{\overline{\alpha}} = \alpha t^{\overline{\alpha-1}}$.

(ii) $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.

(iii) $\nabla_1^{-\nu} t^{\overline{\mu}} = \Gamma(\mu+1)/\Gamma(\mu+\nu+1)t^{\overline{\mu+\nu}}$.

(iv) $\nabla_0^{-\nu}(t+1)^{\overline{\mu}} = \Gamma(\mu+1)/\Gamma(\mu+\nu+1)(t+1)^{\overline{\mu+\nu}}$.

(v) $\sum_{n=0}^{k} t^{\overline{n}}/\Gamma(n+1) = (t+1)^{\overline{k}}/\Gamma(k+1)$.

Proof. (i) and (ii) are easily observed by applying standard Gamma function identities. (iii) is proved in [**5**] and is referred to as the power rule. (iv) is another form of the power rule and is stated for clarity since applications of nabla fractional calculus to composite functions is not fully understood. (v) is obtained by induction on k and is in fact one of the identities found in Pascal's triangle since $(t+1)^{\overline{k}}/\Gamma(k+1) = \binom{t+k}{t}$. Thus, by induction on k,

$$\sum_{n=0}^{k+1} \frac{t^{\overline{n}}}{\Gamma(n+1)} = \binom{t+k}{t} + \frac{t^{\overline{k+1}}}{\Gamma(k+2)}$$
$$= \binom{t+k}{t} + \binom{t+k}{t-1} = \binom{t+k+1}{t}. \qquad \square$$

The following definition and the derivation of properties are found in [**5**]. Define the discrete Laplace transform ($\mathcal{N}$-transform) by

$$\mathcal{N}_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty}(1-s)^{t-1}f(t). \tag{2.1}$$

Lemma 2.2. *For any* $\nu \in \mathbf{R} \setminus \{\dots, -2, -1, 0\}$,

(i) $\mathcal{N}_1(t^{\overline{\nu-1}})(s) = \Gamma(\nu)/s^{\nu}$, $|1-s| < 1$, *and*

(ii) $\mathcal{N}_1(t^{\overline{\nu-1}}\alpha^{-t})(s) = \alpha^{\nu-1}\Gamma(\nu)/(s+\alpha-1)^{\nu}$, $|1-s| < \alpha$.

(iii) $\mathcal{N}_1(t^{\overline{\nu}})(s) = (\nu/s)\mathcal{N}_1(t^{\overline{\nu-1}})$.

(iv) $\mathcal{N}_a(f(\sigma(t))) = (1-s)^{-1}\mathcal{N}_{a+1}f(t)$.

(v) $\mathcal{N}_a(\nabla_a^{-\nu}f(t)) = s^{-\nu}\mathcal{N}_a(f(t))(s)$.

(vi) $\mathcal{N}_{a+1}(\nabla_a^{\nu}f(t))(s) = s^{\nu}\mathcal{N}_a(f(t))(s) - (1-s)^{a-1}f(a)$,

where $0 < \nu < 1$.

(vii) $\mathcal{N}_0((1/(1-a^2))^{t+1}) = 1/[(1-s)(s-a^2)]$.

Example 2.1. Consider the initial value problem

$$\nabla_0^{\nu}y(t) = ay(t) \quad \text{for } t = 1, 2, \dots, \tag{2.2}$$

$$\nabla_0^{-(1-\nu)}y(t)|_{t=0} = y(0) = c, \tag{2.3}$$

where $0 < \nu < 1$ and $|a| < 1$.

Apply the $\mathcal{N}_1$-transform to each side of the equation (2.2) to obtain

$$\mathcal{N}_1(\nabla_0^{\nu}y(t)) = a\mathcal{N}_1y(t).$$

Apply Lemma 2.2 to obtain

$$\begin{aligned} s^{\nu}\mathcal{N}_0y(t) - (1-s)^{-1}y(0) &= a\mathcal{N}_1y(t) \\ &= a\mathcal{N}_0y(t) - \frac{a}{(1-s)}y(0). \\ (s^{\nu}-a)\mathcal{N}_0y(t) &= \frac{1-a}{(1-s)}y(0) \end{aligned}$$

and finally

$$\mathcal{N}_0 y(t) = \frac{1-a}{(1-s)}\frac{1}{s^\nu - a} y(0). \tag{2.4}$$

Expand $1/(1-s^{-\nu}a)$ as a geometric series and obtain

$$\begin{aligned}
\mathcal{N}_0 y(t) &= \frac{1-a}{(1-s)}\frac{s^{-\nu}}{1-s^{-\nu}a} y(0) \\
&= \frac{(1-a)y(0)}{(1-s)} \sum_{k=0}^{\infty} a^k s^{-(k+1)\nu} \\
&= \frac{(1-a)y(0)}{(1-s)} \sum_{k=0}^{\infty} a^k \frac{\mathcal{N}_1(t^{\overline{(k+1)\nu-1}})}{\Gamma((k+1)\nu)} \\
&= (1-a)y(0) \sum_{k=0}^{\infty} a^k \left\{ \frac{\mathcal{N}_0(t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} \right\}.
\end{aligned}$$

Applying the inverse $\mathcal{N}_0$-transform to each side of the above expression, we obtain the solution of the initial value problem (2.2)–(2.3),

$$y(t) = (1-a)y(0) \sum_{k=0}^{\infty} \frac{a^k (t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)}, \quad t = 0, 1, 2, \dots,$$

where $|a| < 1$. Apply Lemma 2.1 (ii) and write

$$\begin{aligned}
y(t) &= (1-a)y(0)(t+1)^{\overline{\nu-1}} \sum_{k=0}^{\infty} \frac{a^k (t+\nu)^{\overline{k\nu}}}{\Gamma((k+1)\nu)} \\
&= (1-a)y(0)(t+1)^{\overline{\nu-1}} F_{\nu,\nu}(a(t+\nu)^{\overline{\nu}}).
\end{aligned}$$

In this formula, if we set $\nu = 1$, then by the uniqueness of the solution of the initial value problem (2.2)–(2.3), we have $(1-a)y(0)F_{1,1}(a(t+1)) = y(0)(1/(1-a))^t$. This implies that $F_{1,1}(a(t+1)) = (1/(1-a))^{t+1}$.

Hence, we have that

$$F_{1,1}(at) = \left(\frac{1}{1-a}\right)^t. \tag{2.5}$$

Equation (2.5) has been obtained through the uniqueness of solutions of initial value problems approach. It is also interesting to obtain (2.5) directly. This is done by induction on t and employs Lemma 2.1 (v). Assume (2.5). Then

$$\begin{aligned}\left(\frac{1}{1-a}\right)^{t+1} &= \left(\frac{1}{1-a}\right)^{t}\left(\frac{1}{1-a}\right) = \sum_{k=0}^{\infty}\frac{t^{\overline{k}}}{\Gamma(k+1)}a^k\sum_{k=0}^{\infty}a^k \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}\frac{t^{\overline{n}}}{\Gamma(n+1)}\right)a^k = \sum_{k=0}^{\infty}\frac{(t+1)^{\overline{k}}}{\Gamma(k+1)}a^k.\end{aligned}$$

3. Convolution product. Define the convolution

$$(h *_a g)(t) = \sum_{s=a}^{t} h(t+a-\rho(s))g(s).$$

We now obtain a standard property for $\mathcal{N}_a((h *_a g)(t))(s)$.

Lemma 3.1.

$$\mathcal{N}_a((h *_a g)(t))(s) = \mathcal{N}_1(H(t))(s)\mathcal{N}_a(g(t))(s),$$

where $H(t) = h(t+a)$.

Proof.

$$\begin{aligned}\mathcal{N}_a((h *_a g)(t))(s) &= \sum_{t=a}^{\infty}(1-s)^{t-1}\sum_{r=a}^{t}h(t+a-\rho(r))g(r) \\ &= \sum_{r=a}^{\infty}\sum_{t=r}^{\infty}(1-s)^{t-1}h(t+a-\rho(r))g(r) \\ &= \sum_{r=a}^{\infty}\sum_{u=1}^{\infty}(1-s)^{u+\rho(r)-1}h(u+a)g(r) \\ &= \left(\sum_{u=1}^{\infty}(1-s)^{u-1}h(u+a)\right)\left(\sum_{r=a}^{\infty}(1-s)^{r-1}g(r)\right)\end{aligned}$$

$$= \mathcal{N}_1(H(t))(s)\mathcal{N}_a(g(t))(s),$$

where $H(t) = h(t+a)$. $\square$

Remark 3.1. Note that $(h *_0 g)(t) = \sum_{s=0}^{t} h(t-\rho(s))g(s) = \sum_{s=1}^{t+1} g(t-\rho(s))h(s)$.

Lemma 3.2.

$$\nabla_a^\nu (h *_a g)(t) = ((\nabla_{a+1}^\nu h) *_a g)(t) + \nabla_{a+1}^{-(1-\nu)} h(t)|_{t=a} g(t).$$

Proof.

$$\begin{aligned}
\nabla_a^\nu (h *_a g)(t) &= \nabla \nabla_a^{-(1-\nu)} \sum_{s=a}^{t} h(t+a-\rho(s))g(s) \\
&= \frac{1}{\Gamma(1-\nu)} \nabla \sum_{\tau=a}^{t} (t-\rho(\tau))^{\overline{-\nu}} \sum_{s=a}^{\tau} h(\tau+a-\rho(s))g(s) \\
&= \frac{1}{\Gamma(1-\nu)} \nabla \sum_{s=a}^{t} \sum_{\tau=s}^{t} (t-\rho(\tau))^{\overline{-\nu}} h(\tau+a-\rho(s))g(s) \\
&= \frac{1}{\Gamma(1-\nu)} \nabla \sum_{s=a}^{t} \sum_{u=a+1}^{t-\rho(s)+a} (t-\rho(s)+a-\rho(u))^{\overline{-\nu}} h(u)g(s) \\
&= \nabla \sum_{s=a}^{t} \nabla_{a+1}^{-(1-\nu)} h(t-\rho(s)+a)g(s) \\
&= \sum_{s=a}^{t} \nabla_{a+1}^{\nu} h(t-\rho(s)+a)g(s) + \nabla_{a+1}^{-(1-\nu)} h(t)|_{t=a} g(t) \\
&= ((\nabla_{a+1}^\nu h) *_a g)(t) + \nabla_{a+1}^{-(1-\nu)} h(t)|_{t=a} g(t). \qquad \square
\end{aligned}$$

4. The νth order nonhomogeneous system of equations. We begin this section by solving a homogeneous equation and constructing the fundamental matrix.

Theorem 4.1. *If A is a 2×2 constant matrix with eigenvalues of modulus less than one, then the solution of the initial value problem*

$$\nabla_0^\nu y(t) = Ay(t) \quad t = 1, 2, \dots, \tag{4.1}$$

$$\nabla_0^{-(1-\nu)} y(t)|_{t=0} = y(0) = C, \tag{4.2}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, $0 < \nu < 1$, *is given by*

$$\begin{aligned} y(t) &= (t+1)^{\overline{\nu-1}} \sum_{k=0}^{\infty} \frac{A^k (t+\nu)^{\overline{k\nu}}}{\Gamma((k+1)\nu)} (I-A)C \\ &= (t+1)^{\overline{\nu-1}} F_{\nu,\nu}(A(t+\nu)^{\overline{\nu}})(I-A)C \end{aligned} \tag{4.3}$$

where $F_{\nu,\nu}(At^{\overline{\nu}})$ is the matrix Mittag-Leffler function and C is a 2×1 vector.

Proof. For the sake of self-containment, we provide a proof by directly calculating $\nabla_0^\nu = \nabla \nabla_0^{\nu-1}$ of the right hand side of (4.3) by repeated use of the power rule, Lemma 2.1 (iv).

$$\nabla_0^{\nu-1} \sum_{k=0}^{\infty} \frac{A^k (t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} = \sum_{k=0}^{\infty} \frac{A^k (t+1)^{\overline{k\nu}}}{\Gamma(k\nu+1)}.$$

Thus,

$$\nabla \sum_{k=0}^{\infty} \frac{A^k (t+1)^{\overline{k\nu}}}{\Gamma(k\nu+1)} = \sum_{k=1}^{\infty} \frac{A^k (t+1)^{\overline{k\nu-1}}}{\Gamma(k\nu)} = A \sum_{k=0}^{\infty} \frac{A^k (t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu+1)},$$

and

$$\nabla_0^\nu \sum_{k=0}^{\infty} \frac{A^k (t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} = A \sum_{k=0}^{\infty} \frac{A^k (t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)}.$$

One multiplies on the right by $(I - A)C$ to satisfy the initial condition. □

Note that in Example 2.1 an explicit form of the solution of the scalar problem is given in terms of the Mittag-Leffler functions. Hence, if A is a diagonal matrix, we have an explicit form for the matrix Mittag-Lefler function and moreover, if A is diagonalizable, we can produce an explicit form for fundamental matrix of (1.1). In the next theorem, we apply Putzer's algorithm to provide an alternate construction of the fundamental matrix.

Definition 4.2 (Matrix exponential function). Let A be a 2×2 constant matrix. The unique matrix valued solution of the IVP

$$\nabla_0^{\nu} Y(t) = AY(t) \quad \text{for } t = 1, 2, \dots , \tag{4.4}$$

$$\nabla_0^{-(1-\nu)} Y(t)|_{t=0} = Y(0) = I, \tag{4.5}$$

where I denotes the 2×2 identity matrix, is called the matrix exponential function.

Theorem 4.3. *If λ_1 and λ_2 are (not necessarily distinct) eigenvalues of the 2×2 matrix A, then*

$$(t+1)^{\overline{\nu-1}} F_{\nu,\nu}(A(t+\nu)^{\overline{\nu}}) = p_1(t) M_0 + p_2(t) M_1,$$

where $M_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_1 = A - \lambda_1 I = \begin{bmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{bmatrix}$, and the vector valued function p is defined by

$$p(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix},$$

the solution of the initial value problem

$$\nabla_0^{\nu} y(t) = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} y(t) \quad \text{for } t = 1, 2, \dots , \tag{4.6}$$

$$\nabla_0^{-(1-\nu)} y(t)|_{t=0} = y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4.7}$$

Proof. Let $\Phi(t) = p_1(t) M_0 + p_2(t) M_2$. We first show that Φ solves the IVP (4.4)–(4.5).

First note that

$$\begin{aligned}
\nabla_0^{-(1-\nu)}\Phi(0) &= \nabla_0^{-(1-\nu)}p_1(0)M_0 + \nabla_0^{-(1-\nu)}p_2(0)M_1 = I.\\
\nabla_0^{\nu}\Phi(t) - A\Phi(t) &= \nabla_0^{\nu}[p_1(t)M_0 + p_2(t)M_1] - A[p_1(t)M_0 + p_2(t)M_1]\\
&= \lambda_1 p_1(t)M_0 + [p_1(t) + \lambda_2 p_2(t)]M_1\\
&\quad - A[p_1(t)M_0 + p_2(t)M_1]\\
&= \lambda_2 p_2 M_1 - (M_1 + \lambda_1 I)p_2(t)M_1\\
&= (\lambda_2 I - M_1 - \lambda_1 I)p_2(t)M_1\\
&= -p_2(t)(A - \lambda_2 I)(A - \lambda_1 I)\\
&= 0.
\end{aligned}$$

Since $(t+1)^{\overline{\nu-1}}F_{\nu,\nu}(A(t+\nu)^{\overline{\nu}})(I-A)$ satisfies the IVP (4.4)–(4.5) with $C = I$,

$$\Phi(t) = (t+1)^{\overline{\nu-1}}F_{\nu,\nu}(A(t+\nu)^{\overline{\nu}})(I-A)$$

by the unique solvability of initial value problems. □

Theorem 4.4. *Let A be an $n\times n$ constant matrix and suppose f is a vector valued function. Then the initial value problem*

$$\nabla_0^{\nu}y(t) = Ay(t) + f(t) \quad \textit{for } t = 1, 2, \dots, \tag{4.8}$$

$$\nabla_0^{-(1-\nu)}y(t)|_{t=0} = y(0), \tag{4.9}$$

has a unique solution. Moreover, this solution is given by

$$\begin{aligned}
y(t) &= (t+1)^{\overline{\nu-1}}F_{\nu,\nu}(A(t+\nu)^{\overline{\nu}})(I-A)y(0)\\
&\quad + \sum_{s=1}^{t}(t-\rho(s))^{\overline{\nu-1}}F_{\nu,\nu}(A(t+\nu-1-\rho(s))^{\overline{\nu}})f(s).
\end{aligned}$$

Proof. The details are very similar to those provided in Example 2.1. The introduction of the nonhomogeneous term requires the application of the convolution product. Applying the $\mathcal{N}_1$-transform to each side of the equation (4.8), we have

$$\mathcal{N}_1(\nabla_0^{\nu}y(t)) = A\mathcal{N}_1 y(t) + \mathcal{N}_1 f(t).$$

Employ Lemma 2.2 and obtain

$$s^{\nu}\mathcal{N}_0 y(t) - (1-s)^{-1} y(0) = A\mathcal{N}_1 y(t) + \mathcal{N}_1 f(t).$$

$$s^{\nu}\mathcal{N}_0 y(t) - (1-s)^{-1} y(0) = A\mathcal{N}_0 y(t) - A\frac{1}{(1-s)} y(0) + \mathcal{N}_1 f(t).$$

$$(s^{\nu} I - A)\mathcal{N}_0 y(t) = (I-A)\frac{1}{(1-s)} y(0) + \mathcal{N}_0 f(t) - \frac{f(0)}{(1-s)}.$$

Again employ the geometric series expansion to obtain

$$\begin{aligned}
\mathcal{N}_0 y(t) &= s^{-\nu}(I - s^{-\nu}A)^{-1}\left[(I-A)\frac{1}{(1-s)} y(0) + \mathcal{N}_0 f(t) - \frac{f(0)}{(1-s)}\right] \\
&= \{s^{-\nu} I + s^{-2\nu} A + s^{-3\nu} A^2 + \dots\} \\
&\quad \times \left[(I-A)\frac{1}{(1-s)} y(0) + \mathcal{N}_0 f(t) - \frac{f(0)}{(1-s)}\right] \\
&= \left\{\frac{\mathcal{N}_1(t^{\overline{\nu-1}})}{\Gamma(\nu)} I + \frac{\mathcal{N}_1(t^{\overline{2\nu-1}})}{\Gamma(2\nu)} A + \dots\right\} \\
&\quad \times \left[(I-A)\frac{1}{(1-s)} y(0) + \mathcal{N}_0 f(t) - \frac{f(0)}{(1-s)}\right] \\
&= \left\{\frac{\mathcal{N}_0(t+1)^{\overline{\nu-1}}}{\Gamma(\nu)} I + \frac{\mathcal{N}_0(t+1)^{\overline{2\nu-1}}}{\Gamma(2\nu)} A + \dots\right\} \\
&\quad \times [(I-A)y(0) - f(0)] \\
&\quad + \left\{\frac{\mathcal{N}_1(t^{\overline{\nu-1}})}{\Gamma(\nu)} I + \frac{\mathcal{N}_1(t^{\overline{2\nu-1}})}{\Gamma(2\nu)} A + \dots\right\}\mathcal{N}_0 f(t) \\
&= \mathcal{N}_0\left\{\sum_{k=0}^{\infty} \frac{A^k (t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)}\right\}[(I-A)y(0) - f(0)] \\
&\quad + \mathcal{N}_1\left\{\sum_{k=0}^{\infty} \frac{A^k t^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)}\right\}\mathcal{N}_0 f(t).
\end{aligned}$$

Next we use Lemma 3.1 to obtain

$$\begin{aligned}
\mathcal{N}_0 y(t) = \mathcal{N}_0&\left\{\sum_{k=0}^{\infty} \frac{A^k (t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)}\right\}[(I-A)y(0) - f(0)] \\
&+ \mathcal{N}_0\left\{\sum_{k=0}^{\infty} \frac{A^k t^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} *_0 f(t)\right\}.
\end{aligned}$$

Applying the inverse $\mathcal{N}_0$ transform to each side of the above last equalities, we obtain the desired result.

$$\begin{aligned} y(t) &= \sum_{k=0}^{\infty} \frac{A^k (t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)}(I-A)y(0) \\ &\quad + \sum_{s=1}^{t} (t-\rho(s))^{\overline{\nu-1}} F_{\nu,\nu}(A(t+\nu-1-\rho(s))^{\overline{\nu}}) f(s). \qquad \square \end{aligned}$$

We now present a formal argument that

$$\begin{aligned} y_p &= \sum_{s=1}^{t} (t-\rho(s))^{\overline{\nu-1}} F_{\nu\nu}(A(t-\rho(s)-1+\nu)^{\overline{\nu}}) f(s) \\ &= \sum_{k=0}^{\infty} A^k \sum_{s=1}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}} (t-\rho(s)+\nu-1)^{\overline{k\nu}}}{\Gamma((k+1)\nu)} f(s) \end{aligned}$$

is a particular solution of the nonhomogeneous equation. Apply Lemma 2.1 (ii) to see that

$$(t-\rho(s))^{\overline{\nu-1}}(t-\rho(s)+\nu-1)^{\overline{k\nu}} = (t-\rho(s))^{\overline{(k+1)\nu-1}}.$$

Thus,

$$y_p = \sum_{k=0}^{\infty} A^k \sum_{s=1}^{t} \frac{(t-\rho(s))^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} f(s) = \sum_{k=0}^{\infty} A^k \nabla^{-((k+1)\nu)} f(t)$$

and

$$\begin{aligned} \nabla^{\nu} y_p &= \nabla \nabla^{\nu-1} y_p \\ &= \nabla \sum_{k=0}^{\infty} A^k \nabla^{\nu-1} \nabla^{-((k+1)\nu)} f \\ &= \nabla \sum_{k=0}^{\infty} A^k \nabla^{-k\nu-1} f(t) \\ &= \sum_{k=1}^{\infty} A^k \nabla^{-k\nu} f(t) + f(t) \\ &= A \sum_{k=0}^{\infty} A^k \nabla^{-(k+1)\nu} f(t) + f(t) \\ &= A y_p + f(t). \end{aligned}$$

Note that these calculations agree with the Neumann series solution

$$\begin{aligned} y_p &= (I - A\nabla^{-\nu})^{-1}\nabla^{-\nu} f(t) \\ &= \sum_{k=0}^{\infty} (A\nabla^{-\nu})^k \nabla^{-\nu} f(t) \\ &= \sum_{k=0}^{\infty} A^k \nabla^{-((k+1)\nu)} f(t). \end{aligned}$$

Thus, the term

$$H(t,s) = (t-\rho(s))^{\overline{\nu-1}} F_{\nu,\nu}(A(t+\nu-1-\rho(s))^{\overline{\nu}}) \tag{4.10}$$

will play a role as the causal Green's function for problems related to (1.1).

Last, we also note that if we apply ∇_0^{ν} operator to each side of the solution

$$\begin{aligned} y(t) &= \sum_{k=0}^{\infty} \frac{A^k (t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} (I-A) y(0) \\ &\quad + \sum_{s=1}^{t} (t-\rho(s))^{\overline{\nu-1}} F_{\nu,\nu}(A(t+\nu-1-\rho(s))^{\overline{\nu}}) f(s) \\ &= \sum_{k=0}^{\infty} \frac{A^k (t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} (I-A) y(0) \\ &\quad + \sum_{k=0}^{\infty} \frac{A^k t^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} *_0 f(t) - (t+1)^{\overline{\nu-1}} F_{\nu,\nu}(A(t+\nu)^{\overline{\nu}}) f(0), \end{aligned}$$

we obtain the equation $\nabla_0^{\nu} y(t) = Ay(t) + f(t)$ by the use of Theorem 4.1, Lemma 3.2 and the fact that

$$\nabla_1^{-(1-\nu)} \left(\sum_{k=0}^{\infty} \frac{A^k t^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)} \right) \Bigg|_{t=0} = 1.$$

5. An asymptotic result. In this section, we shall study the behavior, as $t \to \infty$, of a solution $y(t)$ of the equation $\nabla_0^{1/2} y(t) = ay(t)$ with initial condition $y(0) > 0$. We continue to assume that $|a| < 1$.

Set $\nu = 1/2$ in (2.4) to obtain

$$\mathcal{N}_0 y(t) = \frac{1-a}{(1-s)}\frac{1}{(s^{1/2}-a)}y(0).$$

Applications of the Convolution theorem and Lemma 2.2 (ii) and (iv) imply that

$$\begin{aligned}
&\mathcal{N}_0 y(t)\\
&= \frac{1-a}{(1-s)}\frac{s^{1/2}+a}{(s-a^2)}y(0)\\
&= (1-a)y(0)\left\{\frac{s^{1/2}}{(s-a^2)(1-s)} + \frac{a}{(1-s)(s-a^2)}\right\}\\
&= (1-a)y(0)\left\{\mathcal{N}_1\frac{t^{\overline{-3/2}}}{\Gamma(-1/2)} * \mathcal{N}_0\left(\frac{1}{1-a^2}\right)^{t+1} + a\mathcal{N}_0\left(\frac{1}{1-a^2}\right)^{t+1}\right\}\\
&= (1-a)y(0)\left\{\mathcal{N}_0\left(\frac{t^{\overline{-3/2}}}{\Gamma(-1/2)} *_0 \left(\frac{1}{1-a^2}\right)^{t+1}\right) + a\mathcal{N}_0\left(\frac{1}{1-a^2}\right)^{t+1}\right\}.
\end{aligned}$$

Applying the inverse transform to each side of the above expression, we have

$$y(t) = (1-a)y(0)\left\{\frac{t^{\overline{-3/2}}}{\Gamma(-1/2)} *_0 \left(\frac{1}{1-a^2}\right)^{t+1} + a\left(\frac{1}{1-a^2}\right)^{t+1}\right\}.$$

It follows from the definition of the convolution product that

$$\begin{aligned}
y(t) &= (1-a)y(0)\left\{a\left(\frac{1}{1-a^2}\right)^{t+1}\right.\\
&\qquad\left. + \frac{1}{\Gamma(-0.5)}\sum_{s=0}^{t}\left(\frac{1}{1-a^2}\right)^{s+1}\frac{\Gamma(t-s-0.5)}{\Gamma(t-s+1)}\right\}\\
&= (1-a)y(0)\left(\frac{1}{1-a^2}\right)^{t+1}\\
&\quad\times\left\{a + \frac{1}{\Gamma(-0.5)}\sum_{s=0}^{t}(1-a^2)^s\frac{\Gamma(s-0.5)}{\Gamma(s+1)}\right\}.
\end{aligned} \tag{5.1}$$

Remark 5.1. (i) $y(t)$ is the solution of the initial value problem for the first order finite difference equation,

$$\nabla x(t) = a^2 x(t) + (1-a)y(0)\frac{1}{\Gamma(-0.5)}(t+1)^{\overline{-3/2}}, \quad x(0) = y(0).$$

(ii) By the uniqueness of the solution of the half order initial value problem, we have

$$\left(\frac{1}{1-a^2}\right)^{t+1}\left\{a + \frac{1}{\Gamma(-0.5)}\sum_{s=0}^{t}(1-a^2)^s\frac{\Gamma(s-0.5)}{\Gamma(s+1)}\right\} = \sum_{k=0}^{\infty}\frac{a^k(t+1)^{\overline{(k+1)(1/2)-1}}}{\Gamma((k+1)(1/2))}.$$

Lemma 5.1. *Let $t > 0$ be an integer. $f(x) = (t+1)^{\overline{x-1}}/\Gamma(x)$ is an increasing function on the interval $[1,\infty)$.*

Proof. Let $x, y \in [1,\infty)$ be such that $x \geq y$. Then we have

$$\begin{aligned} f(x) &= \frac{(t+1)^{\overline{x-1}}}{\Gamma(x)} = \frac{\Gamma(t+x)}{t\Gamma(x)\Gamma(t)} = \frac{1}{t\int_0^1 u^{t-1}(1-u)^{x-1}du} \\ &\geq \frac{1}{t\int_0^1 u^{t-1}(1-u)^{y-1}du} \\ &= \frac{\Gamma(t+y)}{t\Gamma(y)\Gamma(t)} = \frac{(t+1)^{\overline{y-1}}}{\Gamma(y)} = f(y). \qquad \square \end{aligned}$$

Lemma 5.2.

$$\lim_{t\to\infty}\left(a + \frac{1}{\Gamma(-0.5)}\sum_{s=0}^{t}(1-a^2)^s\frac{\Gamma(s-0.5)}{\Gamma(s+1)}\right) = 2a > 0. \tag{5.2}$$

Proof. Note that

$${}_2F_1(-0.5,1,1,1-a^2) = \lim_{t\to\infty}\frac{1}{\Gamma(-0.5)}\sum_{s=0}^{t}(1-a^2)^s\,\frac{\Gamma(s-0.5)}{\Gamma(s+1)}$$

and ${}_2F_1(-0.5,1,1,1-a^2)$ converges since $1-a^2<1$.

However, we write

$$a+\frac{1}{\Gamma(-0.5)}\sum_{s=0}^{t}(1-a^2)^s\frac{\Gamma(s-0.5)}{\Gamma(s+1)})$$

$$=a+1+\frac{1}{\Gamma(-0.5)}\sum_{s=1}^{t}(1-a^2)^s\frac{\Gamma(s-0.5)}{\Gamma(s+1)}$$

$$=a+1+\frac{1-a^2}{\Gamma(-0.5)}\sum_{s=0}^{t-1}(1-a^2)^s\frac{\Gamma(s+0.5)}{\Gamma(s+2)}$$

$$=a+1+\frac{1-a^2}{2\Gamma(0.5)}\sum_{s=0}^{t-1}(1-a^2)^s\frac{\Gamma(s+0.5)}{\Gamma(s+2)}$$

and consider

$${}_2F_1(0.5,1,2,1-a^2)=\lim_{t\to\infty}\frac{1}{\Gamma(0.5)}\sum_{s=0}^{t-1}(1-a^2)^s\frac{\Gamma(s+0.5)}{\Gamma(s+2)}.$$

Apply the identity [**4**] or [**1**, page 556]

$${}_2F_1(b,0.5+b,1+2b,z)=2^{2b}(1+\sqrt{(1-z)})^{-2b}$$

to see that

$${}_2F_1(0.5,1,2,1-a^2)=\frac{2}{1+a}.$$

Then

$$\lim_{t\to\infty}\left(a+\frac{1}{\Gamma(-0.5)}\sum_{s=0}^{t}(1-a^2)^s\frac{\Gamma(s-0.5)}{\Gamma(s+1)}\right)$$

$$=\lim_{t\to\infty}\left(a+1-\frac{1-a^2}{2\Gamma(0.5)}\sum_{s=0}^{t-1}(1-a^2)^s\frac{\Gamma(s+0.5)}{\Gamma(s+2)}\right)$$

$$=a+1-\frac{1-a^2}{2}\frac{2}{a+1}=2a. \qquad \square$$

Theorem 5.1. *Let* $0.5\le\nu\le 1$, $0<a<1$. *Then the solution of* $\nabla_0^\nu y(t)=ay(t)$, $y(0)>0$ *diverges to infinity as* $t\to\infty$.

Proof. It follows from Lemma 5.1 that

$$\begin{aligned}(1-a)y(0)\sum_{k=2}^{\infty}\frac{a^k(t+1)^{\overline{(k+1)(1/2)-1}}}{\Gamma((k+1)(1/2))}&\\ &\le (1-a)y(0)\sum_{k=2}^{\infty}\frac{a^k(t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)}\\ &\le (1-a)y(0)\sum_{k=0}^{\infty}\frac{a^k(t+1)^{\overline{(k+1)\nu-1}}}{\Gamma((k+1)\nu)}.\end{aligned}$$

Hence it is sufficient to show that the solution of $\nabla_0^{1/2}y(t) = ay(t)$ diverges to infinity as $t\to\infty$.

Apply (5.1) and (5.2) to see that

$$\begin{aligned}\lim_{t\to\infty} y(t) = (1-a)y(0)\lim_{t\to\infty}\left(\frac{1}{1-a^2}\right)^{t+1}&\\ \times\left\{a+1-\frac{(1-a^2)}{2}\,{}_2F_1(1,0.5,2,1-a^2)\right\} = \infty. \qquad \square\end{aligned}$$

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