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SEMIDEFINITENESS WITHOUT HERMITICITY

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ABSTRACT. Let $A \in M_n(\mathbf{C})$. We provide a rank characterization of the semidefiniteness of Hermitian A in two ways. We show that A is semidefinite if and only if $\operatorname{rank}[X^*AX] = \operatorname{rank}[AX]$, for all $X \in M_n(\mathbf{C})$, and that A is semidefinite if and only if $\operatorname{rank}[X^*AX] = \operatorname{rank}[AXX^*]$, for all $X \in M_n(\mathbf{C})$. We show that, if A has semidefinite Hermitian part and A^2 has positive semidefinite Hermitian part, then A satisfies row and column inclusion. Let $B \in M_n(\mathbf{C})$, and let k be an integer with $k \geq 2$. If $B^*BA, B^*BA^2, \dots, B^*BA^k$ each has positive semidefinite Hermitian part; we show that $\operatorname{rank}[BAX] = \operatorname{rank}[X^*B^*BAX] = \cdots = \operatorname{rank}[X^*B^*BA^{k-1}X]$, for all $X \in M_n(\mathbf{C})$. These results generalize or strengthen facts about real matrices known earlier.

1. Introduction. In [**6**], a number of results about real matrices, not necessarily symmetric, with semidefinite real quadratic form were given. In some cases these results generalize to complex matrices with semidefinite Hermitian part, and, in some, they do not. Here, we sort out what happens in the complex case, and in some instances give new or stronger results. If the proofs in the complex case extend naturally from the real case, by merely changing "transpose" to "transpose complex conjugate," we skip the proof and only refer to [**6**]. However, we have allowed some overlap of material, for the purpose of clarity.

Let $A \in M_n(\mathbf{C})$. The Hermitian part of A is defined in [**3**] by $H(A) = (A + A^*)/2$. A matrix $A \in M_n(\mathbf{C})$ is called positive semidefinite if it is Hermitian ($A^* = A$) and $x^*Ax \geq 0$ for all $x \in \mathbf{C}^n$. $A \in M_n(\mathbf{C})$ is said to have positive semidefinite Hermitian part if $H(A)$ is positive semidefinite. We say $A \in M_n(\mathbf{C})$ is semidefinite if either A or $-A$ is positive semidefinite.

The principal submatrix of A lying in rows and columns $\alpha \subseteq \{1, \dots, n\}$ will be denoted by $A[\alpha]$, and the submatrix lying in rows α

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and columns β will be denoted $A[\alpha, \beta]$. A matrix $A \in M_n(\mathbf{C})$ satisfies row (column) inclusion if $A[\{i\}, \alpha]$ lies in the row space of $A[\alpha]$ for each $i = 1, \ldots, n$ (if $A[\alpha, \{j\}]$ lies in the column space of $A[\alpha]$ for each $j = 1, \ldots, n$) and each $\alpha \subseteq \{1, \ldots, n\}$.

It is well known that if $A \in M_n(\mathbf{C})$ is semidefinite then A satisfies row and column inclusion. This follows easily as a corollary of Theorem 1's rank characterization of the semidefiniteness of $A \in M_n(\mathbf{C})$.

Theorem 1. *Let $A \in M_n(\mathbf{C})$ be Hermitian. Then the following are equivalent*:

(a) *A is semidefinite*;

(b) $\operatorname{rank}[X^*AX] = \operatorname{rank}[AX]$, *for all $X \in M_n(\mathbf{C})$*;

(c) *$x^*Ax = 0$ implies $Ax = 0$, for $x \in \mathbf{C}^n$.*

Proof. Straightforward.

Corollary 2. *Let $A \in M_n(\mathbf{C})$ be Hermitian. If A is semidefinite, then A satisfies row and column inclusion.*

Proof. Take $X \in M_n(\mathbf{C})$ diagonal with ones and zeros on the diagonal in $\operatorname{rank}[X^*AX] = \operatorname{rank}[AX] = \operatorname{rank}[X^*A]$. □

Theorem 1 and Lemma 3 imply another rank characterization, in Theorem 4, of the semidefiniteness of $A \in M_n(\mathbf{C})$.

Lemma 3. *Let $A \in M_n(\mathbf{C})$. Then* $\ker[X^*XA] = \ker[XA]$, *for all $X \in M_n(\mathbf{C})$.*

Proof. For $u \in \mathbf{C}^n$, $X^*XAu = 0$ implies $0 = u^*A^*X^*XAu = (XAu)^*(XAu)$, so $XAu = 0$. □

Theorem 4. *Let $A \in M_n(\mathbf{C})$ be Hermitian. Then A is semidefinite if and only if we have* $\operatorname{rank}[X^*AX] = \operatorname{rank}[AXX^*]$, *for all $X \in M_n(\mathbf{C})$.*

Proof. Starring the terms in square brackets in the statement of Lemma 3 we have $\operatorname{rank}[AX^*X] = \operatorname{rank}[AX^*]$, for all $X \in M_n(\mathbf{C})$, since $A = A^*$. With $Y = X^*$ we have $\operatorname{rank}[Y^*AY] = \operatorname{rank}[AYY^*]$, for all $Y \in M_n(\mathbf{C})$, if and only if A is semidefinite. □

We return to the issue of finding sufficient conditions for not necessarily Hermitian $A \in M_n(\mathbf{C})$ to satisfy row and column inclusion. It is routine to check that the (a) ⇒ (b) and (b) ⇒ (c) parts of the proof of Theorem 4 in [**6**] extend naturally from the real to the complex case to give Theorem 5 below, although statement (b) says more than in [**6**].

Theorem 5. it Let $A \in M_n(\mathbf{C})$. Consider the following statements:

(a) *A has semidefinite Hermitian part and A^2 has positive semidefinite Hermitian part*;

(b) $\operatorname{rank}[X^*H(A)X] = \operatorname{rank}[H(A)X] = \operatorname{rank}[AX] = \operatorname{rank}[X^*AX]$, *for all* $X \in M_n(\mathbf{C})$;

(c) $x^*Ax = 0$ *implies* $Ax = 0$, *for* $x \in \mathbf{C}^n$.

Then (a) *implies* (b), *and* (b) *implies* (c).

Proof. Assuming (a) to prove the equalities in (b), let $u \in \ker[H(A)X]$. Then $H(A)Xu = 0$, so $(A + A^*)Xu = 0$, and $AXu = -A^*Xu$. Then $A^*AXu = -(A^*)^2Xu$, and therefore $(AXu)^*(AXu) = u^*X^*A^*AXu = -u^*X^*(A^*)^2Xu$, which implies $AXu = 0$. Thus, $\ker[H(A)X] \subseteq \ker[AX]$, so $\operatorname{rank}[H(A)X] \geq \operatorname{rank}[AX] \geq \operatorname{rank}[X^*AX]$.

$u \in \ker[X^*AX]$ implies $X^*AXu = 0$, so $u^*X^*AXu = 0$ and $u^*X^*A^*Xu = 0$. Adding these two equations we have $u^*X^*(A + A^*)Xu = 0$, so $H(A)Xu = 0$. Thus, $\ker[X^*AX] \subseteq \ker[H(A)X]$, and so $\operatorname{rank}[X^*AX] \geq \operatorname{rank}[H(A)X]$.

Combining the inequalities of the last two paragraphs we have that $\operatorname{rank}[H(A)X] = \operatorname{rank}[AX] = \operatorname{rank}[X^*AX]$. Since $H(A)$ is positive semidefinite we have $\operatorname{rank}[X^*H(A)X] = \operatorname{rank}[H(A)X]$, from Theorem 1. □

An example of a matrix that satisfies (c), but does not imply either of the two hypotheses of (a) in Theorem 5, is

$$A = \begin{pmatrix} 1+2i & 0 & 0 \\ 0 & -1+2i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Our next theorem gives us a better understanding of the rank statements of Theorem 1 and Theorem 5. For $A \in M_n(\mathbf{C})$, $F(A)$ denotes the classical field of values [**4**] defined by $F(A) = \{x^*Ax \mid x \in \mathbf{C}, x^*x = 1\}$.

Theorem 6. *For $A \in M_n(\mathbf{C})$, the following statements are equivalent*:

(i) $x^*Ax = 0$ *implies* $Ax = 0$, *for* $x \in \mathbf{C}^n$;

(ii) *there is a unitary* $V \in M_n(\mathbf{C})$ *so that*

$$V^*AV = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$$

with $0 \notin F(A_1)$, *where* $A_1 \in M_k(\mathbf{C})$ *and* $k \leq n$;

(iii) $\operatorname{rank}[AX] = \operatorname{rank}[X^*AX]$, *for all* $X \in M_n(\mathbf{C})$.

Proof. Suppose (i). There is a unitary $V \in M_n(\mathbf{C})$ which upper triangularizes A so that

$$V^*AV = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},$$

where $A_1 \in M_k(\mathbf{C})$ is nonsingular and $A_3 \in M_{n-k}(\mathbf{C})$ has all eigenvalues 0, with $k \leq n$. Taking $x = Ve_i$ we have $e_i^*V^*AVe_i = 0$ which implies $AVe_i = 0$ and $V^*AVe_i = 0$, for $k+1 \leq i \leq n$, so

$$V^*AV = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

If $0 \in F(A_1)$, then for some $x \in \mathbf{C}^k$, $x \neq 0$, we have $0 = x^*A_1x = y^*Ay$, where $y = V\,(\,x\ 0\,)^* \in \mathbf{C}^n$. But then $Ay = 0$ and so $A_1x = 0$, which is not possible since A_1 is nonsingular, so (ii) holds.

Suppose (ii). If $X^*AXu = 0$, then

$$0 = u^*X^*AXu = (V^*Xu)^* \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} V^*Xu.$$

If we write $V^*Xu = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$, we have $v_1^* A_1 v_1 = 0$, which implies $v_1 = 0$. Writing

$$V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix},$$

gives

$$AXu = V \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} V^*Xu = \begin{pmatrix} V_1A_1 & 0 \\ V_3A_1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = 0,$$

so that $u \in \ker[AX]$. Then $\ker[AX] = \ker[X^*AX]$ implies (iii). □

Corollary 7. *If $A \in M_n(\mathbf{C})$ satisfies* (i), (ii) *or* (iii) *in Theorem* 6, *then A satisfies row and column inclusion.*

Corollary 8. *If $A \in M_n(\mathbf{C})$ has semidefinite Hermitian part and A^2 has positive semidefinite Hermitian part, then* (i), (ii), (iii) *of Theorem* 6 *hold.*

Proof. Follows from Theorem 5. □

When $A = \left(\begin{smallmatrix} 1+2i & -1 \\ -1 & 1-2i \end{smallmatrix}\right)$, we have A with semidefinite Hermitian part, A^2 has negative semidefinite Hermitian part, but we do not have statement (i) when $x = (\begin{smallmatrix} 1 & 1 \end{smallmatrix})^*$.

We can extend Theorem 5 even further as follows.

Theorem 9. *Let $A, B \in M_n(\mathbf{C})$. Given the following statements*:

(a) *B^*BA has semidefinite Hermitian part and B^*BA^2 has positive semidefinite Hermitian part*;

(b) $\operatorname{rank}[BAX] = \operatorname{rank}[X^*B^*BAX]$, *for all $X \in M_n(\mathbf{C})$*;

(c) *$x^*B^*BAx = 0$ implies $BAx = 0$, for $x \in \mathbf{C}^n$*;

then (a) *implies* (b), *and* (b) *implies* (c).

Proof. Similar to the proof of Theorem 5.

Theorem 12 uses Lemmas 10 and 11, which improve on the corresponding lemma in [**6**].

Lemma 10. *Let $A, B \in M_n(\mathbf{C})$ with B^*BA having semidefinite Hermitian part. Then* $\ker[BA] = \ker[BA^2]$.

Proof. $BA^2x = 0$ implies $x^*A^*B^*BAAx = 0$, $x^*A^*A^*B^*BAx = 0$, and $x^*A^*(B^*BA + A^*B^*B)Ax = 0$. But then, since $B^*BA + A^*B^*B = \pm C^*C$, for some $C \in M_n(\mathbf{C})$, so $x^*A^*(C^*C)Ax = 0$, $(CAx)^*(CAx) = 0$, $CAx = 0$, $C^*CAx = 0$, so $(B^*BA + A^*B^*B)Ax = 0$. Rewriting this as $B^*BA^2x + A^*B^*BAx = 0$, and using $BA^2x = 0$, we have $A^*B^*BAx = 0$. So $0 = x^*A^*B^*BAx = (BAx)^*(BAx)$, and $BAx = 0$. This shows that $\ker[BA^2] \subseteq \ker[BA]$. Evidently, $\operatorname{rank}[BA^2] \leq \operatorname{rank}[BA]$, and so $\dim(\ker[BA^2]) \geq \dim(\ker[BA])$. □

Lemma 11. *Let $A, B \in M_n(\mathbf{C})$, and let B^*BA have semidefinite Hermitian part. Then* $\ker[BA] = \ker[BA^m]$, *for any positive integer m.*

Proof. By induction on m. We will show that $\ker[BA^m] \subseteq \ker[BA]$. Now $BA^mx = 0$ implies $x^*(A^*)^{m-1}B^*BAA^{m-1}x = 0$ and $x^*(A^*)^{m-1}A^*B^*BA^{m-1}x = 0$, which imply $x^*(A^*)^{m-1}(B^*BA + A^*B^*B)A^{m-1}x = 0$, so $B^*BAA^{m-1}x + A^*B^*BA^{m-1}x = 0$. Using $BA^mx = 0$, we must have that $A^*B^*BA^{m-1}x = 0$. But then $0 = x^*(A^*)^{m-1}B^*BA^{m-1}x = (BA^{m-1}x)^*(BA^{m-1}x)$, so that $BA^{m-1}x = 0$. From $\ker[BA] = \ker[BA^{m-1}]$, by induction, we conclude that $x \in \ker[BA]$. Finally, $\operatorname{rank}[BA^m] \leq \operatorname{rank}[BA]$ implies $\dim(\ker[BA^m]) \geq \dim(\ker[BA])$. □

Theorem 12 generalizes the (a) $\Rightarrow$ (b) part of Theorem 9, as well as generalizing Theorem 7 in [**6**].

Theorem 12. *Let $A, B \in M_n(\mathbf{C})$, and let k be an integer with $k \geq 2$. If B^*BA, $B^*BA^2, \ldots, B^*BA^k$ each have positive semidefinite Hermitian part, then*

$$\operatorname{rank}[BAX] = \operatorname{rank}[X^*B^*BAX] = \cdots = \operatorname{rank}[X^*B^*BA^{k-1}X],$$

for all $X \in M_n(\mathbf{C})$.

Proof. By induction on k. Assume the result is true for $k-1$, in other words that $\operatorname{rank}[BAX] = \operatorname{rank}[X^*B^*BAX] = \cdots = \operatorname{rank}[X^*B^*BA^{k-2}X]$, for all $X \in M_n(\mathbf{C})$. Let $u \in \ker[X^*B^*BA^{k-1}X]$, so $X^*B^*BA^{k-1}Xu = 0$. Then we have $u^*X^*B^*BA^{k-1}Xu = 0$, but since $B^*BA^{k-1} + (A^{k-1})^*B^*B = C^*C$, we also have $u^*X^*C^*CXu = 0$, which implies $CXu = 0$, so $0 = C^*CXu = (B^*BA^{k-1} + (A^{k-1})^*B^*B)Xu$. Now $B^*BA^{k-1}Xu = -(A^*)^{k-1}B^*BXu$ gives us that $A^*B^*BA^{k-1}Xu = -(A^*)^kB^*BXu$, but then we have that $u^*X^*A^*B^*BA^{k-2}AXu = -u^*X^*(A^*)^kB^*BXu$. That is, $(AXu)^*B^* \times BA^{k-2}(AXu) = -u^*X^*B^*BA^kXu$. Since B^*BA^{k-2} and B^*BA^k have positive semidefinite Hermitian part, we must have $(AXu)^*B^*BA^{k-2} \times (AXu) = 0$, so by induction $BA(AXu) = 0$. This means that $Xu \in \ker(BA^2)$, and from Lemma 10 this implies that $Xu \in \ker(BA)$, so $BAXu = 0$. We have just shown that $\ker[X^*B^*BA^{k-1}X] \subseteq \ker[BAX]$. Suppose now that $BAXu = 0$; then $BA^{k-1}Xu = 0$ from Lemma 11, so $X^*B^*BA^{k-1}Xu = 0$, and $\ker[BAX] \subseteq \ker[X^*B^*BA^{k-1}X]$. □

The bibliography in [**6**] has further references to results about matrices with positive semidefinite Hermitian part. See also [**1, 4, 5**].

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