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ON ORTHOGONAL CHORDS IN NORMED PLANES

JAVIER ALONSO, HORST MARTINI AND ZOKHRAB MUSTAFAEV

ABSTRACT. It is known that a convex plate of diameter 1 in the Euclidean plane is of constant width 1 if and only if any two perpendicular intersecting chords have total length at least 1. We show that, in general, this result cannot be extended to normed (or Minkowski) planes when the type of orthogonality is defined in the sense of Birkhoff. Inspired by this, we present also further results on intersecting chords in normed planes that are orthogonal in the sense of Birkhoff and in the sense of James.

1. Introduction. A convex body in Euclidean space $\mathbf{R}^d$, $d \geq 2$, is called *of constant width* if the distance between any two parallel supporting hyperplanes is constant. There is a large variety of noncircular and, for $d \geq 3$, nonspherical convex bodies of constant width (see, e.g., the surveys [**6, 8**]). The most famous one is the *Reuleaux triangle* in the Euclidean plane. It is representable as the intersection of three circles of radius $r > 0$ which are centered at the vertices of an equilateral triangle of side-length r.

The notion of convex body of constant width is naturally extended to normed linear (or Minkowski) spaces, and so one can also define Minkowskian analogues of Reuleaux triangles (see [**6, 12, 13**] and [**14**, subsection 4.2]).

Makai and Martini [**11**] proved that in the Euclidean plane a convex body of diameter 1 is of constant width 1 if and only if any two perpendicular intersecting chords of it have total length greater than or equal to 1. Soltan has posed the question of characterizing the Minkowski geometries for which the analogue of this result holds (see [**11, 13**]). In this case, instead of perpendicularity, we consider orthogonality in the sense of Birkhoff.

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Our purpose is to show that, in general, the result from [**11**] cannot be extended to Minkowski planes. More precisely, we will construct unit circles of Minkowski planes for which already this result does not hold. Furthermore, we will prove new results on orthogonal chords of unit circles of Minkowski planes, when their orthogonality is defined in the sense of Birkhoff and James; cf. [**5, 9, 10**].

2. Definitions. A generalized orthogonality in a real normed linear space X is a binary relation that coincides with the usual orthogonality if the norm is induced by an inner product. In this paper we shall deal with Birkhoff and isosceles (or James) orthogonalities.

Given $x, y \in X$, x is said to be *Birkhoff orthogonal* to y ($x \perp_B y$) if $\|x+\lambda y\| \geq \|x\|$ for every $\lambda \in \mathbf{R}$ (i.e., if the line $x+\lambda y$, $\lambda \in \mathbf{R}$, supports the ball of center 0 and radius $\|x\|$ at x); and x is said to be *isosceles orthogonal* to y ($x \perp_I y$) if $\|x+y\| = \|x-y\|$.

A generalized orthogonality $\perp$ is said to be *symmetric* if $x \perp y$ implies $y \perp x$, and it is said to be *homogeneous* if $x \perp y$ implies $\alpha x \perp \beta y$ for every $\alpha, \beta \in \mathbf{R}$. Birkhoff orthogonality is always homogeneous but, in general, not symmetric; and isosceles orthogonality is always symmetric, but homogeneous only in inner product spaces. More about these orthogonalities can be found in the papers [**1–5, 9, 10**].

From now on, $X = (\mathbf{R}^2, \|\cdot\|)$ will denote a real two-dimensional normed linear space (i.e., a Minkowski plane) with unit sphere $S_X = \{x \in X : \|x\| = 1\}$ and unit ball $B_X = \{x \in X : \|x\| \leq 1\}$. In view of our examples below we recall (although this is obvious) that B_X is a convex body (i.e., a compact, convex set with nonempty interior) of constant width 2.

For $u, v \in S_X$, $u \neq v$, we denote by $[u, v]$ the *chord* of S_X with endpoints u and v. We say that $[u, v]$ is *maximal* if, for every $\lambda > 0$, $\|v + \lambda(v-u)\| > 1$ and $\|u + \lambda(u-v)\| > 1$. Thus, we can say that S_X is *strictly convex* if all the chords with endpoints in S_X are maximal.

We should emphasize that in the formulation of the question of Soltan orthogonal chords stand for maximal orthogonal chords. As we will see, the maximality of the chords plays a crucial role.

3. Counterexamples. Examples 1 and 2 show that if we consider Birkhoff orthogonality then the result of Makai and Martini cannot be

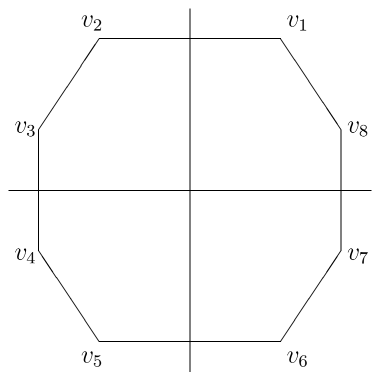

FIGURE 1.

extended to Minkowski planes. In fact, we will present unit circles of Minkowski planes that have two maximal and intersecting orthogonal chords with total length less than 2. For that purpose it is sufficient to consider maximal chords with a common endpoint.

Example 3 shows that there are unit circles all of whose maximal and intersecting chords that are Birkhoff orthogonal have total length greater than or equal to 2 and that this bound can be attained by non-degenerate maximal chords.

In Example 4 all the non-degenerate maximal and intersecting chords that are Birkhoff orthogonal have total length greater than 2, showing that this property is not characteristic for Euclidean circles.

Example 1. Let $1/2 < a < 1$ and consider the norm whose unit circle is the octagon with vertices $v_1 = (a, 1)$, $v_2 = (-a, 1)$, $v_3 = (-1, 2 - 1/a)$, $v_4 = (-1, 1/a - 2)$ and $v_i = -v_{i-4}$, $i = 5, \dots, 8$ (see Figure 1). Then the chord $[v_2, v_3]$ is parallel to the diagonal $[v_1, v_5]$, which implies that the chord $[v_2, v_3]$ is Birkhoff orthogonal to the chord $[v_1, v_2]$. A simple calculation shows that $\|v_1 - v_2\| = 2a$ and $\|v_2 - v_3\| = 1/a - 1$. Thus the sum of the lengths of these two maximal chords is less than 2.

One could also round off the sides v_2v_3, v_4v_5, v_6v_7, and v_8v_1 in such a way that the unit circle would be smooth, not strictly convex, and

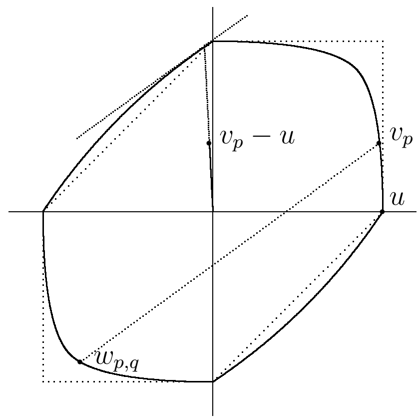

FIGURE 2.

the maximal chord $[v_2, v_3]$ would still be Birkhoff orthogonal to the maximal chord $[v_1, v_2]$.

Example 2. Let X be the Minkowski plane $\mathbf{R}^2$ endowed with the $l_\infty - l_1$ norm defined by

$$\|x\| = \begin{cases} \max\{|x_1|, |x_2|\} & \text{if } x_1x_2 \geq 0, \\ |x_1| + |x_2| & \text{if } x_1x_2 \leq 0, \end{cases}$$

where $x = (x_1, x_2)$. That is, the unit circle is the hexagon with vertices $\pm(1,0)$, $\pm(1,1)$, $\pm(0,1)$. Let $u = (1,0)$, $v = (1,\alpha)$, $w = (-\alpha,-1)$ with $0 < \alpha < 1/2$. Then $u, v, w \in S_X$, $\|u - v\| = \alpha$, $\|v - w\| = 1 + \alpha$, and $v - u \perp_B v - w$, but

$$\|u - v\| + \|v - w\| = 1 + 2\alpha < 2.$$

However, $[u, v]$ is not a maximal chord. To avoid this, consider (following the same idea) $\mathbf{R}^2$ endowed with the $l_p - l_q$ norm

$$\|x\|_{p,q} = \begin{cases} \|x\|_p & \text{if } x_1x_2 \geq 0, \\ \|x\|_q & \text{if } x_1x_2 \leq 0, \end{cases}$$

with $1 < q < p < +\infty$, q being "small" and p being "large" (see Figure 2). Then we get a unit circle similar to the hexagon above but strictly convex. Let $0 < \alpha < 1/2$, $u = (1,0)$, and $v_p = ((1-\alpha^p)^{1/p}, \alpha)$. Then $\|u\|_{p,q} = \|v_p\|_{p,q} = 1$ and

$$\|v_p - u\|_{p,q} = \|v_p - u\|_q < \|v_p - u\|_1 = 1 - (1-\alpha^p)^{1/p} + \alpha \underset{p \to +\infty}{\longrightarrow} \alpha.$$

Let $w_{p,q}$ be the unit vector that satisfies $v_p - u \perp_B v_p - w_{p,q}$. Then $w_{p,q} \underset{q \to 1}{\longrightarrow} w_{p,1} = (-\alpha, -(1-\alpha^p)^{1/p})$ and

$$\|v_p - w_{p,q}\|_{p,q} = \|v_p - w_{p,q}\|_p \underset{q \to 1}{\longrightarrow} \|v_p - w_{p,1}\|_p = 2^{1/p}\big(\alpha + (1-\alpha^p)^{1/p}\big).$$

Therefore,

$$\begin{aligned}\|v_p - u\|_{p,q} + \|v_p - w_{p,q}\|_{p,q} &< 1 - (1-\alpha^p)^{1/p} + \|v_p - w_{p,q}\|_p \\ &\underset{(p,q) \to (+\infty,1)}{\longrightarrow} 2\alpha + 1 < 2,\end{aligned}$$

which implies that we can find p and q such that $\|v_p - u\|_{p,q} + \|v_p - w_{p,q}\|_{p,q} < 2$. Moreover, if we take p and q such that $1/p + 1/q = 1$, then $\|\cdot\|_{p,q}$ is a Radon norm, i.e., a norm for which Birkhoff orthogonality is symmetric; see [**7**].

Example 3. Let X be the Minkowski plane $\mathbf{R}^2$ endowed with the maximum norm. Then S_X is the square of vertices $c_1 = (1,-1)$, $c_2 = (1,1)$, $c_3 = (-1,1)$ and $c_4 = (-1,-1)$. Assume that $u, v, w \in S_X$ are different points such that $[u,v]$ and $[v,w]$ are maximal chords and $u - v \perp_B v - w$. We will show that $\|v-u\| + \|v-w\| \geq 2$. For this purpose we can assume, without loss of generality, that $u \in [c_1, c_2]$ and $v \in [c_2, c_3]$. Since $u - v \perp_B v - w$, we have that $w \in [c_3, c_4] \cup [c_4, c_1]$. Let $u = (1, \alpha)$, $-1 \leq \alpha \leq 1$ and $v = (\beta, 1)$, $-1 \leq \beta \leq 1$. If $w \in [c_3, c_4]$, then $w = (-1, \gamma)$ with $-1 \leq \gamma \leq 1$ and so $\|v-u\| + \|v-w\| = \max\{|\beta - 1|, |1-\alpha|\} + \max\{|\beta+1|, |1-\gamma|\} \geq |\beta - 1| + |\beta + 1| = 2$. On the other hand, if $w \in [c_4, c_1]$ then $\|v-u\| + \|v-w\| = \|v-u\| + 2 > 2$. Finally, to see that the bound 2 can be attained by non-degenerate chords, we can consider $u = (1,0)$, $v = (0,1)$ and $w = (-1,\gamma)$ with $0 \leq \gamma \leq 1$.

Example 4. Let X be the Minkowski plane $\mathbf{R}^2$ endowed with the norm whose unit circle S_X is the regular octagon with vertices $u_1 = (1,0)$, $u_2 = (1/\sqrt{2}, 1/\sqrt{2})$, $u_3 = (0,1)$, $u_4 = (-1/\sqrt{2}, 1/\sqrt{2})$ and $u_i = -u_{i-4}$, $i = 5, \dots, 8$. Assume that $u, v, w \in S_X$ are different points such that $[u,v]$ and $[v,w]$ are maximal chords and $v-u \perp_B v-w$. We will show that $\|v-u\| + \|v-w\| > 2$. For this purpose we need to consider several cases according to the position of u, v, and w in S_X. For the sake of brevity we will reproduce only three representative cases. The others follow in a similar way.

Case 1. Assume that $u = u_1$ and $v \in [u_2, u_3)$. Then $v = \lambda u_2 + (1-\lambda)u_3$ with $0 < \lambda \leq 1$, and $v - u = \rho(\mu u_3 + (1-\mu)u_4)$ with $0 < \mu \leq \frac{1}{2}$ and $\rho = \lambda(\sqrt{2}-2) + \sqrt{2}$, which implies that $\|v-u\| = \rho$. Since $v-u \perp_B v-w$, the vector $v-w$ must be parallel to the chord $[u_3, u_4]$, which implies that $w = \lambda u_5 + (1-\lambda)u_4$ and then $v - w = \overline{\rho}(\frac{1}{2}u_1 + \frac{1}{2}u_2)$, with $\overline{\rho} = 2\big(\sqrt{2} - 1 + \lambda(2-\sqrt{2})\big)$. Thus we get $\|v-u\| + \|v-w\| = \rho + \overline{\rho} > 2$.

Case 2. Assume that $u = u_1$ and $v = u_3$. Then $v - u = \sqrt{2}\, u_4$. Since $[v,w]$ is maximal and $v-u \perp_B v-w$, we have that $w \in [u_4, u_5] \cup [u_5, u_6]$. Let $0 \leq \mu \leq 1$. If $w = \mu u_4 + (1-\mu)u_5$, then $v - w = \rho(\gamma u_1 + (1-\gamma)u_2)$ with $0 \leq \gamma \leq \frac{1}{2}$ and $\rho = \mu(\sqrt{2}-2) + \sqrt{2}$. On the other hand, if $w = \mu u_5 + (1-\mu)u_6$, then $v - w = \rho(\gamma u_2 + (1-\gamma)u_3)$ with $\frac{1}{2} \leq \gamma \leq 1$ and $\rho = \mu(\sqrt{2}-2)+2$. Thus, in both cases $\|v-u\| + \|v-w\| = \sqrt{2} + \rho > 2$.

Case 3. Assume that $u = \mu u_1 + (1-\mu)u_2$, $0 < \mu < 1$ and $v = \lambda u_2 + (1-\lambda)u_3$, $0 \leq \lambda < 1$. Then,

$$v - u = \left(\frac{\lambda+\mu-1}{\sqrt{2}} - \mu\right)u_1 + \left(\frac{\lambda+\mu-1}{\sqrt{2}} + 1 - \lambda\right)u_3.$$

If $\lambda + \mu > 1$, then $v - u = \rho(\alpha u_3 + (1-\alpha)u_4)$ with $0 < \alpha < \frac{1}{2}$ and $\rho = (\sqrt{2}-2)(\lambda-1) + 2(\sqrt{2}-1)\mu$. Moreover, $w = \lambda u_5 + (1-\lambda)u_4$ and $v - w = \overline{\rho}(\frac{1}{2}u_1 + \frac{1}{2}u_2)$ with $\overline{\rho} = 2(\sqrt{2}-1) + 2\lambda(2-\sqrt{2})$. Then, $\|v-u\| + \|v-w\| = \rho + \overline{\rho} = \sqrt{2} + (2-\sqrt{2})\lambda + 2(\sqrt{2}-1)\mu > \sqrt{2} + (2-\sqrt{2})(1-\mu) + 2(\sqrt{2}-1)\mu = 2 + \mu(3\sqrt{2}-4) > 2$. If $\lambda + \mu = 1$, then $\|v-u\| = \mu\sqrt{2}$. In this case w can vary from $w_1 = \lambda u_5 + (1-\lambda)u_4$ to $w_2 = \lambda u_7 + (1-\lambda)u_6$. Therefore $\|v-w\| \geq \|v-w_1\|$ and, as in the above case, $\|v-w_1\| = \overline{\rho}$. Thus $\|v-u\| + \|v-w\| \geq \mu\sqrt{2} + \overline{\rho} = 2 + \mu(3\sqrt{2}-4) > 2$. Finally, if $\lambda + \mu < 1$, the line $\langle v-u \rangle$ cuts $[u_4, u_5]$, which implies that $w = \lambda u_7 + (1-\lambda)u_6$ and, in consequence, $\|v-u\| + \|v-w\| = \|v-u\| + 2 > 2$.

4. Further results. Examples 1 and 2 show that there are unit circles where we can find two intersecting maximal chords that are Birkhoff orthogonal whose total length is less than 2. Our next theorem shows that, nevertheless, one of the two chords always has length greater than or equal to 1. Its proof gives more information than that contained in the enunciation.

Theorem 5. *Let u, v and w be three different points in S_X such that $v-u \perp_B v-w$. If $\|v-u\| \le 1$ and $[v,w]$ is maximal, then $\|v-w\| \ge 1$.*

Proof. Since $0 < \|v-u\| \le 1$, we can assume that u and v are linearly independent. Let $\alpha, \beta \in \mathbf{R}$ be such that $w = \alpha u + \beta v$. We can assume that $\alpha \ne 0$, otherwise $w = -v$ and $\|v-w\| = 2$. The proof will be divided into two steps. In the first step we show that $\alpha + \beta \le 0$, and in the second that $\|v - w\| \ge 1$.

Step 1. Assume that $\alpha + \beta > 0$. We shall get a contradiction to the hypothesis. For that aim we need to consider several cases according to the signs of α and β.

1.1. Assume first that $\beta \ge 0$. Since $v - u \perp_B v - w$, we have

$$\begin{aligned}|\alpha|\|v-u\| &= \|(\alpha+\beta)v - w\| \ge \big||\alpha+\beta| - 1\big| \\ &= |\alpha+\beta-1| \\ &= \|\alpha(v-u) + w - v\| \ge |\alpha|\|v-u\|.\end{aligned}$$

Therefore,

$$|\alpha|\|v-u\| = |\alpha+\beta-1|. \tag{1}$$

Now, if $\alpha > 0$, then $1 = \|w\| \le \alpha + \beta$, and from (1) we get

$$\begin{aligned}\left\|v + \frac{1}{\alpha\|v-u\|}(v-w)\right\| &= \frac{\|\alpha\|v-u\|v + v - w\|}{\alpha\|v-u\|} \\ &= \frac{\|(\alpha+\beta-1)v + v - w\|}{\alpha\|v-u\|} \\ &= 1,\end{aligned}$$

which contradicts the maximality of the chord $[v, w]$. On the other hand, if $\alpha < 0$, then $1 = \|w\| \geq |\beta| - |\alpha| = \alpha + \beta$, and from (1) it follows

$$\begin{aligned}
&\left\| w + \left(\frac{\alpha+\beta}{-\alpha\|v-u\|} \right)(w-v) \right\| \\
&\qquad = \frac{\big\| -\alpha\|v-u\|w + (\alpha+\beta)(w-v) \big\|}{-\alpha\|v-u\|} \\
&\qquad = \frac{\|(1-\alpha-\beta)w + (\alpha+\beta)(w-v)\|}{-\alpha\|v-u\|} \\
&\qquad = 1,
\end{aligned}$$

again contradicting the maximality of $[v, w]$.

1.2. Assume that $\beta \leq 0$. Then $\alpha > 0$, since we are assuming $\alpha+\beta > 0$. Since $1 = \|w\| \geq |\alpha| - |\beta| = \alpha + \beta$ and $v - u \perp_B v - w$, we get

$$\begin{aligned}
(1-\beta)\|v-u\| &\leq \|(1-\beta)(v-u) + w - v\| \\
&= \|(\alpha+\beta-1)u\| \qquad (2)\\
&= 1 - \alpha - \beta.
\end{aligned}$$

Moreover,

$$1 = \|w\| = \|\beta(v-u) + (\alpha+\beta)u\| \leq -\beta\|v-u\| + \alpha + \beta. \qquad (3)$$

Adding the inequalities (2) and (3), we get $\|v-u\| \leq 0$, which is absurd. The same absurdity would follow by assuming $\alpha + \beta = 0$.

Step 2. From Step 1 we know that $\alpha+\beta \leq 0$. Again we shall consider several cases according to the signs of α and β.

2.1. Assume that $\alpha < 0$ and $\beta \geq 1$. Then

$$\|w - v\| = \|\alpha u + (\beta-1)v\| \geq |\alpha| - |\beta - 1| = 1 - \alpha - \beta \geq 1.$$

Moreover, if $\|w - v\| = 1$, then $w = (v-u)/\|v-u\|$ and $[v, w] \subset S_X$. If, in addition, $\|v - u\| < 1$, then also $[v - w, v] \subset S_X$.

2.2. Assume that $\alpha < 0$ and $0 < \beta < 1$. Then

$$1 = \|w\| = \|\beta(v-u) + (\alpha+\beta)u\| \leq \beta\|v-u\| - \alpha - \beta,$$

and therefore

$$\begin{aligned}\|w-v\| &= \frac{\|\alpha u+(\beta-1)w\|}{\beta} \geq \frac{|\alpha|-|\beta-1|}{\beta} \\ &= \frac{\beta-\alpha-1}{\beta} \geq \frac{\beta+\beta(1-\|v-u\|)}{\beta} \\ &= 2-\|v-u\| \geq 1.\end{aligned}$$

Moreover, if $\|w-v\|=1$, then $\|v-u\|=1$ and $\alpha=-1$, which implies that $[u-v,u]\subset S_X$.

2.3. Assume that $\alpha<0$ and $\beta\leq 0$. Then

$$\begin{aligned}\|w-v\| &= \|(\alpha+\beta-1)v+\alpha(u-v)\| \\ &\geq |\alpha+\beta-1|-|\alpha|\|u-v\| \\ &= 1-\alpha-\beta+\alpha\|u-v\| \\ &= 1-\beta-\alpha(1-\|u-v\|) \geq 1.\end{aligned}$$

Moreover, $\|w-v\|=1$ only if $\beta=0$ and $\|u-v\|=1$, which implies that $w=-u$, and the unit sphere S_X is the square with vertices $\pm u$, $\pm v$.

2.4. Assume that $\alpha>0$ and $\beta\leq 0$. Then

$$\|w-v\|=\|(\beta-1)v+\alpha u\| \geq |\beta-1|-|\alpha| = 1-\alpha-\beta>1,$$

since (recall case 1.2) $\alpha+\beta<0$. □

The next theorem shows that any triangle inscribed to S_X with two sides that are isosceles orthogonal has the third side limited by opposite points. Thus, we can say that "isosceles rectangular triangles" inscribed to circles in a Minkowski plane behave like rectangular triangles inscribed to Euclidean circles. Examples 1 and 2 showed that the same is not true with "Birkhoff rectangular triangles."

Theorem 6. *Let u, v and w be three different points in S_X such that $[v,u]$ and $[v,w]$ are maximal. If $v-u\perp_I v-w$, then $w=-u$.*

Proof. Throughout the proof we must have in mind that, due to $v-u\perp_I v-w$, then

$$\|w-u\|=\|2v-u-w\|. \tag{4}$$

Moreover, $2\|v-u\| = \|2v-u-w-(u-w)\| \leq 2\|w-u\|$, and then

$$\|v-u\| \leq \|w-u\|. \tag{5}$$

If u and v were linearly dependent, then necessarily $v=-u$. Then, by (4), $2 \geq \|w+v\| = \|3v-w\| \geq 2$, which implies that $\|v+\frac{1}{2}(v-w)\| = 1$, contradicting the maximality of the chord $[v,w]$. Thus we can assume, without loss of generality, that u and v are linearly independent. Let $\alpha, \beta \in \mathbf{R}$ be such that $w = \alpha u + \beta v$. If $\beta = 0$, then $w = -u$. Moreover, if $\alpha = 0$, then $w = -v$, and from (4) we have that $2 \geq \|u+v\| = \|3v-u\| \geq 2$, contradicting the maximality of $[v,u]$. Next we will show that the other possibilities also contradict the hypothesis.

Case 1. Assume that $\alpha > 0$ and $\beta > 0$. Then $0 = \|w\|-1 \leq \alpha+\beta-1$. If $\alpha \leq 1$, then from (5) we have that

$$\begin{aligned} \|v-u\| \leq \|w-u\| &= \|(1-\alpha)(v-u)+(\alpha+\beta-1)v\| \\ &\leq (1-\alpha)\|v-u\| + \alpha+\beta-1, \end{aligned}$$

and we get

$$\alpha\|v-u\| \leq \alpha+\beta-1. \tag{6}$$

On the other hand, if $\alpha \geq 1$, then, again from (5), we have that

$$\alpha\|v-u\| \leq \alpha\|w-u\| = \|(\alpha-1)w+\beta v\| \leq \alpha+\beta-1,$$

and we also obtain (6). Therefore,

$$\left\| v + \frac{1}{\alpha+\beta-1}(v-w) \right\| = \frac{\alpha\|v-u\|}{\alpha+\beta-1} \leq 1,$$

which contradicts the maximality of the chord $[v,w]$.

Case 2. Assume that $\alpha < 0$ and $\beta > 0$. Then $1 = \|w\| \geq |\beta| - |\alpha| = \beta + \alpha$, which gives

$$1-\alpha-\beta \geq 0. \tag{7}$$

2.1. Assume that $-1 < \alpha < 0$. Then, from (5) and (7), it follows that

$$\begin{aligned} \|2v-u-w\| &= \|(1+\alpha)(v-u)+(1-\alpha-\beta)v\| \\ &\leq (1+\alpha)\|v-u\| + 1-\alpha-\beta \\ &\leq (1+\alpha)\|w-u\| + 1-\alpha-\beta, \end{aligned}$$

and from (4) we get that $-\alpha\|w-u\| \leq 1-\alpha-\beta$. Then

$$\left\| w + \left(\frac{\beta}{1-\alpha-\beta} \right)(w-v) \right\| = \frac{-\alpha\|w-u\|}{1-\alpha-\beta} \leq 1,$$

contradicting the maximality of the chord $[v, w]$.

2.2. Assume that $\alpha \leq -1$. If $\beta \leq 2$, then

$$\|u-w\| = \|2v-u-w\| = \|-(1+\alpha)u + (2-\beta)v\| \leq 1-\alpha-\beta.$$

On the other hand, if $\beta \geq 2$, then from (7) we get

$$\begin{aligned} \beta\|2v-u-w\| &= \|(\beta-2)(u-w) + 2(1-\alpha-\beta)u\| \\ &\leq (\beta-2)\|u-w\| + 2(1-\alpha-\beta), \end{aligned}$$

and again we obtain that $\|u-w\| \leq 1-\alpha-\beta$. Thus,

$$\left\| u + \frac{\beta}{1-\alpha-\beta}(u-v) \right\| = \frac{\|u-w\|}{1-\alpha-\beta} \leq 1,$$

contradicting the maximality of the chord $[u, v]$.

Case 3. Assume that $\alpha < 0$ and $\beta < 0$. Then

$$2 \geq \|u+w-2v\| = \|(\beta-2)v + (1+\alpha)u\| \geq 2-\beta-|1+\alpha| \geq 2,$$

where the last inequality follows because $-\alpha+\beta \leq 1 \leq -\alpha-\beta$. Therefore, $\|u+w-2v\| = 2$. Moreover, if $1+\alpha \geq 0$, then $\alpha+\beta+1 = 0$ and

$$\left\| v - \frac{\beta}{2}(v-u) \right\| = \frac{\|2v-u-w\|}{2} = 1,$$

contradicting the maximality of $[u, v]$. On the contrary, if $1+\alpha \leq 0$, then $1+\alpha-\beta = 0$ and

$$\left\| v + \frac{1+\alpha}{2\alpha}(v-w) \right\| = \frac{\|2v-u-w\|}{2} = 1,$$

contradicting the maximality of $[v, w]$.

Case 4. Assume that $\alpha > 0$ and $\beta < 0$. Then $1-\alpha-\beta = \|w\| - |\alpha| + |\beta| \geq 0$ and

$$\begin{aligned}(2-\beta)\|w-u\| &= \|\beta(2v-u-w) - 2(1-\alpha-\beta)u\| \\ &\leq -\beta\|2v-u-w\| + 2(1-\alpha-\beta).\end{aligned}$$

Thus we get from (4) that $\|2v-u-w\| \leq 1-\alpha-\beta$, which implies that

$$\left\| v + \frac{1+\alpha}{1-\alpha-\beta}(v-u) \right\| = \frac{\|2v-u-w\|}{1-\alpha-\beta} \leq 1,$$

contradicting the maximality of $[u, v]$. □

Remarks. 1) Theorem 6 is not true if the chords are not maximal: consider, in Example 3, $u = (1, -1)$, $v = (1, 1)$ and $w = (0, 1)$.

2) If $u, v \in S_X$, then $\|v-u\| + \|v+u\| \geq 2$. Therefore, from Theorem 6 it follows that any two maximal chords of S_X that are isosceles orthogonal, with a common endpoint, have total length greater than or equal to 2. The bound 2 can be attained by non-degenerate chords: consider, in Example 3, $u = (1, 0)$ and $v = (0, 1)$.

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Department of Mathematics, University of Extremadura, 06006 Badajoz, Spain
Email address: jalonso@unex.es

Faculty of Mathematics, University of Technology Chemnitz, 09107 Chemnitz, Germany
Email address: martini@mathematik.tu-chemnitz.de

Department of Mathematics, University of Houston-Clear Lake, Houston, TX 77058
Email address: mustafaev@uhcl.edu