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MULTIPLIER HOPF ALGEBRAS IMBEDDED IN LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. Let (A, Δ) be a locally compact quantum group and (A_0, Δ_0) a regular multiplier Hopf algebra. We show that if (A_0, Δ_0) can in some sense be imbedded in (A, Δ), then A_0 will inherit some of the analytic structure of A. Under certain conditions on the imbedding, we will be able to conclude that (A_0, Δ_0) is actually an algebraic quantum group with a full analytic structure. The techniques used to show this can be applied to obtain the analytic structure of a *-algebraic quantum group *in a purely algebraic fashion*. Moreover, the *reason* that this analytic structure exists at all is that one-parameter groups, such as the modular group and the scaling group, are diagonalizable. In particular, we will show that necessarily the scaling constant μ of a *-algebraic quantum group equals 1. This solves an open problem posed in [**13**].

1. Introduction. In [**20**], the second author introduced *multiplier Hopf algebras*, generalizing the notion of a Hopf algebra to the case where the underlying algebra is not necessarily unital. In [**21**], he considered those multiplier Hopf algebras that have a nonzero left invariant functional. It turned out that these objects, termed *algebraic quantum groups*, possess a rich structure, allowing for example a duality theory. These objects seemed to form an algebraic model of locally compact quantum groups, which at the time had no generally accepted definition.

In [**13**], Kustermans showed that a *-algebraic quantum group (which is an algebraic quantum group with a well-behaving *-structure) naturally gives rise to a *C^*-algebraic quantum group*, which was a proposed definition for a locally compact quantum group by Masuda, Nakagami and Woronowicz [**16**]. Kustermans showed, however, that there was one discrepancy with the proposed definition, in that the invariance of the scaling group with respect to the left Haar weight was only relative.

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These investigations culminated in the, by now acknowledged, definition of a *locally compact quantum group* by Kustermans and Vaes, as laid down in [**11**]. This definition was (up to the relative invariance of the scaling group) equivalent with the one proposed by Woronowicz, Masuda and Nakagami, but the set of axioms was smaller and simpler. These axioms were very much inspired by those of *-algebraic quantum groups, but introducing analysis made it much harder to show that they were sufficiently powerful to carry a theory of locally compact quantum groups with the desired properties.

In this article, we examine a converse of the problem studied in [**9, 13**]. Namely, instead of starting with a *-algebraic quantum group and imbedding it into a locally compact quantum group, we start with an imbedding of a general regular multiplier Hopf algebra in a locally compact quantum group and look at whether the multiplier Hopf algebra inherits some structural properties.

The study of this problem led us to an enhanced structure theory for *-algebraic quantum groups. For example, the analytic structure of these objects is a consequence of the fact that all the actions at hand are diagonalizable. This has as a nice corollary that the scaling constant of a *-algebraic quantum group is necessarily 1. It is odd that *-algebraic quantum groups, which provided a motivation for allowing relative invariance of the scaling group under the Haar weight, turn out to have proper invariance after all.

The paper is organized as follows. In the first part we introduce definitions of the objects at play and introduce notations. We will not always use the definitions as given in the fundamental papers, but use equivalent ones which are better suited for our purposes.

In the second part we investigate the following problem: if a multiplier Hopf algebra A_0 can be imbedded in a locally compact quantum group, does this give us information about the multiplier Hopf algebra? Firstly, we must specify what we mean by *imbedded in*: A_0 has to be a subalgebra of the locally compact quantum group, and the respective comultiplications Δ_0 and Δ have to satisfy formulas of the form $\Delta_0(a)(1 \otimes b) = \Delta(a)(1 \otimes b)$ for a and b in A_0. Secondly, we must specify whether we imbed A_0 in the von Neumann algebra M or in the C*-algebra A associated to the locally compact quantum group. Already in the first situation, the objects of A_0 will behave nicely with respect to analyticity of the various one-parameter groups. But only in the second case can we conclude, under a mild extra condition, that A_0 is invariant under these one-parameter groups. Moreover, A_0 will then automatically have the structure of an algebraic quantum group.

In the third part we apply the techniques of the previous section to obtain structural properties of *-algebraic quantum groups. We want to stress that this section is entirely of an algebraic nature. For example, we prove in a purely algebraic fashion the existence of a positive right invariant functional on the *-algebraic quantum group. Up to now, some involved analysis was necessary to arrive at this.

In the fourth part we consider some special cases. We also look at a concrete example, namely the discrete quantum group $U_q(su(2))$.

Some of the motivation for this paper comes from [**14**], where similar questions are investigated in the commutative and co-commutative case. For example, it is shown that the function space $C_0(G)$ of a locally compact group contains a dense multiplier Hopf *-algebra if and only if G contains a compact open subgroup. The multiplier Hopf *-algebra will be the space spanned by translates of regular (equivalently, polynomial) functions on this compact group.

1. Preliminaries. In this article we use the concepts of a regular multiplier Hopf (*-)algebra, a (*-)algebraic quantum group, a (reduced) C*-algebraic quantum group and a von Neumann-algebraic quantum group, as introduced respectively in [**11, 12, 20, 21**] (see also [**24**]). Since these objects stem from quite different backgrounds, we will give an overview of their definitions and properties. As mentioned in the introduction, we take those forms of the definitions which are most suited for our purpose.

1.1. Regular multiplier Hopf (*-)algebras. We recall the notion of the *multiplier algebra of an algebra*. Let A be a nondegenerate algebra (over the field $\mathbf{C}$), with or without a unit. The nondegeneracy condition means that, if $ab = 0$ for all $b \in A$, or $ba = 0$ for all $b \in A$, then $a = 0$. As a set, the multiplier algebra $M(A)$ of A consists of couples (λ, ρ), where λ and ρ are linear maps $A \to A$ obeying the following law:

$$a\lambda(b) = \rho(a)b, \quad \text{for all } a, b \in A.$$

In practice we write m for (λ, ρ), and denote $\lambda(a)$ by $m \cdot a$ or ma, and $\rho(a)$ by $a \cdot m$ or am. Then the above law becomes an associativity condition. Now $M(A)$ is an algebra, called the *multiplier algebra* of A, by the composition $(\lambda, \rho) \cdot (\lambda', \rho') = (\lambda \circ \lambda', \rho' \circ \rho)$. Moreover, if A is a *-algebra, $M(A)$ also carries a *-operation: for $m \in M(A)$ and $a \in A$, we define m^* by $m^* \cdot a = (a^* \cdot m)^*$ and $a \cdot m^* = (m \cdot a^*)^*$. Note that when A is a C*-algebra, this definition coincides with the usual definition of the multiplier algebra.

There is a natural map $A \to M(A)$, letting an element a correspond with left and right multiplication by it. Because of nondegeneracy, this (*-)algebra morphism will be injective. In this way nondegeneracy compensates the possible lack of a unit. Note that, when A is unital, $M(A)$ is equal to A.

Let B be another nondegenerate (*-)algebra. In our paper, a *morphism* between A and B is a nondegenerate (*-)algebra homomorphism $f : A \to M(B)$. The nondegeneracy of a map f means that $f(A)B = B$ and $Bf(A) = B$, where $f(A) \subseteq B = \{\sum_{i=1}^{n} f(a_i)b_i \mid a_i \in A, b_i \in B\}$ (and likewise for $Bf(A)$). If f is a morphism from A to B, then f can be extended to a unital (*-)algebra morphism from $M(A)$ to $M(B)$. A *proper morphism* between A and B is a morphism f such that $f(A) \subseteq B$. Note that when A and B are C*-algebras, a *-algebra homomorphism $f : A \to M(B)$ will be nondegenerate in this sense if and only if it is nondegenerate in the ordinary sense, i.e., $f(A)B$ and $Bf(A)$ are *dense* in B. This follow, for example, by an application of the Cohen-Hewitt factorization theorem, see e.g., [**16**, Theorem A.1].

We can now state the definition of a regular multiplier Hopf (*-)algebra ([**20**]). It is the appropriate generalization of a Hopf (*-)algebra to the case where the underlying algebra need not be unital. A *regular multiplier Hopf* (*-)*algebra* consists of a couple (A, Δ), with A a nondegenerate (*-)algebra and Δ, the *comultiplication*, a morphism (in the above sense) from A to $A \otimes A$, where $\otimes$ denotes the algebraic tensor product. Moreover, (A, Δ) has to satisfy the following conditions:

M.1. $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ (coassociativity).

M.2. The maps

$$T_{\Delta 2} : A \otimes A \longrightarrow M(A \otimes A) : a \otimes b \longrightarrow \Delta(a)(1 \otimes b),$$

$$T_{1\Delta} : A \otimes A \longrightarrow M(A \otimes A) : a \otimes b \longrightarrow (a \otimes 1)\Delta(b),$$
$$T_{\Delta 1} : A \otimes A \longrightarrow M(A \otimes A) : a \otimes b \longrightarrow \Delta(a)(b \otimes 1),$$
$$T_{2\Delta} : A \otimes A \longrightarrow M(A \otimes A) : a \otimes b \longrightarrow (1 \otimes a)\Delta(b)$$

all induce linear bijections $A \otimes A \to A \otimes A$.

Here, and elsewhere in the text, we will use ι to denote the identity map. We remark that we use the regularity of the maps $(\Delta \otimes \iota)$ and $(\iota \otimes \Delta)$ to make sense of M.1.

The T-maps can be used to define a co-unit $\varepsilon : A \to \mathbf{C}$ and an antipode $S : A \to A$, determined by the formulas

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab,$$
$$S(a)b = (\varepsilon \otimes \iota)(T_{\Delta 2}^{-1}(a \otimes b)),$$

with $a, b \in A$. The co-unit will be a (*-)morphism, while S will be a bijective proper anti-morphism $A \to A$, satisfying $S(S(a)^*)^* = a$ for all $a \in A$ when A is a multiplier Hopf *-algebra.

We will need an identity concerning the antipode (cf. [**20**, Lemma 5.5]): if a, b in A satisfy $a \otimes b = \sum_{i=1}^{n} (p_i \otimes 1)\Delta(q_i)$ for certain $p_i, q_i \in A$, then $\Delta(a)(1 \otimes S(b)) = \sum \Delta(p_i)(q_i \otimes 1)$. We can give a quick motivation for this result if we look at the special case where A is a Hopf algebra. Using Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$, we have that if $a \otimes b = \sum_{i=1}^{n} p_i q_{i(1)} \otimes q_{i(2)}$, then

$$\begin{aligned}
\Delta(a)(1 \otimes S(b)) &= a_{(1)} \otimes a_{(2)} S(b) \\
&= \sum (p_i q_{i(1)})_{(1)} \otimes (p_i q_{i(1)})_{(2)} S(q_{i(2)}) \\
&= \sum p_{i(1)} q_{i(1)} \otimes p_{i(2)} q_{i(2)} S(q_{i(3)}) \\
&= \sum p_{i(1)} q_i \otimes p_{i(2)} \\
&= \sum \Delta(p_i)(q_i \otimes 1).
\end{aligned}$$

It is also interesting to note that if A carries a multiplier Hopf algebra structure, it must necessarily satisfy some nice algebraic properties. Namely, local units will exist in the following sense: for any finite collection of elements $a_i \in A$, elements e, f will exist in A such that $ea_i = a_i$ and $a_i f = a_i$, for all i (see [**4**]).

1.2. (*-)Algebraic quantum groups. Our second object forms an intermediate step between the former, purely algebraic notion of a regular multiplier Hopf algebra and the analytic set-up of a locally compact quantum group. An *algebraic quantum group* [**21**] is a regular multiplier Hopf algebra (A, Δ) for which there exists a nonzero left invariant functional φ. This means that φ satisfies

$$(\iota \otimes \varphi) \circ \Delta = \varphi.$$

This should be interpreted as follows: the lefthand side makes sense as a map $A \to M(A)$, by sending $a \in A$ to the multiplier $m = (\iota \otimes \varphi)\Delta(a)$, determined by $mb = (\iota \otimes \varphi)(\Delta(a)(b \otimes 1))$ and $bm = (\iota \otimes \varphi)((b \otimes 1)\Delta(a))$ for $b \in A$. Also the righthand side can be seen as a map $A \to M(A)$, namely, the map sending $a \in A$ to $\varphi(a)1$. Then the assumption is that in fact both these maps are the same.

An **-algebraic quantum group* is a multiplier Hopf *-algebra for which there exists a *positive* nonzero left invariant functional φ, i.e., $\varphi(a^*a) \geq 0$ for all $a \in A$. This extra condition is in fact very restrictive, as we shall see.

We can prove that a nonzero left invariant functional φ is unique up to multiplication with a scalar. It will be faithful in the following sense: if $\varphi(ab) = 0$ for all $b \in A$, or $\varphi(ba) = 0$ for all $b \in A$, then $a = 0$. Then (A, Δ) will also have a nonzero functional ψ, again unique up to a scalar, such that

$$(\psi \otimes \iota) \circ \Delta = \psi.$$

If A is a *-algebraic quantum group, we can still choose ψ to be positive. We note however, that to arrive at this functional, a detour into an analytic landscape (with aid of the GNS-device for φ) seemed inevitable. The problem is that it is not obvious that the evident right invariant functional $\psi = \varphi \circ S$ is positive. To create the right ψ, some analytic machinery was needed, namely, the square root of the modular element, or a polar decomposition of the antipode (see [**13**]). In this paper we show that it is possible to arrive at the positivity of $\psi = \varphi \circ S$ by *purely algebraic means* (see Corollary 3.6). This means also that *-algebraic quantum groups are appropriate objects of study for algebraists with a fear of analysis.

Algebraic quantum groups have some nice features. For example, there exists a unique automorphism σ of the algebra A, satisfying

$\varphi(ab) = \varphi(b\sigma(a))$ for all $a, b \in A$. We call it the *modular automorphism*, a notion we borrow from the theory of weights on von Neumann algebras. Note that in pure algebra, σ is rather called the Nakayama automorphism of φ. A unique multiplier $\delta \in M(A)$ also exists such that

$$(\varphi \otimes \iota)(\Delta(a)(1 \otimes b)) = \varphi(a)\delta b,$$
$$(\varphi \otimes \iota)((1 \otimes b)\Delta(a)) = \varphi(a)b\delta,$$

for all $a, b \in A$. It is called the *modular element*, as it is the noncommutative analogue of the modular function in the theory of locally compact groups, and indeed it can be shown that $\psi = \varphi(\cdot\,\delta)$ for $\psi = \varphi \circ S$. When A is a *-algebraic quantum group, δ will be a positive element in the strong sense, i.e., $\delta = q^*q$ for some $q \in M(A)$.

There also is a particular number that can be associated with an algebraic quantum group. Since $\varphi \circ S^2$ is a left invariant functional, the uniqueness of φ implies that there exists a $\mu \in \mathbf{C}$ such that $\varphi(S^2(a)) = \mu\varphi(a)$, for all $a \in \mathbf{C}$. This number μ is called the *scaling constant* of (A, Δ). In an early stage, examples of algebraic quantum groups were found where $\mu \neq 1$, see [**21**]. However, it remained an open question whether *-algebraic quantum groups existed with $\mu \neq 1$. We will show in this paper that in fact $\mu = 1$ for all *-algebraic quantum groups, see Theorem 3.4.

To any algebraic quantum group (A, Δ), one can associate another algebraic quantum group $(\widehat{A}, \widehat{\Delta})$ which is called its dual. As a set, it consists of functionals on A of the form $\varphi(\cdot\, a)$ with $a \in A$, where φ is the left invariant functional on A. Note that this is the same as the set of functionals of the form $\varphi(a\,\cdot)$ with $a \in A$, or with the set of functionals of the form $\psi(\cdot\, a)$ or $\psi(a\,\cdot)$ with $\psi = \varphi \circ S$: we can use the modular automorphism or the modular element to switch between the different representations of these functionals. The multiplication and comultiplication of $\widehat{A}$ are dual to, respectively, the comultiplication and multiplication on A. Intuitively, this means that

$$\widehat{\Delta}(\omega_1)(a \otimes b) = \omega_1(ab),$$
$$(\omega_1 \cdot \omega_2)(a) = (\omega_1 \otimes \omega_2)(\Delta(a)),$$

for $a, b \in A$ and $\omega_1, \omega_2 \in \widehat{A}$, but some care is needed in giving sense to these formulas.

The counit on $\widehat{A}$ is defined by evaluation in 1, while the antipode is the dual of the antipode of A: if $\widehat{S}$ denotes the antipode of $\widehat{A}$, then

$$\widehat{S}(\omega_1)(a) = \omega_1(S(a)),$$

for $\omega_1 \in \widehat{A}$ and $a \in A$. A left invariant functional $\widehat{\varphi}$ of $\widehat{A}$ is determined by $\widehat{\varphi}(\psi(a\,\cdot)) = \varepsilon(a)$, while a right invariant functional is determined by $\widehat{\psi}(\varphi(\cdot\, a)) = \varepsilon(a)$, with $a \in A$.

If A is a *-algebraic quantum group, then also $\widehat{A}$ will be *-algebraic. The *-structure on $\widehat{A}$ is given by $\omega_1^*(x) = \overline{\omega_1(S(x)^*)}$, where $\omega_1 \in \widehat{A}$ and $x \in A$. More concretely, we have $\varphi(\cdot\, a)^* = \psi(\cdot\, S(a)^*)$ for $a \in A$, where $\psi = \varphi \circ S$. Then it can be computed that $\widehat{\psi}(\omega_1^* \cdot \omega_1) = \varphi(a^*a)$ if $\omega_1 = \varphi(\cdot\, a)$ with $a \in A$. So $\widehat{\psi}$ is positive, and by a small adaptation of the results mentioned, also $\widehat{\varphi} = \widehat{\psi} \circ \widehat{S}^{-1}$ will be positive.

1.3. Von Neumann-algebraic quantum groups. We now enter the analytic arena and pose the definition of a *von Neumann algebraic quantum group* [**12, 24**]. A von Neumann algebraic quantum group consists of a quadruple $(M, \Delta, \varphi, \psi)$ (which we mostly denote by just (M, Δ)), with M a von Neumann algebra, Δ a normal unital *-homomorphism from M to $M \otimes M$ which satisfies the coassociativity condition

$$(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta,$$

and φ and ψ normal, semi-finite faithful (nsf) weights which satisfy

$$(\iota \otimes \varphi) \circ \Delta = \varphi,$$
$$(\psi \otimes \iota) \circ \Delta = \psi.$$

Here $\otimes$ denotes the ordinary von Neumann algebraic tensor product, and then the maps $(\Delta \otimes \iota)$ and $(\iota \otimes \Delta)$ are well-defined maps from $M \otimes M$ to $M \otimes M \otimes M$. The identities concerning the weights should be interpreted as follows: for any $\omega \in M_*^+$, the weight $\psi \circ (\iota \otimes \omega)\Delta$ should equal the weight $\omega(1)\psi$, and similarly for φ. Note that these are the *strong forms* of invariance and that they in fact follow from weaker ones, see [**12**, Proposition 3.1]. It can be shown also that here the weights φ and ψ are unique up to multiplication with a positive scalar.

The most important objects associated with (M, Δ) are the *multiplicative unitaries*, which essentially carry all information about (M, Δ). To introduce them, we first recall the notion of the GNS-representation associated to the nsf weight ψ. Denote by $\mathscr{N}_\psi$ the left ideal $\{x \in M \mid \psi(x^* x) < \infty\}$ of square-integrable elements. The GNS-space associated to ψ is the closure $\mathscr{H}_\psi$ of the pre-Hilbert space $\mathscr{N}_\psi$, with scalar product defined by $\langle a, b\rangle = \psi(b^* a)$ for $a, b \in \mathscr{N}_\psi$. The injection of $\mathscr{N}_\psi$ into $\mathscr{H}_\psi$ will be denoted by Λ_ψ. We can construct a faithful normal representation of M on $\mathscr{H}_\psi$ via left multiplication. We can do the same for φ, obtaining a Hilbert space $\mathscr{H}_\varphi$ and an injection $\Lambda_\varphi : \mathscr{N}_\varphi \to \mathscr{H}_\varphi$. Moreover, there exists a unitary from $\mathscr{H}_\psi$ to $\mathscr{H}_\varphi$, intertwining the representations of M. We will identify both Hilbert spaces by this unitary and denote it simply as $\mathscr{H}$. We will let elements of M act directly on $\mathscr{H}$ as operators (suppressing the representation). Now we can define the multiplicative unitary W, also called the *left regular representation*: it is the unitary operator on $\mathscr{H} \otimes \mathscr{H}$, characterized by

$$(\omega \otimes \iota)(W^*)\Lambda_\varphi(x) = \Lambda_\varphi((\omega \otimes \iota)\Delta(x)), \quad x \in \mathscr{N}_\varphi \text{ and } \omega \in B(\mathscr{H})_*.$$

We want to remark that it is not too difficult to show that the W^* defined by this equation is an isometry, but that proving surjectivity of W^* requires some subtle but beautiful arguments. The unitary W will implement comultiplication as follows:

$$\Delta(x) = W^*(1 \otimes x)W \quad \text{for all } x \in M.$$

Moreover, the σ-weak closure of the set $\{(\iota \otimes \omega)(W) \mid \omega \in B(\mathscr{H})_*\}$ will be equal to M.

We can also define the multiplicative unitary V, called the *right regular representation*: it is the unitary operator on $\mathscr{H} \otimes \mathscr{H}$, determined by

$$(\iota \otimes \omega)(V)\Lambda_\psi(x) = \Lambda_\psi((\iota \otimes \omega)\Delta(x)), \quad x \in \mathscr{N}_\psi \text{ and } \omega \in B(\mathscr{H})_*.$$

It also implements the comultiplication:

$$\Delta(x) = V(x \otimes 1)V^* \quad \text{for all } x \in M.$$

Again, the σ-weak closure of the set $\{(\omega \otimes \iota)(V) \mid \omega \in B(\mathscr{H})_*\}$ will be equal to M.

The regular representations can be used to define the *antipode* S on (M,Δ). It is the (possibly unbounded) σ-weakly closed linear map S from M to M, with a core consisting of elements of the form $(\omega\otimes\iota)(V)$, $\omega\in B(\mathscr{H})_*$, such that

$$S((\omega\otimes\iota)(V))=(\omega\otimes\iota)(V^*).$$

Then also $(\iota\otimes\omega)(W)\in\mathscr{D}(S)$ for all $\omega\in B(\mathscr{H})_*$, and

$$S((\iota\otimes\omega)(W))=(\iota\otimes\omega)(W^*).$$

This map S has a *polar decomposition*, consisting of a (point-wise) σ-weakly continuous one-parameter group (τ_t) of automorphisms of M (called the *scaling group*) and a *-anti-automorphism R of M (called the *unitary antipode*). Then the antipode equals the map $R\circ\tau_{-i/2}$, where $\tau_{-i/2}$ is the analytic continuation of (τ_t) to the point $-i/2$. We recall here that for $z\in\mathbf{C}$, τ_z is the (possibly unbounded) map $M\to M$ which has as its domain the elements $x\in M$ such that for any $\omega\in M_*$ the function $f_\omega^x: t\to\omega(\tau_t(x))$ can be extended to a bounded continuous function on the closed strip $\{w\in\mathbf{C}\mid \operatorname{Im}(w)$ lies between 0 and $\operatorname{Im}(z)\}$, analytic in the interior of this strip, and that then $\tau_z(x)$ is the element of M determined by the functional $\omega\to f_\omega^x(z)$ on M_*. An element x is called *analytic* for (τ_t) if it is in the domain of all τ_z.

Now by noncommutative integration theory (see, e.g., [**17**]), we can associate to φ and ψ two other σ-weakly continuous one-parameter groups of automorphisms, denoted respectively by (σ_t) and (σ_t'). They are called the *modular one-parameter groups* (associated with φ and ψ). Then $(\sigma_t),(\sigma_t')$ and (τ_t) will all commute with each other. It can also be shown that there exists a (possibly unbounded) nonsingular positive operator δ affiliated with M, called the *modular element* of (M,Δ), such that $\sigma_t'(x)=\delta^{it}\sigma_t(x)\delta^{-it}$ for $x\in M$. As we shall see further on, δ^{it} is a cocycle for (σ_t) up to a scalar one-parameter group, and then ψ can be seen as a cocycle perturbation of φ by δ. We can denote this intuitively as $\psi=\varphi(\delta^{1/2}\cdot\delta^{1/2})$. This δ will be invariant under the scaling group (τ_t) and satisfy $R(\delta)=\delta^{-1}$. In the sequel, we will also frequently use the commutation rules between these objects and Δ: we have $\Delta(\delta^{it})=\delta^{it}\otimes\delta^{it}$, and further,

$$\Delta\tau_t=(\tau_t\otimes\tau_t)\Delta \qquad \Delta\sigma_t=(\tau_t\otimes\sigma_t)\Delta$$
$$\Delta\tau_t=(\sigma_t\otimes\sigma_{-t}')\Delta \qquad \Delta\sigma_t'=(\sigma_t'\otimes\tau_{-t})\Delta.$$

Now, just as for algebraic quantum groups, we can associate a certain number to (M,Δ). Namely, there exists a $\nu \in \mathbf{R}^+$ such that $\sigma_t(\delta) = \nu^t\delta$ (and hence $\nu^{it^2/2}\delta^{it}$ is the proper cocycle for σ_t). This constant ν is called the *scaling constant.* It was an important question whether there exist locally compact quantum groups where this constant is not trivially 1. Such quantum groups do indeed exist: an interesting example is the quantum $az+b$-group, see [**23**].

With the aid of a multiplicative unitary, it is also possible to construct a dual von Neumann-algebraic quantum group $(\widehat{M},\hat{\Delta})$. Namely, it can be shown that the σ-weak closure of the *second leg* of W, by which we mean the set $\{(\omega\otimes\iota)(W) \mid \omega \in B(\mathscr{H})_*\}$, is a von Neumann-algebra $\widehat{M}$. It has a natural comultiplication by defining $\hat{\Delta}(x) = \Sigma W(x\otimes 1)W^*\Sigma$ for $x \in \widehat{M}$, where Σ denotes the flip map (taking $\xi\otimes\eta$ to $\eta\otimes\xi$). Also a left and right invariant nsf weight can be constructed on $\widehat{M}$. For example, the left invariant nsf weight $\widehat{\varphi}$ is determined by the following: if $a \in \mathscr{N}_\varphi$ happens to be such that $\varphi(\cdot a)$ is bounded on $\mathscr{N}_\varphi^*$, then denoting its closure by ω_a and $(\omega_a\otimes\iota)(W)$ by $\widehat{a}$, we have that $\widehat{a} \in \mathscr{N}_{\widehat{\varphi}}$ and $\widehat{\varphi}(\widehat{a}^*\widehat{a}) = \varphi(a^*a)$. In particular, we can again make our GNS-construction for $\widehat{\varphi}$ in the old Hilbert space $\mathscr{H}$, by identifying $\Lambda_{\widehat{\varphi}}(\widehat{a})$ with $\Lambda_\varphi(a)$. Then the left regular representation of $\widehat{M}$ has $\widehat{W} = \Sigma W^*\Sigma$ as its multiplicative unitary.

Note that we have essentially shown already that $W \in M\otimes\widehat{M}$. It can also be shown that $V \in \widehat{M}'\otimes M$, where $\widehat{M}'$ is the commutant of $\widehat{M}$ on $\mathscr{H}$.

1.4. C*-algebraic quantum groups. A (*reduced*) *C*-algebraic quantum group* is the noncommutative version of the space of continuous complex functions vanishing at infinity of a locally compact group. While its theory can be developed by itself (with axioms resembling those of a von Neumann-algebraic quantum group), we will only need to know how to obtain a C*-algebraic quantum group from a von Neumann-algebraic quantum group. This is no restriction, as every (reduced) C*-algebraic quantum group turns out to be of this form.

So assume a von Neumann-algebraic quantum group (M,Δ) is given, and let V,W denote the multiplicative unitaries pertaining to respectively the right and left regular representation. Then the *normclosure* of the space $\{(\omega \otimes \iota)(V) \mid \omega \in B(\mathscr{H})\}$ will equal the normclosure of the space $\{(\iota \otimes \omega)(W^*) \mid \omega \in B(\mathscr{H})\}$, and this can be shown to be a C*-algebra A. The comultiplication Δ will restrict to a morphism $A \to A \otimes A$ (in the previously defined sense), where $\otimes$ denotes the minimal tensor product. Furthermore, all the one-parameter groups (τ_t), (σ_t) and (σ'_t) will now restrict to (point-wise) *norm* continuous one-parameter groups of automorphisms on A.

Our last remark concerns the invariant weights. Namely, it can be shown that if $\mathscr{M}_\varphi^+ = \{x \in M^+ \mid \varphi(x) < \infty\}$ denotes the cone of positive φ-integrable elements, then $A \cap \mathscr{M}_\varphi^+$ will be *norm-dense* in A^+, so that in fact φ can also be seen as a semi-finite weight on A. The same applies for ψ.

2. Multiplier Hopf algebras imbedded in locally compact quantum groups. In this section, we fix a von Neumann-algebraic quantum group (M, Δ) and a regular multiplier Hopf algebra (A_0, Δ_0). The C*-algebraic quantum group associated to (M, Δ) will be denoted by (A, Δ). We will use notations as before, but the structural maps for A_0 will be indexed by 0 (whenever this causes no confusion). We also fix a left invariant nsf weight φ on (M, Δ). We can scale a right invariant nsf weight ψ on (M, Δ) so that $\psi = \varphi \circ R$, with R the unitary antipode. We use the notation $\mathscr{N}_\varphi$ for the square integrable elements in M, and $\mathscr{M}_\varphi = \mathscr{N}_\varphi^* \mathscr{N}_\varphi$ for the algebra of integrable elements.

Assumption 2.1. $A_0 \subseteq M$.

This means that A_0 is a subalgebra of M, not necessarily invariant under the *-involution. We also want to impose a certain compatibility between Δ and Δ_0, but we have to be careful: $M(A_0)$ bears no natural relation to M. For example, denoting by j the inclusion of A_0 in M, the identity $(j \otimes j) \circ \Delta_0 = \Delta \circ j$ can be meaningless if j has no well-defined extension $M(A_0) \to M$. We will therefore assume the following: for all $a, b \in A_0$,

$$\Delta_0(a)(1 \otimes b) = \Delta(a)(1 \otimes b),$$
$$(a \otimes 1)\Delta_0(b) = (a \otimes 1)\Delta(b),$$
$$\Delta_0(a)(b \otimes 1) = \Delta(a)(b \otimes 1),$$
$$(1 \otimes a)\Delta_0(b) = (1 \otimes a)\Delta(b).$$

Remarks. This condition is strictly weaker than the condition $(j \otimes j) \circ \Delta_0 = \Delta \circ j$, when it makes sense. For example, the imbedding of $C(\mathbf{Z})$ in $C(\mathbf{Z})$ sending δ_n to δ_{2n} satisfies the former, but not the latter condition. Moreover, it is possible that unit 1 of M is not in the σ-weak closure of A_0. In fact, it is not difficult to see that if A_0 is a multiplier Hopf *-algebra (with the same *-operation as M), then the projection $p \in M \cap N$ which gives the unit of the σ-weak closure N of A_0 will be a group like projection: $\Delta(p)(1 \otimes p) = p \otimes p$ (see, e.g., [**15**, the first part of the fourth section]). Also, in this case the third and fourth equality will follow from the first two by applying the *-involution. We have not investigated in detail the interdependence of the stated equalities in the general case.

Our first result shows that the antipode S of M restricts to the antipode S_0 of A_0. The difficult part consists of showing that A_0 lies in the domain of S. We will need a lemma which is interesting in its own right. It is a kind of cancelation property involving M and $\widehat{M}'$, the commutant of the dual quantum group $\widehat{M}$.

Lemma 2.1. *Suppose $a \in M$ and $x \in \widehat{M}'$ satisfy $ax = 0$. Then $a = 0$ or $x = 0$.*

Proof. Let W be the left regular representation for M. We recall that $W \in M \otimes \widehat{M}$. So if $ax = 0$, then

$$\begin{aligned} W^*(1 \otimes ax)W &= W^*(1 \otimes a)W(1 \otimes x) \\ &= \Delta(a)(1 \otimes x) \\ &= 0. \end{aligned}$$

Assume $x \neq 0$. Choose $\omega \in B(\mathscr{H})_{*,+}$ such that $\omega(x^*x) = 1$. Then from the foregoing we obtain $(\iota \otimes \omega(x^* \cdot x))\Delta(a^*a) = 0$. Applying ψ and using the strong form of right invariance, we get $\psi(a^*a)\omega(x^*x) = \psi(a^*a) = 0$. Since ψ is faithful, a must be zero. ◻

Remark. By a similar argument, also the following is true: if $a \in M$ and $x \in \widehat{M}$, then $ax = 0$ implies either $a = 0$ or $x = 0$.

We can show now that the antipodes of M and A_0 coincide.

Proposition 2.2. *A_0 lies in the domain of S, and $S_{|A_0}$ will be the antipode of (A_0, Δ_0).*

Proof. Let b be an element of A_0. We will show that $b \in \mathscr{D}(S)$ and $S(b) = S_0(b)$. We start by choosing some fixed a in A_0. We can pick p_i, q_i in A_0 such that

$$a \otimes b = \sum_{i=1}^{n} (p_i \otimes 1)\Delta(q_i).$$

Then we know that this is equivalent with

$$\Delta(a)(1 \otimes S_0(b)) = \sum_{i=1}^{n} \Delta(p_i)(q_i \otimes 1).$$

Let y be $(\omega_{\Lambda_\psi(d),\Lambda_\psi(c)}(a\cdot) \otimes \iota)(V) = (\psi \otimes \iota)((c^* a \otimes 1)\Delta(d))$, where $c, d \in \mathscr{N}_\psi$ and $\omega_{\Lambda_\psi(d),\Lambda_\psi(c)} = \langle \cdot \Lambda_\psi(d), \Lambda_\psi(c)\rangle$. Then

$$\begin{aligned} by &= (\psi \otimes \iota)((c^* a \otimes b)\Delta(d)) \\ &= (\psi \otimes \iota) \sum (c^* p_i \otimes 1)\Delta(q_i d). \end{aligned}$$

We know that this last expression is in $\mathscr{D}(S)$ and that

$$S((\psi \otimes \iota) \sum (c^* p_i \otimes 1)\Delta(q_i d)) = (\psi \otimes \iota) \sum \Delta(c^* p_i)(q_i d \otimes 1).$$

So

$$\begin{aligned} S(by) &= (\psi \otimes \iota) \sum \Delta(c^* p_i)(q_i d \otimes 1) \\ &= (\psi \otimes \iota)(\Delta(c^* a)(d \otimes S_0(b))) \\ &= S(y) S_0(b). \end{aligned}$$

Denote by C the linear span of all such y, with c and d varying. We show that C is a σ-weak core for S. First we remark that functionals of the form $\omega_{\Lambda_\psi(d),\Lambda_\psi(c)}(a\cdot)$ have a norm-dense linear span in $(\widehat{M}')_*$. Indeed, if $z \in \widehat{M}'$ is such that $\langle az\Lambda_\psi(d), \Lambda_\psi(c)\rangle = 0$ for all $c, d \in \mathscr{N}_\psi$, then $az = 0$; hence, $z = 0$ by the previous lemma. Then, since

$V \in \widehat{M}' \otimes M$, there exists for every $\omega \in B(\mathscr{H})_*$ and every $\varepsilon > 0$ a finite number of $c_n, d_n \in \mathscr{N}_\psi$ such that

$$\|\sum_n (\omega_{\Lambda_\psi(d_n),\Lambda_\psi(c_n)}(a\cdot) \otimes \iota)(V) - (\omega \otimes \iota)(V)\| < \varepsilon,$$
$$\|\sum_n (\omega_{\Lambda_\psi(d_n),\Lambda_\psi(c_n)}(a\cdot) \otimes \iota)(V^*) - (\omega \otimes \iota)(V^*)\| < \varepsilon.$$

Since $\{(\omega \otimes \iota)(V) \mid \omega \in B(\mathscr{H})_*\}$ is a σ-weak core for S, the same will be true for C.

By choosing a net y_α in C such that $y_\alpha \to 1$ and $S(y_\alpha) \to 1$ in the σ-weak topology, we can conclude that $b \in \mathscr{D}(S)$ and $S(b) = S_0(b)$, by the closedness of S for the σ-weak topology. □

The previous proposition implies that $A_0 \subseteq \mathscr{D}(\tau_z)$ for every z in $\mathbf{C}$, i.e., every $a \in A_0$ is analytic with respect to (τ_t). Indeed, $a \in \mathscr{D}(S)$ means that $a \in \mathscr{D}(\tau_{-i/2})$. Since $S(S_0^{-1}(a)) = a$ for $a \in A_0$, we also have that $a \in \mathscr{D}(S^{-1}) = \mathscr{D}(\tau_{i/2})$. So $A_0 \subseteq \mathscr{D}(\tau_{ni})$ for every integer $n \in \mathbf{Z}$.

This again illustrates the lack of analytical structure of a general algebraic quantum group: if its antipode S satisfies $S^{2n} = \iota$, but $S^2 \neq \iota$, then it cannot be imbedded in a locally compact quantum group. Such algebraic quantum groups do indeed exist (see, e.g., [**21**]).

We can also use Lemma 2.1 to prove that actually $A_0 \subseteq M(A)$. Fix $a \in A_0$. Choose $b \in A_0$ and $\omega \in B(\mathscr{H})_*$. Then

$$\begin{aligned} a \otimes b &= \sum (q_i \otimes 1)\Delta(p_i) \\ &= \sum (q_i \otimes 1)V(p_i \otimes 1)V^*, \end{aligned}$$

for some p_i, q_i in A_0. Multiplying from the right with V and applying $\omega \otimes \iota$, we get $b(\omega(a\cdot) \otimes \iota)(V) \in A$. But, as we have shown, the set $\{(\omega(a\cdot) \otimes \iota)(V) \mid \omega \in B(\mathscr{H})_*\}$ is norm-dense in A. Hence, $bA \subseteq A$. Similarly $Ab \subseteq A$, and thus $A_0 \subseteq M(A)$.

As a second result, we show that A_0 consists of analytic elements for (σ_t). This follows easily from the following proposition, which elucidates the behavior of A_0 with respect to the one-parameter group (κ_t), with $\kappa_t = \sigma_t \tau_{-t}$. It will be decisive in obtaining some structural properties of *-algebraic quantum groups, as we will show in the third section.

Proposition 2.3. *$A_0 \subseteq \mathscr{D}(\kappa_z)$ for all $z \in \mathbf{C}$, and $\kappa_z(A_0) \subseteq A_0$. Here κ_z denotes the analytic continuation of the one-parameter group (κ_t) to the point $z \in \mathbf{C}$, and $\mathscr{D}(\kappa_z)$ denotes its domain.*

Proof. Let b be a fixed element of A_0. Choose a nonzero $a \in A_0$, and write

$$a \otimes b = \sum_{i=1}^{n} \Delta(p_i)(1 \otimes q_i),$$

with $p_i, q_i \in A_0$. Using the commutation relations between Δ, (τ_t), (σ_t) and (σ'_t), we get that

$$\kappa_{-t}(a) \otimes \rho_t(b) = \sum \Delta(p_i)(1 \otimes \rho_t(q_i)) \text{ for all } t \in \mathbf{R},$$

where $\rho_t = \sigma'_t \tau_t$. Choose $c \in A_0$ such that $cb \neq 0$, and multiply this equation to the left with $1 \otimes c$ to get

$$\kappa_{-t}(a) \otimes c\rho_t(b) = \sum ((1 \otimes c)\Delta(p_i))(1 \otimes \rho_t(q_i)).$$

Choose $a_{ij}, b_{ij} \in A_0$ such that

$$(1 \otimes c)\Delta(p_i) = \sum_{j=1}^{m_i} a_{ij} \otimes b_{ij},$$

and let L be the finite-dimensional space spanned by the a_{ij}. We see that $\kappa_{-t}(a) \otimes c\rho_t(b) \in L \otimes M$, for every $t \in \mathbf{R}$. Since $c\rho_0(b) = cb \neq 0$ and ρ_t is strongly continuous, we get that there exists a $\delta > 0$ such that $c\rho_t(b) \neq 0$ for all t with $|t| < \delta$. This means $\kappa_t(a) \in L$ for all $|t| < \delta$.

For every $\varepsilon > 0$, let $K_\varepsilon = \text{span}\{\kappa_t(a) \mid |t| < \varepsilon\}$ and $n_\varepsilon = \dim(K_\varepsilon)$. For small ε, we have $n_\varepsilon \in \mathbf{N}$. Choose an $\varepsilon > 0$ where this dimension reaches a minimum. Then $K := K_\varepsilon = K_{\varepsilon/2}$ will be a finite-dimensional space containing a, invariant under κ_t for all $t \in \mathbf{R}$.

Now (κ_t) induces a continuous homomorphism $\widetilde{\kappa} : \mathbf{R} \to \mathrm{GL}(K)$. It is then well known that it must necessarily be analytic (see, e.g., [**6**]).

Hence, for any $\omega \in M_*$, the map $t \to \omega(\kappa_t(a))$ is analytic. Thus, $a \in \mathscr{D}(\kappa_z)$ for any $z \in \mathbf{C}$, and $\kappa_z(a) \in K \subseteq A_0$. This concludes the proof. $\square$

Remark. The lemma remains true if we replace κ_t by $\rho_t = \tau_t \sigma'_t$ or $\sigma_t \sigma'_t$.

Corollary 2.4. *A_0 consists of analytic elements for (σ_t).*

Proof. This follows easily from the previous two statements. If $a \in A_0$, we know that a is analytic for τ_t and $\kappa_t = \sigma_t \tau_{-t}$. If $z \in \mathbf{C}$, then $\tau_z(\kappa_z(a))$ makes sense, since A_0 is invariant under κ_z. Since σ_z is the closure of $\tau_z \circ \kappa_z$ (a fact for which we have found no concrete reference in the von Neumann-algebra case, but which is true here anyway because $A_0 \subseteq M(A)$, so that we can use Proposition 3.11 of [**8**]), we arrive at $a \in \mathscr{D}(\sigma_z)$. $\square$

As a consequence, A_0 is invariant under σ_{ni} and σ'_{ni}, with $n \in \mathbf{Z}$.

Remark. We do not know if A_0, or even the von Neumann-algebra N generated by it, has to be invariant under the one-parameter groups (σ_t) and (τ_t). There seems to be an analytic obstruction to be able to conclude this. It is however easy to see that if N is invariant under either (σ_t), (τ_t) or $\delta^{-it} \cdot \delta^{it}$, then it is invariant under all of them (see, e.g., Proposition 2.9). That this is an important problem is shown by the following: suppose that A_0 is a multiplier Hopf *-algebra and that N contains the unit of M. Then $\Delta(N) \subseteq N \otimes N$, and invariance under τ_t and R would give, by Proposition 10.5 of [**1**], that N is in fact itself a von Neumann-algebraic quantum group (possibly with a different left invariant weight). Thus, this would show that such multiplier Hopf *-algebras are intimately related to von Neumann-algebraic quantum groups.

Next, we impose a stronger condition on A_0:

Assumption 2.2. $A_0 \subseteq A$.

We will say then that A_0 has a proper imbedding in A. Because A_0 is now a subspace of the C^*-algebra A, we can say more about its connection to φ. We first need a simple lemma, which also appears in some form in [**14**]:

Lemma 2.5. *Suppose that $a \in A \cap \mathscr{D}(\sigma_{i/2})$ and $e \in A$ satisfy $ea = a$. Then $a \in \mathscr{N}_\varphi$.*

Proof. Choose c in $A \cap \mathscr{M}_\varphi^+$ such that $\|c - e^*e\| \leq 1/2$. This is possible because $\mathscr{M}_\varphi^+ \cap A$ is *normdense* in A^+. Then

$$\frac{1}{2}a^*a \leq a^*(1 + c - e^*e)a = a^*ca.$$

Since $a \in \mathscr{D}(\sigma_{i/2})$, we know that $a^*ca \in \mathscr{M}_\varphi^+$ by a fundamental result in noncommutative integration theory. Thus, $a^*a \in \mathscr{M}_\varphi^+$ since it is bounded from above by an integrable element. □

Proposition 2.6. *A_0 belongs to the Tomita algebra of φ: $A_0 \subseteq \mathscr{T}_\varphi = \{x \in \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^* \mid x \in \mathscr{D}(\sigma_z)$ and $\sigma_z(x) \in \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*$ for all $z \in \mathbf{C}\}$.*

Proof. We know that A_0 has local units: for every $a \in A_0$ there exist $e, f \in A_0$ such that $a = ea$ and $a = af$. So, since A_0 consists of analytic elements for (σ_t), we can apply the previous lemma to each element of $(\cup_{z\in\mathbf{C}}\sigma_z(A_0))$ and A_0^* since also for each of these we can supply local units. This implies that $A_0 \subseteq \mathscr{T}_\varphi$. □

Remark. The converse is also true. Suppose A_0 consists of square integrable elements in M. Then A_0 will be a subset of A. Namely, let b be a fixed element in A_0 such that $\varphi(b^*b) = 1$. Choose a in A_0. Then $a \otimes b = \sum \Delta(q_i)(p_i \otimes 1)$ with $p_i, q_i \in A_0$. Multiply to the left with $1 \otimes b^*$ and apply $\iota \otimes \varphi$. Then $a = \sum(\iota \otimes \langle \cdot \Lambda_\varphi(q_i), \Lambda_\varphi(b)\rangle)(W^*)p_i$. Since $A_0 \subseteq M(A)$ and each $(\iota \otimes \omega_{\Lambda_\varphi(q_i),\Lambda_\varphi(b)}(W^*) \in A$, this is an element of A.

The previous proposition has the interesting corollary that the scaling constant of A is necessarily trivial. We will come back to this fact in the third section, where we apply our techniques to *-algebraic quantum groups (see Theorem 3.4).

Corollary 2.7. *The scaling constant ν of (A, Δ) equals 1.*

Proof. We have that $\nu^{-t/2}\kappa_t$, where $\kappa_t = \sigma_t \circ \tau_{-t}$, induces a one-parameter unitary group u_t on $\mathscr{H}$ by the formula $u_t\Lambda_\varphi(x) = \nu^{-t/2}\Lambda_\varphi(\kappa_t(x))$ for $x \in \mathscr{N}_\varphi$. As in the proof of Lemma 2.3, there is a nontrivial, finite-dimensional subspace K of A_0 that is invariant under (κ_t). Therefore, the space $L = \Lambda_\varphi(K)$ is invariant under (u_t). This means that there exists a nonzero $x \in A_0$ such that $\xi = \Lambda_\varphi(x) \in L$ and $u_t\xi = e^{it\lambda}\xi$, for some $\lambda \in \mathbf{R}$. Hence, $\nu^{-t/2}\kappa_t(x) = e^{it\lambda}x$. But, since κ_t is a one-parameter group of *-automorphisms, we get

$$\|x\| = \|\kappa_t(x)\| = \|e^{it\lambda}\nu^{t/2}x\| = \nu^{t/2}\|x\|.$$

So $\nu = 1$. □

From the previous proposition, it follows that $A_0 \subseteq \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*$. Because $A_0^2 = A_0$, we also have $A_0 \subseteq \mathscr{M}_\varphi$, so every element of A_0 is integrable with respect to φ. However, we cannot conclude that (A_0, Δ_0) is an algebraic quantum group, because we do not know if the restriction φ_0 of φ to A_0 is nonzero. In any case, it will be left invariant: If $a, b \in A_0$, then $\Delta(a)(b \otimes 1) \in \mathscr{M}_{\iota\otimes\varphi}$, and

$$(\iota \otimes \varphi_0)(\Delta_0(a)(b \otimes 1)) = (\iota \otimes \varphi)(\Delta(a)(b \otimes 1)) = \varphi(a)b = \varphi_0(a)b.$$

Assumption 2.3. *$A_0 \subseteq A$ and $\varphi_{|A_0} \neq 0$.*

The assumption is sufficient to conclude that (A_0, Δ_0) is an algebraic quantum group, as we have shown. We remark that the second condition is automatically fulfilled if (A_0, Δ_0) is a multiplier Hopf *-algebra (with the same *-involution as in A).

We now show that A_0 itself possesses an analytic structure, thus generalizing the results in [**9**].

Proposition 2.8. *Let δ be the modular element of (A, Δ), and δ_0 the modular element of (A_0, Δ_0). Then every a in A_0 is a left and a right multiplier for δ such that $a\delta = a\delta_0$ and $\delta a = \delta_0 a$. Moreover, we have that $\delta^z A_0 = A_0$ and $A_0\delta^z = A_0$.*

Proof. Choose a fixed b in A_0 with $\varphi(b) \neq 0$. Choose a in A_0. Then $a \otimes b = \sum \Delta(p_i)(q_i \otimes 1)$ for certain p_i, q_i in A_0. Multiplying to the left with $\delta^{it} \otimes \delta^{it}$, we get that

$$\delta^{it}a \otimes \delta^{it}b = \sum \Delta(\delta^{it}p_i)(q_i \otimes 1).$$

Now, for any $c \in A_0$, we have that $\delta^{it}c$ will be in $\mathscr{M}_\varphi$. Namely, choose $e \in A_0$ with $ce = c$. Then $d = c^*\delta^{-it}$ will be in $A \cap \mathscr{D}(\sigma_{i/2})$, and $e^*d = d$. Hence, by Lemma 2.5, $d \in \mathscr{N}_\varphi$ and so $\delta^{it}c = \delta^{it}ce \in \mathscr{N}_\varphi^* \mathscr{N}_\varphi = \mathscr{M}_\varphi$. So in particular, we can apply $\iota \otimes \varphi$ to each side of the above equation, and we get

$$\varphi(\delta^{it}b)\delta^{it}a = \sum \varphi(\delta^{it}p_i)q_i \in A_0.$$

Denote by L the finite-dimensional vector space spanned by the q_i. Since $t \to \varphi(\delta^{it}b)$ is a continuous function and $\varphi(b) = 1$, we can choose t small such that $\varphi(\delta^{it}b) \neq 0$. For such t we have $\delta^{it}a \in L$. A similar argument as in Proposition 2.3 lets us conclude that the linear span of the $\delta^{it}a$ with $t \in \mathbf{R}$ is a finite-dimensional subspace of A_0. This easily implies that a is a right multiplier of δ and $\delta a \in A_0$, since $t \to \delta^{it}a$ has an analytic extension to the complex plane. So every element of A_0 lies in the domain of left multiplication with any δ^z, $z \in \mathbf{C}$, and also $\delta^z A_0 = A_0$. Since the space $B_0 = \{a \in A \mid a^* \in A_0\}$ also has the structure of a multiplier Hopf algebra, and $\varphi_{|B_0} \neq 0$, we can conclude that B_0 consists of right multipliers for δ. So A_0 consists of left multipliers for δ, and $A_0\delta^z = A_0$.

Choose a fixed $a \in A_0$ with $\varphi(a) \neq 0$. Let b, c be elements in A_0. By Result 7.6 of [**11**], we know that

$$\begin{aligned}\varphi((\iota \otimes \langle \cdot \Lambda_\varphi(b), \Lambda_\varphi(c^*)\rangle)\Delta(a)) &= \varphi(a)\langle \delta^{1/2}\Lambda_\varphi(b), \delta^{1/2}\Lambda_\varphi(c^*)\rangle \\ &= \varphi(a)\langle \Lambda_\varphi(b), \Lambda_\varphi(\delta c^*)\rangle \\ &= \varphi_0(a)\varphi_0(c\delta b).\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi((\iota \otimes \langle \cdot \Lambda_\varphi(b), \Lambda_\varphi(c^*)\rangle)\Delta(a)) &= (\varphi \otimes \varphi)((1 \otimes c)\Delta(a)(1 \otimes b)) \\ &= (\varphi_0 \otimes \varphi_0)((1 \otimes c)\Delta_0(a)(1 \otimes b)) \\ &= \varphi_0(a)\varphi_0(c\delta_0 b).\end{aligned}$$

Since φ_0 is faithful, $\delta_0 b = \delta b$ and $b\delta_0 = b\delta$ for all $b \in A_0$. □

Remark. In general, i.e., when $A_0 \subseteq M(A)$, we do not have to expect nice behavior of A_0 with respect to δ. Consider, for example, the trivial quantum group $\mathbf{C}1$ in $M(A)$.

As we have remarked, invariance under the one-parameter groups of A_0 follows easily.

Proposition 2.9. *$\tau_z(A_0) = \sigma_z(A_0) = R(A_0) = A_0$, for all $z \in \mathbf{C}$.*

Proof. We have $\sigma_{2z}(a) = \delta^{-iz}(\sigma'_z\sigma_z(a))\delta^{iz}$. But $\sigma'_z\sigma_z$ and $\delta^{-iz} \cdot \delta^{iz}$ leave A_0 invariant. Hence $\sigma_z(A_0) \subseteq A_0$. Then also $\tau_z = (\tau_z\sigma_{-z}) \circ \sigma_z$ leaves A_0 invariant. Since $R = S \circ \tau_{i/2}$, we have that R leaves A_0 invariant. □

Gathering all we have proven so far, we obtain the following theorem:

Theorem 2.10. *Let (A, Δ) be a reduced C^*-algebraic quantum group with left invariant weight φ. Let (A_0, Δ_0) be a regular multiplier Hopf algebra in A, such that*

$$\begin{aligned}
\Delta_0(a)(1 \otimes b) &= \Delta(a)(1 \otimes b),\\
\Delta_0(a)(b \otimes 1) &= \Delta(a)(b \otimes 1),\\
(a \otimes 1)\Delta_0(b) &= (a \otimes 1)\Delta(b),\\
(1 \otimes a)\Delta_0(b) &= (1 \otimes a)\Delta(b),
\end{aligned}$$

for all $a, b \in A_0$. Then A_0 will consist of integrable elements for φ. If $\varphi_{|A_0} \neq 0$, then (A_0, Δ_0) will be an algebraic quantum group with left invariant functional $\varphi_0 = \varphi_{|A_0}$. Moreover, A_0 will consist of analytic elements for the modular automorphism group, the scaling group, the unitary antipode and left and right multiplication with the modular element of (A, Δ). A_0 will be invariant under all these actions. Further, σ_{-i} will then restrict to the modular automorphism σ for φ_0, τ_{-i} will restrict to S_0^2, and δ, considered as a multiplier for A_0, will coincide with δ_0. In particular, ψ will restrict to a right invariant functional ψ_0 on A_0.

As a corollary, we have

Corollary 2.11. *Let (A, Δ) be a reduced C^*-algebraic quantum group with a dense, properly imbedded regular multiplier Hopf *-algebra*

(A_0, Δ_0). Then (A_0, Δ_0) is a *-algebraic quantum group, with associated C^*-algebraic quantum group (A, Δ).

Proof. From the foregoing, we know that (A_0, Δ_0) is a *-algebraic quantum group with left invariant functional $\varphi_0 = \varphi_{|A_0}$. The only difficult step left to show is that A_0 is actually a core for the GNS-map Λ_φ. The proof of this follows along the lines of Theorem 6.12 of [**13**].

Let Λ_0 be the closure of the restriction of Λ_φ to A_0. Choose a bounded net (e_j) in A_0 converging strictly to 1. We can replace e_j by $1/\sqrt{\pi}\int \exp(-t^2)\sigma_t(e_j)\,dt$, since each will be an element of A_0 (because $\{\sigma_t(e_j) \mid t \in \mathbf{R}\}$ only spans a finite-dimensional space in A_0), and the net will still be bounded, converging strictly to 1. Moreover, now also $\sigma_{i/2}(e_j)$ will be a bounded net, converging strictly to 1.

Let x be an element of $\mathscr{N}_\varphi$. Then $xe_j \to x$ in norm. Moreover, $\Lambda_\varphi(xe_j) = J\sigma_{i/2}(e_j)^* J\Lambda_\varphi(x)$, where J denotes the modular conjugation operator for φ. Because $\sigma_{i/2}(e_j)$ also converges *-strongly to 1, we have $\Lambda_\varphi(xe_j) \to \Lambda_\varphi(x)$. Now if x is the norm-limit of (a_i), with $a_i \in A_0$, then $\Lambda_0(a_ie_j) = a_i\Lambda_\varphi(e_j) \to x\Lambda_\varphi(e_j) = \Lambda_\varphi(xe_j)$ for each e_j. Since Λ_0 is closed, each xe_j and hence x is in the domain of Λ_0. So $\Lambda_0 = \Lambda_\varphi$, and A_0 is a core for Λ_φ.

The corollary follows, since the multiplicative unitary of A and the multiplicative unitary of A_0 on $\mathscr{H}_\varphi \otimes \mathscr{H}_\varphi = \mathscr{H}_{\varphi_0} \otimes \mathscr{H}_{\varphi_0}$ coincide, and their first leg constitutes, respectively, A and the C*-algebraic quantum groups associated to A_0. □

In our last proposition we will say something about the dual of (A_0, Δ_0) when $A_0 \subseteq A$ is a properly imbedded regular multiplier Hopf algebra with $\varphi_{|A_0} \neq 0$.

Proposition 2.12. *Let $(\widehat{A}, \widehat{\Delta})$ be the dual locally compact quantum group of (A, Δ), and let $(\widehat{A_0}, \widehat{\Delta_0})$ be the dual algebraic quantum group of (A_0, Δ_0). Then*

$$j : \widehat{A_0} \longrightarrow \widehat{A} : \varphi_0(\cdot\, a) \longrightarrow (\varphi(\cdot\, a) \otimes \iota)(W)$$

is an injective (*-)algebra homomorphism, such that

$$
\begin{aligned}
(j\otimes j)(\hat{\Delta}_0^{\mathrm{op}}(\omega_1)(1\otimes\omega_2)) &= \widehat{\Delta}(j(\omega_1))(1\otimes j(\omega_2)),\\
(j\otimes j)((\omega_1\otimes 1)\widehat{\Delta}_0^{\mathrm{op}}(\omega_2)) &= (j(\omega_2)\otimes 1)\widehat{\Delta}(j(\omega_1)),\\
(j\otimes j)(\widehat{\Delta}_0^{\mathrm{op}}(\omega_1)(\omega_2\otimes 1)) &= \widehat{\Delta}(j(\omega_1))(j(\omega_2)\otimes 1),\\
(j\otimes j)((1\otimes\omega_1)\widehat{\Delta}_0^{\mathrm{op}}(\omega_2)) &= (1\otimes j(\omega_1))\widehat{\Delta}(j(\omega_2)),
\end{aligned}
$$

for all $\omega_1,\omega_2\in\widehat{A_0}$.

Proof. We remark that by $\widehat{\Delta}_0^{\mathrm{op}}$ we mean the comultiplication on the dual space $\widehat{A}$ determined by $(\widehat{\Delta}_0^{\mathrm{op}}(\omega))(x\otimes y)=\omega(yx)$. The only reason why it appears is by a difference in convention about the dual for algebraic quantum groups and locally compact quantum groups.

Recall that W denotes the multiplicative unitary of the left regular representation. We remark that the expression

$$(\varphi(\cdot\, a)\otimes\iota)(W)$$

makes sense, since a is in the square of the Tomita algebra; hence, $\varphi(\cdot a)$ has weak*-continuous extension from $\mathscr{N}_\varphi$ to M. It is also easily seen that j is injective.

We first check that j preserves the *-operation, in case A_0 is a *-algebraic quantum group. First we remark that if $\omega\in M_*$ is such that $\omega\circ S^{-1}$ is bounded on $\mathscr{D}(S^{-1})$, then by definition of the antipode, $(\omega\otimes\iota)(W)^*=(\omega^*\otimes\iota)(W)$ where ω^* is the closure of $x\to\overline{\omega(S^{-1}(x^*))}$ with $x\in\mathscr{D}(S)$. But if we denote by ω_a the functional $\varphi(\cdot a)$, then ω_a satisfies this condition, since for $x\in\mathscr{D}(S^{-1})$, we have, using that the scaling constant of A equals 1 and that thus φ is τ_t-invariant,

$$
\begin{aligned}
\omega_a(S^{-1}(x)) &= \varphi(S^{-1}(x)a)\\
&= \varphi(R(x)\tau_{-i/2}(a)),
\end{aligned}
$$

so $\omega_a\circ S^{-1}$ is bounded. So we are left to prove that $j(\varphi(\cdot\, a)^*)=\omega_a^*$. But for $x\in M$ with x a right multiplier of $\delta^{1/2}$, we have, again using that the scaling constant equals 1,

$$
\begin{aligned}
\omega_a^*(x) &= \varphi(\tau_{-i/2}(a)^*R(x))\\
&= \varphi(\delta^{1/2}xS(a)^*\delta^{1/2})\\
&= \varphi(xS(a)^*\delta),
\end{aligned}
$$

and so, because such x are σ-weakly dense in M, we get, since $\varphi_0(\cdot\, a)^* = \varphi_0(\cdot\, S_0(a)^*\delta_0)$, that j preserves the *-operation.

Now we show that j is an algebra morphism. Choose $a, b \in A_0$. Choose p_i, q_i in A_0 such that $a \otimes b = \sum \Delta_0(p_i)(q_i \otimes 1)$. Then $\varphi_0(\cdot\, a) \cdot \varphi_0(\cdot\, b) = \sum \varphi(q_i)\varphi_0(\cdot\, p_i)$. Now as for $\omega_1, \omega_2 \in M_*$, we have $(\omega_1 \otimes \iota)(W)(\omega_2 \otimes \iota)(W) = (\omega_1 \cdot \omega_2 \otimes \iota)(W)$, where $\omega_1 \cdot \omega_2 = (\omega_1 \otimes \omega_2) \circ \Delta$, we only have to check if $\varphi(\cdot\, a) \cdot \varphi(\cdot\, b)$ equals $\sum \varphi(q_i)\varphi(\cdot\, p_i)$. But, evaluating this last functional in $x \in M$, we get $(\varphi \otimes \varphi)(\Delta(x)(a \otimes b))$, which equals $\sum (\varphi \otimes \varphi)(\Delta(xp_i)(q_i \otimes 1)) = \sum \varphi(q_i)\varphi(xp_i)$, so indeed both functionals are equal.

Now we show that j flips the comultiplication. Denoting $\widehat{a} = j(\varphi_0(\cdot a))$ for $a \in A_0$, we have

$$\begin{aligned}(\Lambda_{\widehat{\varphi}} \otimes \Lambda_{\widehat{\varphi}})(\widehat{\Delta}(\hat{b})(\widehat{a} \otimes 1)) &= \Sigma W \Sigma(\Lambda_{\widehat{\varphi}}(\widehat{a}) \otimes \Lambda_{\widehat{\varphi}}(\widehat{b})) \\ &= \Sigma W \Sigma(\Lambda_{\varphi}(a) \otimes \Lambda_{\varphi}(b)),\end{aligned}$$

with Σ denoting the flip. Writing $b \otimes a$ as $\sum \Delta(p_i)(q_i \otimes 1)$ with $p_i, q_i \in A_0$, this reduces to $\sum \Lambda_\varphi(p_i) \otimes \Lambda_\varphi(q_i)$. As $\widehat{\Delta}_0^{\mathrm{op}}(\varphi_0(\cdot\, b))(\varphi_0(\cdot\, a) \otimes 1) = \sum \varphi_0(\cdot\, p_i) \otimes \varphi_0(\cdot\, q_i)$, we have proven the third equality of the proposition. The other equalities can be proven in a similar way (by using the appropriate representation). □

Remark. The previous proposition says that the dual $\widehat{A_0}$ will be properly imbedded in $\widehat{A}$ if A_0 is properly imbedded in (A, Δ). This implies that, under the given conditions, the dual $\widehat{A_0}$ of A_0 will also have an analytic structure.

3. Structure of *-algebraic quantum groups. We apply the techniques of the above section to obtain some interesting structural properties of *-algebraic quantum groups. While many of the results follow easily from the previous section, we have decided to give new proofs, using only algebraic machinery. As such, we can give a purely algebraic proof of the existence of a *positive* right invariant functional on a *-algebraic quantum group.

We fix a *-algebraic quantum group (A, Δ) with antipode S, positive left invariant functional φ, modular automorphism σ and modular element δ. As a right invariant functional (not assumed to be positive) we take $\psi = \varphi \circ S$, with modular automorphism σ'. We adapt the proof of Lemma 2.1 to show that A is spanned by eigenvectors for $\kappa = \sigma^{-1}S^2$. We need an easy lemma.

Lemma 3.1. *If b is a nonzero element in A and n is an even integer, then $b^*((\sigma')^n S^{2n})(b) \neq 0$.*

Proof. Suppose that $b \in A$ and $n \in 2\mathbf{Z}$ are such that

$$b^*((\sigma')^n S^{2n}(b)) = 0.$$

Then

$$((\sigma')^{n/2} S^n(b))^*((\sigma')^{n/2} S^n(b)) = 0.$$

So $(\sigma')^{n/2} S^n(b) = 0$ by the faithfulness of φ, and hence $b = 0$. □

Lemma 3.2. *If $a \in A$, then the linear span of the $\kappa^n(a)$, with $n \in \mathbf{Z}$, is finite-dimensional.*

Proof. We can follow the proof as in Lemma 2.1:

Let b be a fixed element of A. Choose a nonzero $a \in A$, and write

$$a \otimes b = \sum_{i=1}^{n} \Delta(p_i)(1 \otimes q_i),$$

with $p_i, q_i \in A$. Then

$$\kappa^n(a) \otimes \rho^{-n}(b) = \sum \Delta(p_i)(1 \otimes \rho^{-n}(q_i)), \text{ for all } n \in \mathbf{Z},$$

where $\rho = \sigma' S^2$. Multiply this equation to the left with $1 \otimes b^*$ to get

$$\kappa^n(a) \otimes b^* \rho^{-n}(b) = \sum ((1 \otimes b^*)\Delta(p_i))(1 \otimes \rho^{-n}(q_i)).$$

Choose $a_{ij}, b_{ij} \in A$ such that

$$(1 \otimes b^*)\Delta(p_i) = \sum_{j=1}^{m_i} a_{ij} \otimes b_{ij},$$

and let L be the finite-dimensional space spanned by the a_{ij}. We see that $\kappa^n(a) \otimes b^*\rho^{-n}(b) \in L \otimes A$, for every $n \in \mathbf{Z}$. Using the previous lemma, we can conclude that $\kappa^n(a) \in L$ for all $n \in 2\mathbf{Z}$. But this easily implies that the linear span K of all $\kappa^n(a)$, with $n \in \mathbf{Z}$, is a finite-dimensional, κ-invariant linear subspace of A. □

Denote by $(\widehat{A}, \widehat{\Delta})$ the dual *-algebraic quantum group of (A, Δ). We can regard $\widehat{A}$ and $M(\widehat{A})$ as functionals on A. We know from [**9**] that $\widehat{\delta} = \varepsilon \circ \kappa$, which is proven in an algebraic (and more general) setting in [**3**]. Then

$$\langle \omega\widehat{\delta}, x\rangle = \langle \omega \otimes (\varepsilon\kappa), \Delta(x)\rangle = \langle \omega, \kappa(x)\rangle,$$

for each $\omega \in \widehat{A}$ and $x \in A$. If ω is of the form $\varphi(\cdot a)$, this means $\varphi(\cdot a)\widehat{\delta}$ is a scalar multiple of $\varphi(\cdot \kappa^{-1}(a))$. By the previous lemma, this implies that, for ω fixed, the linear span of the $\omega\widehat{\delta}^n$ is finite-dimensional. The same is of course true for left multiplication with $\widehat{\delta}$.

By duality, we conclude that for each a in A, the linear span of the $\delta^n a$ is a finite-dimensional space K. (We could also prove this along the lines of Proposition 2.8.) Since δ is a self-adjoint operator on K, with Hilbert space structure induced by φ, we can diagonalize δ. Hence, we arrive at

Proposition 3.3. *Let (A, Δ) be a *-algebraic quantum group. Then A is spanned by elements which are eigenvectors for left multiplication by δ.*

We can use this to settle an open question, cf. [**13**]:

Theorem 3.4. *Let (A, Δ) be a *-algebraic quantum group. Then the scaling constant μ equals 1.*

Proof. Choose a nonzero element $b \in A$ with $\delta b = \lambda b$, for some $\lambda \in \mathbf{R}_0$. Then $\varphi(bb^*\delta) = \lambda\varphi(bb^*)$. But the lefthand side equals $\mu\varphi(\delta bb^*) = \mu\lambda\varphi(bb^*)$. Since $\varphi(bb^*) \neq 0$, we arrive at $\mu = 1$. □

Proposition 3.3 can be strengthened:

Theorem 3.5. *Let (A, Δ) be a *-algebraic quantum group. Then A is spanned by elements which are simultaneously eigenvectors for S^2, σ and σ', and left and right multiplication by δ. Moreover, the eigenvalues of these actions are all positive.*

Proof. We know that A is spanned by eigenvectors for left multiplication with δ, and the same is easily seen to be true for κ and $\rho = \sigma' S^2$. But all these actions commute. Hence, we can find a basis of A consisting of simultaneous eigenvectors. Since σ, σ' and S^2 can be written as compositions of the maps κ, ρ and left and right multiplication with δ, the first part of the theorem is proven.

We show that left multiplication with δ has positive eigenvalues. Fix $a \in A_0$. If λ is an eigenvalue, choose an eigenvector b. Consider $x = \Delta(a)(1 \otimes b)$. Then $(\varphi \otimes \varphi)(x^* x)$ will be a positive number. But this is equal to $\varphi(a^* a)\varphi(b^* \delta b) = \lambda \varphi(a^* a)\varphi(b^* b)$. Hence, λ must be positive. As before, duality implies that κ and ρ have positive eigenvalues, hence the same is true of σ, σ' and S^2. $\square$

With a little more effort, this result can be shown to hold true also for the *-algebraic quantum *hyper*groups, introduced in [**3**].

This theorem *explains* why there exists an analytic structure on a *-algebraic quantum group (A, Δ): the actions are all diagonal with positive entries. Hence $\sigma_z, \sigma'_z, \tau_z$ and multiplication with δ^{iz} are all well-defined on A.

Corollary 3.6. *The functional $\psi = \varphi \circ S$ is a positive right invariant functional.*

Proof. We already know that it is a right invariant functional. As for positivity, note that $\psi(a^* a) = \varphi(a^* a \delta) = \varphi((a\delta^{1/2})^* a \delta^{1/2}) \geq 0$ for any $a \in A$. Here we use that $\sigma(\delta^{1/2}) = \delta^{1/2}$, which is easily proven using an eigenvector argument. $\square$

Finally remark that the extension of φ to M, with M the von Neumann-algebraic quantum group associated with A, is an almost periodic weight, since the modular operator ∇ implementing σ on $\mathscr{H}_\varphi$ is diagonalizable.

4. Special cases.

4.1. Compact and discrete quantum groups. Let (A, Δ) be a discrete locally compact quantum group (see, e.g., [**25**]). Then A is the C*-algebraic direct sum of matrix algebras $M_{n_\alpha}(\mathbf{C})$. The algebraic direct sum $\mathscr{A} = \oplus_\alpha M_{n_\alpha}(\mathbf{C})$ has the structure of a multiplier Hopf *-algebra. So it is easy to see that δ, being a positive element in $\prod M_{n_\alpha}$, is diagonalizable with respect to $\mathscr{A}$. Then the same will be true for S^2, the square of the antipode, since in a discrete quantum group we have $S^2(a) = \delta^{-1/2} a \delta^{1/2}$. Lastly, σ is diagonalizable since $\sigma = S^2$ in a discrete quantum group.

Suppose now that (A_0, Δ_0) is a *-algebraic quantum group, properly imbedded in (A, Δ). Suppose a is a nonzero element in A_0 such that $a \notin \mathscr{A}$. We know that A_0 has local units, so there exists an $e \in A_0$ with $ae = a$. Then $aee^*a^* = aa^*$, and this implies that infinitely many components of ee^* have norm greater than 1. But this is impossible, since $ee^* \in A$. So $A_0 \subseteq \mathscr{A}$.

The same argument implies that A_0 is again a *-algebraic quantum group of discrete type, since A_0 itself will be an algebraic direct sum of matrix algebras. In particular, A_0 has a co-integral h_0, which will be a group like projection in $\mathscr{A}$. (A grouplike projection in a *-algebraic quantum group is a (self-adjoint) projection p satisfying $\Delta(p)(1 \otimes p) = p \otimes p$. See [**15**] for more details.)

The dual side is also easy to treat. Namely, let (A, Δ) be a (reduced) compact locally compact quantum group. We know then that A contains a dense Hopf *-algebra $\mathscr{A}$. Suppose that (A_0, Δ_0) is a multiplier Hopf *-algebra imbedded in (A, Δ). Since the left invariant weight φ is everywhere defined, the elements of A_0 are automatically integrable. Then (A_0, Δ_0) will be a *-algebraic quantum group. We know that $\widehat{A_0}$ is a discrete quantum group properly imbedded in $\widehat{A}$. Hence $A_0 \subseteq \mathscr{A}$ and (A_0, Δ_0) is a *-algebraic quantum group of compact type, i.e., a *-algebraic quantum group with unit. Note that we could also have used Theorem 5.1 of [**2**], by considering the Hopf algebra generated by $\mathscr{A}$ and A_0. The dual p of the co-integral h_0 of $\widehat{A_0}$ in $\widehat{\mathscr{A}}$ will be a group like projection in $\mathscr{A}$. It will be a unit for A_0.

4.2. Locally compact groups. Suppose G is a locally compact group. Let (A_0, Δ_0) be a regular multiplier Hopf *-algebra imbedded in $(\mathscr{L}^\infty(G), \Delta)$, where Δ is the usual comultiplication determined by $\Delta(f)(g,h) = f(gh)$. Then $A_0 \subseteq M(C_0(G)) = C_b(G)$, so A_0 consists of bounded continuous functions on G. Let $\overline{A_0}$ be the normclosure of A_0 in $\mathscr{L}^\infty(G)$. Then Δ restricts to a *-algebra morphism $\overline{A_0} \to M(\overline{A_0} \otimes \overline{A_0})$. Since $\overline{A_0}$ is abelian, this induces a locally compact semigroup structure on the spectrum X of $\overline{A_0}$. Since S is now just an involutive *-morphism $\mathscr{L}^\infty(G) \to \mathscr{L}^\infty(G)$, coinciding with S_0 on A_0, it restricts to a *-morphism $S : \overline{A_0} \to \overline{A_0}$. This induces a continuous map $X \to X : x \to \overline{x}$. Using the fact that $m((S_0 \otimes \iota)(\Delta(f)(1 \otimes g))) = \varepsilon(f)g$ for $f, g \in A_0$, we see that $f(\overline{x}x)g(x) = \varepsilon(f)g(x)$ for all $f, g \in A_0$. Since A_0 separates points, $\varepsilon(f) = f(\overline{x}x)$ for all $f \in A_0$ and $x \in X$, so that there exists $e \in X$ with $\overline{x}x = e$ for all $x \in X$. Now e is easily seen to be a unit for the semi-group X, and $\overline{x}$ will be an inverse for x. This makes X a locally compact group. But this means that X has a Haar measure. So (A_0, Δ_0) is properly imbedded in the C*-algebraic quantum group $(C_0(X), \Delta)$, hence is itself a *-algebraic quantum group.

We remark however that the invariant functional on A_0 can be different from integration with respect to the Haar measure on G. Consider for example the linear span A_0 of the functions $f_x : t \to e^{itx}$ in $\mathscr{L}^\infty(\mathbf{R})$, with $x \in \mathbf{R}$. Then A_0 is a Hopf *-algebra, but none of its nonzero elements are integrable with respect to the Lebesgue measure. It is easy to see that the left invariant functional φ_0 on A_0 is given by $\varphi_0(f_x) = \delta_{x,0}$, and that the space X equals the Bohr compactification of $\mathbf{R}$, i.e., the dual of $\mathbf{R}$ with the discrete topology, φ_0 being integration with respect to its Haar measure.

The dual case is not so clear: suppose G is a locally compact group, and (A_0, Δ_0) is imbedded in $(\mathscr{L}(G), \Delta)$, where Δ is determined by $\Delta(u_g) = u_g \otimes u_g$ on the generators of $\mathscr{L}(G)$. Will the C^*-algebraic closure $\overline{A_0}$ with the restriction of Δ be of the form $(C^*(X), \Delta)$ for some locally compact group X? This will of course be true if A_0 is properly imbedded in $C_r^*(G)$, since then we can apply the theory of the second section to conclude that $\overline{A_0}$ is a cocommutative C*-algebraic quantum group, hence of the form $(C_r^*(X), \Delta)$.

Let us now look at the results of the third section in the commutative case. Let G be a locally compact group with a compact open subgroup H. Consider the regular functions on H, i.e., the functions generated by the matrix-coefficients of finite-dimensional representations of H. We can see them as functions on G. The linear span of left translates of these functions by elements of G is denoted by $P_0(G)$. In [**14**], it is shown that $P_0(G)$ forms a dense multiplier Hopf *-algebra inside $(C_0(G), \Delta)$, with the usual comultiplication, and that every commutative *-algebraic quantum group is of this form. In this setting, the only nontrivial object is the modular function δ. According to our results, it should be diagonalizable. This is easily seen to be true. For example, the characteristic function of H will be an eigenvector for left multiplication. Indeed, the Haar measure on H is the restriction of the Haar measure on G. Hence $\delta_{|H}$ is the modular function of H. Since H is compact, $\delta_{|H} = 1$. So every regular function on H is invariant for left multiplication. Then the translates by some element g of such functions will be eigenvectors with eigenvalue $\delta(g)$, and the linear span of all such translates equals $P_0(G)$.

4.3. The case of the quantum groups $U_q(su(2))$ and $SU_q(2)$. Finally, we consider a particular, nontrivial example of a Hopf *-algebra (A_0, Δ_0), imbedded in the multiplier algebra of a discrete *-algebraic quantum group $(\mathscr{A}, \Delta)$. This is not a situation we have discussed, since this multiplier algebra contains unbounded operators (when acting on the Hilbert space closure of $\mathscr{A}$ by left multiplication). We will see which of our results are still true in this case.

So as (A_0, Δ_0), we take the quantum enveloping Lie algebra $U_q(su(2))$, with q nonzero in $]-1, 1[$. It is the unital *-algebra generated by two elements E and K, with K invertible and self-adjoint, obeying the following commutation relations:

$$\begin{cases} EK = q^{-1}KE \\ [E, E^*] = 1/(q - q^{-1})(K^2 - K^{-2}). \end{cases}$$

The comultiplication on the generators is given by

$$\begin{cases} \Delta_0(K) = K \otimes K \\ \Delta_0(E) = E \otimes K + K^{-1} \otimes E. \end{cases}$$

To see that this comultiplication is well defined, it is enough to check that it respects the commutation relations, but this is easily done. The antipode is determined by

$$\begin{cases} S_0(K) = K^{-1} \\ S_0(E) = -qE. \end{cases}$$

As our *-algebraic quantum group $(\mathscr{A}, \Delta)$, we take the *-algebraic quantum group $\widehat{\mathscr{B}}$, where $\mathscr{B}$ is the compact *-algebraic quantum group associated with $SU_q(2)$, Woronowicz's twisted $SU(2)$-group. We have that $\mathscr{B}$ is the unital *-algebra generated by two elements a and b, such that

$$\begin{cases} ab = qba \\ ab^* = qb^*a \\ [b, b^*] = 0 \\ a^*a = 1 - q^{-2}b^*b \\ aa^* = 1 - b^*b. \end{cases}$$

The comultiplication $\widehat{\Delta}$ is given by:

$$\begin{cases} \widehat{\Delta}(a) = a \otimes a - q^{-1}b \otimes b^* \\ \widehat{\Delta}(b) = a \otimes b + b \otimes a^*. \end{cases}$$

The antipode $\widehat{S}$ is given by

$$\begin{cases} \widehat{S}(a) = a^* \\ \widehat{S}(a^*) = a \\ \widehat{S}(b) = -q^{-1}b. \end{cases}$$

We will not need the concrete description of the left invariant functional, but we need to know the modular group, which we now denote by (ρ_t). To be complete, we also provide the scaling group, which we will denote by (θ_t):

$$\begin{cases} \rho_z(a) = q^{-2iz}a \\ \rho_z(b) = b \end{cases} \qquad \begin{cases} \theta_z(a) = a \\ \theta_z(b) = q^{-2iz}b. \end{cases}$$

The modular element will of course be trivial, since the quantum group is compact.

The easiest way to see that A_0 can be imbedded in $M(\widehat{\mathscr{B}})$, is by creating a pairing between $\mathscr{B}$ and A_0 (for the notion of a pairing, see e.g., [**26**, Section 4]). For, since $\mathscr{B}$ is compact, it is known that $M(\widehat{\mathscr{B}})$ can be identified with the vector space of *all* linear functionals on $\mathscr{B}$ (see, e.g., the remark after Proposition 4.2 of [**26**]). The fact that there is a *pairing*, implies that the inclusion of A_0 in $M(\widehat{\mathscr{B}})$ will be a morphism. The concrete pairing is as follows:

$$\begin{cases} \langle K, a\rangle = q^{-1/2} \\ \langle K, a^*\rangle = q^{1/2} \\ \langle K, b\rangle = 0 \\ \langle K, b^*\rangle = 0 \end{cases} \qquad \begin{cases} \langle E, a\rangle = 0 \\ \langle E, a^*\rangle = 0 \\ \langle E, b\rangle = 0 \\ \langle E, b^*\rangle = -q. \end{cases}$$

Since on the dual of a compact algebraic quantum group the modular group (σ_t) and the scaling group (τ_t) coincide, we find the following behavior of A_0:

$$\begin{cases} \sigma_z(K) = \tau_z(K) = K \\ \sigma_z(E) = \tau_z(E) = q^{2iz}E. \end{cases}$$

But although there is general invariance under the scaling (and thus the modular) group, we no longer have that A_0 is invariant under left multiplication by δ^z, with δ the modular element of $\widehat{\mathscr{B}}$. For this would imply that actually $\delta^z \in A_0$, since A_0 is a Hopf algebra. We remark that this δ^z is easily computable, for it is given as a functional by $\varepsilon\circ\rho_{iz}$, with ε the co-unit of $\mathscr{B}$. We find that applying δ is the same as pairing with $K^{-4} = (K^*K)^{-2}$, so uniqueness gives us that $\delta^{it} = K^{-4it}$. It is clear that this is no element in A_0. Remark also, that right or left multiplication with δ is no longer diagonal. This is easy to see, using that A_0 has $\{K^l E^m F^n \mid l \in \mathbf{Z}, m, n \in \mathbf{N}\}$ as a basis. In fact, since $\operatorname{span}\{K^{4n}X\}$ has infinite dimension for any $X \in A_0$, we get that $A_0 \cap \widehat{\mathscr{B}} = \{0\}$.

We note that in this example we are in a special situation: $(\widehat{SU_q(2)}, \widehat{\Delta})$ is the C*-algebraic quantum group generated by K, K^{-1} and E, in the sense of Woronowicz. Moreover, the multiplier Hopf *-subalgebra is linked by a pairing to a *-algebraic quantum group. This could explain why we still have invariance under τ_t and σ_t. For example, the same type of behavior occurs with the quantum $az+b$-group. We remark that in these cases, the corresponding Hopf *-algebra can be viewed as the infinitesimal version of the quantum group. We do not know if it is a general fact that the one-parameter groups descend to the Hopf *-algebra associated with the quantum group, if such an object is present. In any case, the connection between a locally compact quantum group and a Hopf *-algebra representing the quantum group at an infinitesimal level, is at present not well understood in a general framework.

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