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FOURIER-FEYNMAN TRANSFORMS, CONVOLUTIONS AND FIRST VARIATIONS ON THE SPACE OF ABSTRACT WIENER SPACE VALUED CONTINUOUS FUNCTIONS

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ABSTRACT. In this paper, we establish various relationships among the Fourier-Feynman transform, convolution and first variation of functionals in some Banach algebra, defined on the space of abstract Wiener space valued continuous functions, which corresponds to the Banach algebra defined on classical Wiener space introduced by Cameron and Storvick.

1. Introduction. The concept of an L_1 analytic Fourier-Feynman transform for functionals on classical Wiener space was introduced by Brue [**2**]. In [**3**] Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform on classical Wiener space. In [**14**] Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ which extended the results in [**2, 3**] and gave various relationships between the L_1 and the L_2 theories. In [**11, 12, 13**], Huffman, Park and Skoug defined a convolution product for functionals on classical Wiener space, and they showed that the analytic Fourier-Feynman transform of convolution product is the product of transforms. In [**18**], Park, Skoug and Storvick investigated various relationships among the first variation, the Fourier-Feynman transform and the convolution product for functionals on classical Wiener space that belong to the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [**4**]. Recently, Ahn, Chang, Kim, Song and Yoo studied the Fourier-Feynman transform theory on abstract Wiener space [**1, 7, 8, 9**]. For a detailed survey of this topic, see [**20**].

Let (H, B, ν) be an abstract Wiener space. Let $C_0(B) \equiv C_0([0,T], B)$ denote the space of abstract Wiener space valued continuous functions $x(t)$ which are defined on $[0,T]$ with $x(0) = 0$. From [**17**] it follows

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that $C_0(B)$ is a real separable Banach space with the norm

$$\|x\|_{C_0(B)} = \sup_{s\in[0,T]} \|x(s)\|_B \tag{1.1}$$

and the minimal σ-algebra making the mapping $x \to x(s)$ measurable consists of the Borel subsets of $C_0(B)$. Moreover, the Brownian motion in B induces a probability measure m_B on $(C_0(B), \mathcal{B}(C_0(B)))$ which is mean-zero Gaussian.

The Feynman integration and Fourier-Feynman transform theory for functionals defined on $C_0(B)$ was studied in [**5, 6, 19, 22**]. In this paper, we continue to study the Fourier-Feynman transform for functionals defined on $C_0(B)$. In particular, we investigate various relationships involving two or three concepts of the Fourier-Feynman transform, the convolution product and the first variation for functionals in the class $\mathcal{S}''_{n,B}$ and $\mathcal{S}''_B$ which correspond to the classes $\mathcal{S}''_n$ and $\mathcal{S}''$ of functionals on classical Wiener space introduced by Cameron and Storvick [**4**].

2. Preliminaries. Let $\vec{s} = (s_1, \dots, s_n)$ be given with $0 = s_0 < s_1 < \cdots < s_n \le T$, and let $T_{\vec{s}} : B^n \to B^n$ be defined by

$$T_{\vec{s}}(x_1, x_2, \dots, x_n) = \Big(\sqrt{s_1 - s_0}\,x_1, \dots, \sum_{k=1}^{n} \sqrt{s_k - s_{k-1}}\,x_k\Big). \tag{2.1}$$

Define a Borel measure $\mu_{\vec{s}}$ on $\mathcal{B}(B^n)$ by $\mu_{\vec{s}}(E) = (\times_1^n \nu)(T_{\vec{s}}^{-1}(E))$ for every $E \in \mathcal{B}(B^n)$. Let $J_{\vec{s}} : C_0(B) \to B^n$ be the function defined by

$$J_{\vec{s}}(x) = (x(s_1), x(s_2), \dots, x(s_n)).$$

For Borel subsets $E_1, E_2, \dots, E_n$ of B, $J_{\vec{s}}^{-1}(\times_{i=1}^n E_i)$ is called I-set with respect to $E_1, E_2, \dots, E_n$. Then the collection $\mathcal{I}$ of all I-sets is a semi-algebra. We define a set function m_B on $\mathcal{I}$ by

$$m_B(J_{\vec{s}}^{-1}(\times_{i=1}^n E_i)) = \mu_{\vec{s}}(\times_{i=1}^n E_i).$$

Then m_B is well defined and countably additive on $\mathcal{I}$. Using the Carathéodory extension process, we have a Borel measure m_B on $C_0(B)$.

Now we introduce an integration formula which plays an important role throughout this paper. This formula is easily obtained by the change of variable theorem [**19**].

Lemma 2.1. *Let $\vec{s} = (s_1, \dots, s_n)$ be given with $0 = s_0 < s_1 < \cdots < s_n \le T$, and let $f : B^n \to \mathbf{C}$ be a Borel measurable function. Then*

$$\int_{C_0(B)} f(x(s_1), \dots, x(s_n))\, dm_B(x) \overset{*}{=} \int_{B^n} (f \circ T_{\vec{s}})(x_1, \dots, x_n)\, d(\times_1^n \nu)(x_1, \dots, x_n) \tag{2.2}$$

where by $\overset{*}{=}$ we mean that if either side exists, then both sides exist and they are equal.

A subset E of $C_0(B)$ is said to be scale-invariant measurable provided αE is measurable for each $\alpha > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m_B(\alpha N) = 0$ for each $\alpha > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., then we write $F \approx G$.

Let $\mathbf{C}$ and $\mathbf{C}_+$ denote respectively the complex numbers and the complex numbers with positive real part.

Next we introduce the analytic Wiener integral and the analytic Feynman integral over $C_0(B)$. Let F be a $\mathbf{C}$-valued measurable functional on $C_0(B)$ such that

$$J_F(z) = \int_{C_0(B)} F(z^{-1/2}x)\, dm_B(x) \tag{2.3}$$

exists as a finite number for all real $z > 0$. If there exists a function $J_F^*(z)$ analytic in $\mathbf{C}_+$ such that $J_F^*(z) = J_F(z)$ for all $z > 0$, then $J_F^*(z)$ is defined to be the analytic Wiener integral of F over $C_0(B)$ with parameter z, and for $z \in \mathbf{C}_+$ we write

$$\int_{C_0(B)}^{\text{anw}_z} F(x)\, dm_B(x) = J_F^*(z). \tag{2.4}$$

If the following limit exists for nonzero real q, then we call it the analytic Feynman integral of F over $C_0(B)$ with parameter q, and we write

$$\int_{C_0(B)}^{\text{anf}_q} F(x)\, dm_B(x) = \lim_{z \to -iq} \int_{C_0(B)}^{\text{anw}_z} F(x)\, dm_B(x) \tag{2.5}$$

where z approaches $-iq$ through $\mathbf{C}_+$.

Notation 2.2. (i) *For $z \in \mathbf{C}_+$ and $y \in C_0(B)$, let*

$$T_z(F)(y) = \int_{C_0(B)}^{\mathrm{anw}_z} F(x+y)\, dm_B(x). \tag{2.6}$$

(ii) *Let $1 < p < \infty$, and let G_n and G be scale-invariant measurable functionals such that, for each $\alpha > 0$,*

$$\lim_{n\to\infty} \int_{C_0(B)} |G_n(\alpha x) - G(\alpha x)|^{p'}\, dm_B(x) = 0 \tag{2.7}$$

where $1/p + 1/p' = 1$. Then we write

$$\underset{n\to\infty}{\mathrm{l.i.m.}}\, G_n \approx G \tag{2.8}$$

and call G the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by a continuously varying parameter.

Now we also introduce the definitions of L_p analytic Fourier-Feynman transform, convolution product and the first variation δF for functionals defined on $C_0(B)$.

Definition 2.3. Let q be a nonzero real number. For $1 < p < \infty$, we define the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F on $C_0(B)$ by the formula, $z \in \mathbf{C}_+$,

$$T_q^{(p)}(F)(y) = \underset{z\to -iq}{\mathrm{l.i.m.}}\, T_z(F)(y), \tag{2.9}$$

whenever this limit exists. We define the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F by $(z \in \mathbf{C}_+)$

$$T_q^{(1)}(F)(y) = \lim_{z\to -iq} T_z(F)(y), \tag{2.10}$$

for s-a.e. $y \in C_0(B)$, whenever this limit exists.

Definition 2.4. Let F_1 and F_2 be functionals on $C_0(B)$. For $z \in \mathbf{C}_+^{\sim}$ and a nonzero real number q, we define their convolution product (if it exists) by

$$(2.11)\quad (F_1 * F_2)_z(y) = \begin{cases} \int_{C_0(B)}^{\mathrm{anw}_z} F_1((y+x)/\sqrt{2})F_2((y-x)/\sqrt{2})\, dm_B(x) & z \in \mathbf{C}_+ \\ \int_{C_0(B)}^{\mathrm{anf}_q} F_1((y+x)/\sqrt{2})F_2((y-x)/\sqrt{2})\, dm_B(x) & z = -iq. \end{cases}$$

Definition 2.5. Let F be a measurable functional on $C_0(B)$, and let $w \in C_0(B)$. Then the first variation of F in the direction w is defined (if it exists) by

$$(2.12)\qquad \delta F(x|w) = \frac{\partial}{\partial t} F(x+tw)\Big|_{t=0}.$$

We next introduce the classes $\mathcal{S}''_{n,B}$ and $\mathcal{S}''_B$ of analytic Feynman integrable functionals on $C_0(B)$ which correspond to the classes $\mathcal{S}''_n$ and $\mathcal{S}''$ introduced by Cameron and Storvick.

Let $\{e_n\}$ be a complete orthonormal system in H such that the e_ns are in B^*, the dual of B. For each $h \in H$ and $x \in B$, a stochastic inner product $(\cdot,\cdot)^{\sim}$ on $H \times B$ is defined by

$$(2.13)\qquad (h,x)^{\sim} = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (x, e_j) & \text{if the limit exists} \\ 0 & \text{otherwise,} \end{cases}$$

where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. For every $h \neq 0$ in H, $(h,\cdot)^{\sim}$ is a Gaussian random variable on B with mean zero, variance $|h|^2$. Also if both h and x are in H, then $(h,x)^{\sim} = \langle h, x\rangle$ [**15, 16**].

Let $\Delta_n = \{(s_1, s_2, \dots, s_n) \in [0,T]^n : 0 = s_0 < s_1 < \cdots < s_n \le T\}$. Let $\mathcal{M}''_n = \mathcal{M}''_n(\Delta_n \times H^n)$ be the class of complex Borel measures on $\Delta_n \times H^n$, and let $\|\mu\| = \operatorname{var}\mu$ be the total variation of $\mu \in \mathcal{M}''_n$.

Definition 2.6. Let $\mathcal{S}''_{n,B} = \mathcal{S}''_{n,B}(\Delta_n \times H^n)$ be the space of functionals of the form

$$(2.14)\qquad F(x) = \int_{\Delta_n \times H^n} \exp\Big\{ i \sum_{j=1}^{n} (h_j, x(s_j))^{\sim} \Big\}\, d\mu(\vec{s}, \vec{h})$$

for s-a.e. $x \in C_0(B)$ where $\mu \in \mathcal{M}''_n$. Here we take $\|F\|''_n = \inf\{\|\mu\|\}$ where the infimum is taken over all μs so that F and μ are related by (2.14).

Let $\mathcal{M}'' = \mathcal{M}''(\sum \Delta_n \times H^n)$ be the class of sequences of measures $\{\mu_n\}$ such that each $\mu_n \in \mathcal{M}''_n$ and $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$.

Definition 2.7. Let $\mathcal{S}''_B = \mathcal{S}''_B(\sum \Delta_n \times H^n)$ be the space of functionals of the form

(2.15)
$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} F_n(x) \\ &= \sum_{n=1}^{\infty} \int_{\Delta_n \times H^n} \exp\left\{ i \sum_{j=1}^{n} (h_j, x(s_j))^{\sim} \right\} d\mu_n(\vec{s}, \vec{h}) \end{aligned}$$

for s-a.e. $x \in C_0(B)$ where each $F_n \in \mathcal{S}''_{n,B}$ and $\sum_{n=1}^{\infty} \|F_n\|''_n < \infty$. The norm of F is defined by $\|F\|'' = \inf\{\sum_{n=1}^{\infty} \|F_n\|''_n\}$ where the infimum is taken over all representation of F given by (2.15).

Note that if n and k are positive integers then $\mathcal{S}''_{n,B} \subset \mathcal{S}''_{n+k,B}$. And if $F \in \mathcal{S}''_{n,B}$ then $\|F\|''_n \geq \|F\|''_{n+k}$ and $|F(x)| \leq \|F\|''_n$ for s-a.e. $x \in C_0(B)$. For completeness, we define $\mathcal{S}''_{0,B}$ to be constant functionals and define their norms to be their absolute values. For $F \in \mathcal{S}''_B$, the series in (2.15) converges absolutely and uniformly over $C_0(B)$. Also if $F \in \mathcal{S}''_B$ then $|F(x)| \leq \|F\|''$ for s-a.e. $x \in C_0(B)$. Moreover, we can show that $\mathcal{S}''_B$ is a Banach algebra with norm $\|\cdot\|''$ which corresponds to Theorem 4.1 in [**4**].

3. Fourier-Feynman transform and convolution.

In this section, we obtain some properties of Fourier-Feynman transform and convolution product for functionals in $\mathcal{S}''_{n,B}$ and $\mathcal{S}''_B$.

The following theorem is due to Chang, Cho, Song and Yoo [**5**].

Theorem 3.1. *Let $1 \leq p < \infty$, and let q be a nonzero real number. Let $F \in \mathcal{S}''_{n,B}$ be given by (2.14). Then the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ exists, belongs to $\mathcal{S}''_{n,B}$ and is given by the formula*

(3.1)
$$T_q^{(p)}(F)(y) = \int_{\Delta_n \times H^n} \exp\left\{ i \sum_{j=1}^{n} (h_j, y(s_j))^{\sim} - \frac{i}{2q} V(\vec{s}, \vec{h}) \right\} d\mu(\vec{s}, \vec{h})$$

for s-a.e. $y \in C_0(B)$ *where*

$$V(\vec{s}, \vec{h}) = \sum_{j=1}^{n} \left[(s_j - s_{j-1}) \left| \sum_{l=j}^{n} h_l \right|^2 \right]. \tag{3.2}$$

Moreover, if $F \in \mathcal{S}''_B$ *is given by* (2.15)*, then the* L_p *analytic Fourier-Feynman transform* $T_q^{(p)}(F)$ *also exists, belongs to* $\mathcal{S}''_B$ *and is given by the formula*

$$T_q^{(p)}(F)(y) = \sum_{n=1}^{\infty} T_q^{(p)}(F_n)(y) \tag{3.3}$$

for s-a.e. $y \in C_0(B)$.

Remark 3.2. We adopt the convention $1/\pm\infty = 0$ throughout this paper. Thus, if $q = \pm\infty$ and F belongs to $\mathcal{S}''_{n,B}$ or $\mathcal{S}_B$, then we mean $T_q^{(p)}(F)$ to be F itself.

By Theorem 3.1 and Remark 3.2, we can extend Theorems 3.3 and 3.7 in [**5**] as follows.

Theorem 3.3. *Let* $1 \le p < \infty$*, and let* q_1, q_2 *be nonzero extended real numbers. Let* F *belongs to* $\mathcal{S}''_{n,B}$ *or* $\mathcal{S}''_B$*. Then*

$$T_{q_1}^{(p)}(T_{q_2}^{(p)}(F)) \approx T_q^{(p)}(F) \tag{3.4}$$

where q *is an extended real number satisfying* $1/q_1 + 1/q_2 = 1/q$.

If $q_1 = -q_2$ in (3.4), then we obtain the following inverse transform theorem for a functional F in $\mathcal{S}''_{n,B}$ or $\mathcal{S}''_B$.

$$(3.5)\qquad T_{-q}^{(p)}(T_q^{(p)}(F)) \approx F$$

for any nonzero extended real number q.

Let $\vec{s} = (s_1, \dots, s_n)$ with $0 = s_0 < s_1 < \cdots < s_n \le T$ and $\vec{t} = (t_1, \dots, t_m)$ with $0 = t_0 < t_1 < \cdots < t_m \le T$. For any nonnegative integers $m_1, \dots, m_{n+1}$ with $m = m_1 + \cdots + m_{n+1}$, let

(3.6)
$$\begin{aligned}\Delta_{m;m_1,\dots,m_{n+1}} = \{\vec{t} \in \Delta_m : 0 = t_0 < t_1 < \cdots < t_{m_1} \le s_1 < t_{m_1+1}\\ < \cdots \\ < t_{m_1+\cdots+m_n} \le s_n < t_{m_1+\cdots+m_n+1} < \cdots < t_m \le T\},\end{aligned}$$

and for $u = 1, \dots, n+1$, let

(3.7)
$$\alpha_{u,v} = \begin{cases} t_{m_1+\cdots+m_{u-1}+1} - s_{u-1} & v = 1 \\ t_{m_1+\cdots+m_{u-1}+v} - t_{m_1+\cdots+m_{u-1}+v-1} & v = 2, 3, \dots, m_u \\ s_u - t_{m_1+\cdots+m_u} & v = m_u + 1. \end{cases}$$

Our next theorem gives a formula for the convolution product of Fourier-Feynman transforms of functionals in $\mathcal{S}''_{n,B}$ and $\mathcal{S}''_{m,B}$.

Theorem 3.4. *Let $F \in \mathcal{S}''_{n,B}$ be given by (2.14), and let $G \in \mathcal{S}''_{m,B}$ be given by*

$$(3.8)\qquad G(x) = \int_{\Delta_m \times H^m} \exp\Big\{ i \sum_{j=1}^{m} (k_j, x(t_j))^{\sim} \Big\}\, d\eta(\vec{t}, \vec{k})$$

for s-a.e. $x \in C_0(B)$ where $\eta \in \mathcal{M}''_m$. Let $1 \le p < \infty$, and let q_1, q_2 be nonzero extended real numbers. Then, for all nonzero real number q, the convolution product $(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q$ exists and is given by the formula

$$\begin{aligned}(3.9)\quad &(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) \\ &= \int_{\Delta_n \times H^n} \sum_{m_1+\cdots+m_{n+1}=m} \int_{\Delta_{m;m_1,\dots,m_{n+1}} \times H^m} \exp\Big\{ \frac{i}{\sqrt{2}} W(y; \vec{s}, \vec{h}; \vec{t}, \vec{k}) \\ &\qquad - \frac{i}{2q_1} V(\vec{s}, \vec{h}) - \frac{i}{2q_2} V(\vec{t}, \vec{k}) - \frac{i}{4q} R(\vec{s}, \vec{h}; \vec{t}, \vec{k}) \Big\}\, d\eta(\vec{t}, \vec{k})\, d\mu(\vec{s}, \vec{h})\end{aligned}$$

for s-a.e. $y \in C_0(B)$ *where*

$$W(y; \vec{s}, \vec{h}; \vec{t}, \vec{k}) = \sum_{j=1}^{n} (h_j, y(s_j))^{\sim} + \sum_{j=1}^{m} (k_j, y(t_j))^{\sim}, \tag{3.10}$$

$$\begin{aligned} R(\vec{s}, \vec{h}; \vec{t}, \vec{k}) = &\sum_{u=1}^{n} \sum_{v=1}^{m_u} \alpha_{u,v} \Big| \sum_{j=u}^{n} h_j - \sum_{l=m_{u-1}+v}^{m} k_l \Big|^2 \\ &+ \sum_{u=1}^{n} \alpha_{u,m_u+1} \Big| \sum_{j=u}^{n} h_j - \sum_{l=m_u+1}^{m} k_l \Big|^2 \\ &+ \sum_{v=1}^{m_{n+1}} \alpha_{n+1,v} \Big| \sum_{l=m_1+\cdots+m_n+v}^{m} k_l \Big|^2, \end{aligned} \tag{3.11}$$

and $V(\vec{s}, \vec{h}), V(\vec{t}, \vec{k})$ *are given by* (3.2) *with corresponding vectors* $(\vec{s}, \vec{h})$ *and* $(\vec{t}, \vec{k})$*, respectively.*

Proof. Because of Theorem 3.1 and Remark 3.2, it is enough to show that

$$(F * G)_q(y) = \int_{\Delta_n \times H^n} \sum_{m_1+\cdots+m_{n+1}=m} \int_{\Delta_{m;m_1,\cdots,m_{n+1}} \times H^m} \exp\Big\{ \frac{i}{\sqrt{2}} W(y; \vec{s}, \vec{h}; \vec{t}, \vec{k}) - \frac{i}{4q} R(\vec{s}, \vec{h}; \vec{t}, \vec{k}) \Big\} \, d\eta(\vec{t}, \vec{k}) \, d\mu(\vec{s}, \vec{h}) \tag{3.12}$$

for s-a.e. $y \in C_0(B)$. By (2.14), (3.6) and the Fubini theorem, we have for all $z > 0$ and s-a.e. $y \in C_0(B)$,

$$\begin{aligned} &(F * G)_z(y) \\ &= \int_{\Delta_n \times H^n} \sum_{m_1+\cdots+m_{n+1}=m} \int_{\Delta_{m;m_1,\cdots,m_{n+1}} \times H^m} \exp\Big\{ \frac{i}{\sqrt{2}} W(y; \vec{s}, \vec{h}; \vec{t}, \vec{k}) \Big\} \\ &\quad \int_{C_0(B)} \exp\Big\{ \frac{i}{\sqrt{2z}} W(x; \vec{s}, \vec{h}; \vec{t}, -\vec{k}) \Big\} \, dm_B(x) \, d\eta(\vec{t}, \vec{k}) \, d\mu(\vec{s}, \vec{h}). \end{aligned}$$

Using the integration formula (2.2) and relabeling $s_j = s_{j,0} = s_{j-1,m_j+1}$, $t_{m_1+\cdots+m_j+l} = s_{j,l}$, $h_j = h_{j,0}$, $k_{m_1+\cdots+m_j+l} = k_{j,l}$, we obtain

$$\int_{C_0(B)} \exp\Big\{ \frac{i}{\sqrt{2z}} W(x; \vec{s}, \vec{h}; \vec{t}, -\vec{k}) \Big\} \, dm_B(x)$$

$$= \int_{B^{n+m}} \exp\bigg\{ \frac{i}{\sqrt{2z}} \bigg[\sum_{j=1}^{n} (h_{j,0}, \sum_{u=1}^{j} \sum_{v=1}^{m_u+1} \sqrt{\alpha_{u,v}} x_{u-1,v})^{\sim}$$
$$- \sum_{j=0}^{n} \sum_{l=1}^{m_{j+1}} (k_{j,l}, \sum_{u=1}^{j} \sum_{v=1}^{m_u+1} \sqrt{\alpha_{u,v}} x_{u-1,v} + \sum_{v=1}^{l} \sqrt{\alpha_{j+1,v}} x_{j,v})^{\sim} \bigg] \bigg\}$$
$$d(\times_1^{n+m} \nu)(x_{0,1}, \dots, x_{n,m_{n+1}})$$

where $\alpha_{u,v} = s_{u-1,v} - s_{u-1,v-1}$. Evaluating the last Wiener integral and restoring the indices to the original ones, we see that it is equal to

$$\exp\Big\{ -\frac{1}{4z} R(\vec{s}, \vec{h}; \vec{t}, \vec{k}) \Big\}.$$

Hence,

$$(F * G)_z(y) = \int_{\Delta_n \times H^n} \sum_{m_1+\cdots+m_{n+1}=m} \int_{\Delta_{m;m_1,\dots,m_{n+1}} \times H^m}$$
$$\exp\Big\{ \frac{i}{\sqrt{2}} W(y; \vec{s}, \vec{h}; \vec{t}, \vec{k}) - \frac{1}{4z} R(\vec{s}, \vec{h}; \vec{t}, \vec{k}) \Big\} \, d\eta(\vec{t}, \vec{k}) \, d\mu(\vec{s}, \vec{h}).$$

But the last expression above is analytic in $z \in \mathbf{C}_+$ and is bounded continuous in $z \in \mathbf{C}_+^{\sim}$ since μ and η are finite Borel measures. Hence $(F * G)_q(y)$ exists and is given by (3.12) for s-a.e. $y \in C_0(B)$ and this completes the proof. □

Our next theorem shows that Fourier-Feynman transform of convolution product is the product of the Fourier-Feynman transforms of functionals in $\mathcal{S}''_{n,B}$ and $\mathcal{S}''_{m,B}$.

Theorem 3.5. *Let F, G, p, q_1 and q_2 be given as in Theorem 3.4. Then, for all nonzero real numbers q,*

$$T_q^{(p)}(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) = T_{q'_1}^{(p)}(F)\Big(\frac{y}{\sqrt{2}}\Big) T_{q'_2}^{(p)}(G)\Big(\frac{y}{\sqrt{2}}\Big) \tag{3.13}$$

for s-a.e. $y \in C_0(B)$, where q'_j are nonzero extended real numbers satisfying $(1/q) + (1/q_j) = 1/q'_j$ for $i = 1, 2$.

Proof. Using expression (3.9) and the Fubini theorem, it is easy to see that for $z > 0$ and s-a.e. $y \in C_0(B)$,

$$\begin{aligned}
&T_z(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) \\
&= \int_{\Delta_n \times H^n} \sum_{m_1+\cdots+m_{n+1}=m} \int_{\Delta_{m;m_1,\dots,m_{n+1}} \times H^m} \\
&\exp\left\{\frac{i}{\sqrt{2}} W(y;\vec{s},\vec{h};\vec{t},\vec{k}) - \frac{i}{2q_1} V(\vec{s},\vec{h}) - \frac{i}{2q_2} V(\vec{t},\vec{k}) - \frac{i}{4q} R(\vec{s},\vec{h};\vec{t},\vec{k})\right\} \\
&\qquad \int_{C_0(B)} \exp\left\{\frac{i}{\sqrt{2z}} W(x;\vec{s},\vec{h};\vec{t},\vec{k})\right\} dm_B(x)\, d\eta(\vec{t},\vec{k})\, d\mu(\vec{s},\vec{h}).
\end{aligned}$$

Comparing the Wiener integral in the last expression with the Wiener integral in the proof of Theorem 3.4 ($-\vec{k}$ is replaced with $\vec{k}$ in this proof), we know that

$$\begin{aligned}
&T_z(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) \\
&\quad = \int_{\Delta_n \times H^n} \sum_{m_1+\cdots+m_{n+1}=m} \int_{\Delta_{m;m_1,\dots,m_{n+1}} \times H^m} \\
&\quad \exp\left\{\frac{i}{\sqrt{2}} W(y;\vec{s},\vec{h};\vec{t},\vec{k}) - \frac{i}{2q_1} V(\vec{s},\vec{h}) - \frac{i}{2q_2} V(\vec{t},\vec{k}) \right. \\
&\quad\quad \left. - \frac{i}{4q} R(\vec{s},\vec{h};\vec{t},\vec{k}) - \frac{1}{4z} R(\vec{s},\vec{h};\vec{t},-\vec{k})\right\} d\eta(\vec{t},\vec{k})\, d\mu(\vec{s},\vec{h}).
\end{aligned}$$

But the last expression above is analytic in $z \in \mathbf{C}_+$ and is bounded continuous in $z \in \mathbf{C}_+^\sim$ since μ and η are finite Borel measures. Moreover, a direct calculation shows that

$$R(\vec{s},\vec{h};\vec{t},\vec{k}) + R(\vec{s},\vec{h};\vec{t},-\vec{k}) = 2[V(\vec{s},\vec{h}) + V(\vec{t},\vec{k})]. \tag{3.14}$$

Hence, we have

$$\begin{aligned}
&T_q^{(p)}(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) \\
&\quad = \int_{\Delta_n \times H^n} \sum_{m_1+\cdots+m_{n+1}=m} \int_{\Delta_{m;m_1,\dots,m_{n+1}} \times H^m} \\
&\exp\left\{\frac{i}{\sqrt{2}} W(y;\vec{s},\vec{h};\vec{t},\vec{k}) - \frac{i}{2q_1'} V(\vec{s},\vec{h}) - \frac{i}{2q_2'} V(\vec{t},\vec{k})\right\} d\eta(\vec{t},\vec{k})\, d\mu(\vec{s},\vec{h})
\end{aligned} \tag{3.15}$$

and, by Theorem 3.1, it is equal to the righthand side of (3.13). □

Now we obtain a version of Parseval's identity for functionals in $\mathcal{S}''_{n,B}$ and $\mathcal{S}''_{m,B}$.

Theorem 3.6. *Let F, G, p, q_1 and q_2 be given as in Theorem 3.4. Then the following Parseval's identity*

$$(3.16)\quad T_{-q}^{(p)}(T_q^{(p)}(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q)(0) = T_q^{(p)}\Big((T_{q_1}^{(p)}(F))\Big(\frac{\cdot}{\sqrt{2}}\Big)(T_{q_2}^{(p)}(G))\Big(-\frac{\cdot}{\sqrt{2}}\Big)\Big)(0)$$

holds for all nonzero real numbers q.

Proof. Having the proofs of Theorems 3.4 and 3.5 in mind, the proof of this theorem is not difficult. Taking the Fourier-Feynman transform of the expression in (3.15) yields the formula (the procedure is the same as the proof of Theorem 3.5 except q is replaced by $-q$)

$$\int_{\Delta_n \times H^n} \sum_{m_1+\cdots+m_{n+1}=m} \int_{\Delta_{m;m_1,\dots,m_{n+1}} \times H^m} \exp\Big\{-\frac{i}{2q_1'}V(\vec{s},\vec{h}) - \frac{i}{2q_2'}V(\vec{t},\vec{k}) + \frac{i}{4q}R(\vec{s},\vec{h};\vec{t},-\vec{k})\Big\}\, d\eta(\vec{t},\vec{k})\, d\mu(\vec{s},\vec{h}).$$

But by relationship (3.14), the above expression is equal to

$$(3.17)\quad \int_{\Delta_n \times H^n} \sum_{m_1+\cdots+m_{n+1}=m} \int_{\Delta_{m;m_1,\dots,m_{n+1}} \times H^m} \exp\Big\{-\frac{i}{2q_1}V(\vec{s},\vec{h}) - \frac{i}{2q_2}V(\vec{t},\vec{k}) - \frac{i}{4q}R(\vec{s},\vec{h};\vec{t},\vec{k})\Big\}\, d\eta(\vec{t},\vec{k})\, d\mu(\vec{s},\vec{h}).$$

On the other hand, by Theorem 3.1,

$$T_{q_1}^{(p)}(F)\Big(\frac{y}{\sqrt{2}}\Big)T_{q_2}^{(p)}(G)\Big(-\frac{y}{\sqrt{2}}\Big) = \int_{\Delta_n \times H^n} \int_{\Delta_m \times H^m} \exp\Big\{\frac{i}{\sqrt{2}}W(y;\vec{s},\vec{h};\vec{t},-\vec{k}) - \frac{i}{2q_1}V(\vec{s},\vec{h}) - \frac{i}{2q_2}V(\vec{t},\vec{k})\Big\}\, d\eta(\vec{t},\vec{k})\, d\mu(\vec{s},\vec{h}).$$

By taking the Fourier-Feynman transform of the above expression, it is easy to see that the righthand side of (3.16) is also expressed as (3.17) and this completes the proof. □

From now on, we extend the results in Theorems 3.4, 3.5 and 3.6 to the functionals in $\mathcal{S}''_B$.

Theorem 3.7. *Let $F \in \mathcal{S}''_B$ be given by* (2.15), *and let $G \in \mathcal{S}''_B$ be given by*

$$G(x) = \sum_{m=1}^{\infty} G_m(x) = \sum_{m=1}^{\infty} \int_{\Delta_m \times H^m} \exp\left\{ i \sum_{j=1}^{m} (k_j, x(t_j))^{\sim} \right\} d\eta_m(\vec{t}, \vec{k}) \tag{3.18}$$

for s-a.e. $x \in C_0(B)$. Let $1 \le p < \infty$, and let q_1, q_2 be nonzero extended real numbers. Then, for all nonzero real numbers q, the convolution product $(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q$ exists and is given by the formula

$$(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) = \sum_{r=0}^{\infty} \sum_{n+m=r} (T_{q_1}^{(p)}(F_n) * T_{q_2}^{(p)}(G_m))_q(y) \tag{3.19}$$

for s-a.e. $y \in C_0(B)$. Of course, the above series can be expressed explicitly using (3.9).

Proof. Using the dominated convergence theorem, we have for $z > 0$

$$(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_z(y) = \sum_{r=0}^{\infty} \sum_{n+m=r} (T_{q_1}^{(p)}(F_n) * T_{q_2}^{(p)}(G_m))_z(y).$$

But the last series above converges uniformly in $z \in \mathbf{C}_+^{\sim}$ since $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$ and $\sum_{m=1}^{\infty} \|\eta_m\| < \infty$. Hence, the series is analytic in $z \in \mathbf{C}_+$ and is bounded continuous in $z \in \mathbf{C}_+^{\sim}$. Hence, the convolution product exists is given by the series in (3.19). □

Theorem 3.8. *Let F, G, p, q_1 and q_2 be given as in Theorem* 3.7. *Then for all nonzero real numbers q,*

$$T_q^{(p)}(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) = T_{q_1'}^{(p)}(F)\left(\frac{y}{\sqrt{2}}\right) T_{q_2'}^{(p)}(G)\left(\frac{y}{\sqrt{2}}\right) \tag{3.20}$$

for s-a.e. $y \in C_0(B)$, where the q_j' are nonzero extended real numbers satisfying $(1/q) + (1/q_j) = (1/q_j')$ for $i = 1, 2$.

Proof. By the dominated convergence theorem, we have for $z > 0$

$$T_z(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) = \sum_{r=0}^{\infty} \sum_{n+m=r} T_z(T_{q_1}^{(p)}(F_n) * T_{q_2}^{(p)}(G_m))_q(y).$$

But, by the same method as in the proof of Theorem 3.7, the series in the last expression is analytic in $z \in \mathbf{C}_+$ and is bounded continuous in $z \in \mathbf{C}_+^{\sim}$. Hence,

$$T_q^{(p)}(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y) = \sum_{r=0}^{\infty} \sum_{n+m=r} T_q^{(p)}(T_{q_1}^{(p)}(F_n) * T_{q_2}^{(p)}(G_m))_q(y).$$

Finally, using Theorems 3.5 and 3.1, we obtain (3.20). □

The proof of the following Parseval's identity for functionals in $\mathcal{S}_B''$ is quite similar to that of Theorem 3.8 and so we will not give it here.

Theorem 3.9. *Let F, G, p, q_1 and q_2 be given as in Theorem* 3.4. *Then the following Parseval's identity*

$$(3.21) \quad T_{-q}^{(p)}(T_q^{(p)}(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q)(0) = T_q^{(p)}\Big((T_{q_1}^{(p)}(F))\Big(\frac{\cdot}{\sqrt{2}}\Big)(T_{q_2}^{(p)}(G))\Big(-\frac{\cdot}{\sqrt{2}}\Big)\Big)(0)$$

holds for all nonzero real numbers q.

4. First variation. In this section we establish various relationships and expressions involving two or three concepts of the Fourier-Feynman transform, the convolution product and the first variation of functionals in $\mathcal{S}_{n,B}''$ and $\mathcal{S}_B''$. We begin with a formula for the first variation of functionals in $\mathcal{S}_{n,B}''$.

Theorem 4.1. *Let $F \in \mathcal{S}_{n,B}''$ be given by* (2.14) *with*

$$\int_{\Delta_n \times H^n} \sqrt{s_j}|h_j|\, d|\mu|(\vec{s}, \vec{h}) < \infty \text{ for all } j = 1, \dots, n.$$

Let $1 \le p < \infty$, and let q be a nonzero extended real number. Then for s-a.e. $w \in C_0(B)$ the first variation $\delta F(y|w)$ exists, belongs to $\mathcal{S}''_{n,B}$ as a function of y and is given by the formula

$$\delta T_q^{(p)}(F)(y|w) = \int_{\Delta_n \times H^n} i\Big(\sum_{j=1}^n (h_j, w(s_j))^\sim\Big) \exp\Big\{i\sum_{j=1}^n (h_j, y(s_j))^\sim - \frac{i}{2q}V(\vec{s},\vec{h})\Big\}\, d\mu(\vec{s},\vec{h}) \tag{4.1}$$

for s-a.e. $y \in C_0(B)$.

Proof. Using (3.1) we have for s-a.e. $w \in C_0(B)$

$$\delta T_q^{(p)}(F)(y|w) = \frac{\partial}{\partial t}\Big(\int_{\Delta_n \times H^n} \exp\Big\{i\sum_{j=1}^n (h_j, y(s_j))^\sim + it\sum_{j=1}^n (h_j, w(s_j))^\sim - \frac{i}{2q}V(\vec{s},\vec{h})\Big\}\, d\mu(\vec{s},\vec{h})\Big)\Big|_{t=0}$$

s-a.e. $y \in C_0(B)$. Then we obtain the expression (4.1) provided we can justify interchanging the differentiation and the integral sign. To do this, we have to show that $\int_{\Delta_n \times H^n} |\sum_{j=1}^n (h_j, w(s_j))^\sim|\, d|\mu|(\vec{s},\vec{h}) < \infty$ for s-a.e. $w \in C_0(B)$. But this can be easily obtained by the fact that

$$\int_{C_0(B)} \int_{\Delta_n \times H^n} \Big|\sum_{j=1}^n (h_j, \alpha w(s_j))^\sim\Big|\, d|\mu|(\vec{s},\vec{h})\, dm_B(w) \le \alpha\Big(\frac{2}{\pi}\Big)^{1/2} \sum_{j=1}^n \int_{\Delta_n \times H^n} \sqrt{s_j}|h_j|\, d|\mu|(\vec{s},\vec{h}) < \infty$$

for all $\alpha > 0$. Finally we can rewrite

$$\delta T_q^{(p)}(F)(y|w) = \int_{\Delta_n \times H^n} \exp\Big\{i\sum_{j=1}^n (h_j, y(s_j))^\sim\Big\}\, d\hat{\mu}(\vec{s},\vec{h})$$

s-a.e. $y \in C_0(B)$ where $\hat{\mu}$ is the measure in $\mathcal{M}''_n$ defined by

$$\hat{\mu}(E) = \int_E i\bigg(\sum_{j=1}^n (h_j, w(s_j))^\sim\bigg) \exp\Big\{-\frac{i}{2q}V(\vec{s},\vec{h})\Big\}\, d\mu(\vec{s},\vec{h}),$$

and this completes the proof. □

In Theorem 4.2, we fix $w \in C_0(B)$ and consider $\delta T_q^{(p)}(F)(y|w)$ as a function of y, while in Theorem 4.3, we fix $y \in C_0(B)$ and consider $\delta T_q^{(p)}(F)(y|w)$ as a function of w.

Theorem 4.2. *Let F and p be given as in Theorem 4.1, and let q_1 and q_2 be nonzero extended real numbers. Then for s-a.e. $w \in C_0(B)$,*

$$(4.2)\quad T_{q_1}^{(p)}(\delta T_{q_2}^{(p)}(F)(\cdot|w))(y) = \delta T_{q_1}^{(p)}(T_{q_2}^{(p)}(F))(y|w) = \delta T_q^{(p)}(F)(y|w)$$

for s-a.e. $y \in C_0(B)$, where q is a nonzero extended real number satisfying $(1/q_1)+(1/q_2)=(1/q)$.

Proof. By Theorem 3.3, the second equality is obvious. Applying Theorems 3.1 and 4.1 in succession to the expression (3.1), we obtain

$$\begin{aligned} &T_{q_1}^{(p)}(\delta T_{q_2}^{(p)}(F)(\cdot|w))(y) \\ &\quad = \int_{\Delta_n\times H^n} i\bigg(\sum_{j=1}^n (h_j, w(s_j))^\sim\bigg) \exp\Big\{i\sum_{j=1}^n (h_j, y(s_j))^\sim \\ &\qquad\qquad - \frac{i}{2q_1}V(\vec{s},\vec{h}) - \frac{i}{2q_2}V(\vec{s},\vec{h})\Big\}\, d\mu(\vec{s},\vec{h}) \end{aligned}$$

which is equal to $\delta T_q^{(p)}(F)(y|w)$ by Theorem 4.1. □

Theorem 4.3. *Let F, p, q_1 and q_2 be given as in Theorem 4.2. Then for s-a.e. $y \in C_0(B)$,*

$$(4.3)\qquad T_{q_1}^{(p)}(\delta T_{q_2}^{(p)}(F)(y|\cdot))(w) = \delta T_{q_2}^{(p)}(F)(y|w)$$

for s-a.e. $w \in C_0(B)$.

Proof. Using Theorem 4.1, we have for all $z > 0$ and s-a.e. $y \in C_0(B)$,

$$T_z(\delta T_{q_2}^{(p)}(F)(y|\cdot))(w) = \int_{C_0(B)} \int_{\Delta_n \times H^n} i\Big(\sum_{j=1}^{n} (h_j, z^{-1/2}x(s_j) + w(s_j))^{\sim} \Big) \exp\Big\{ i \sum_{j=1}^{n} (h_j, y(s_j))^{\sim} - \frac{i}{2q_2} V(\vec{s}, \vec{h}) \Big\} \, dm_B(x) \, d\mu(\vec{s}, \vec{h}).$$

Fubini's theorem and the fact $\int_{C_0(B)} (h_j, x(s_j))^{\sim} \, dm_B(x) = 0$ enable us to conclude that

$$T_{q_1}^{(p)}(\delta T_{q_2}^{(p)}(F)(y|\cdot))(w) = \int_{\Delta_n \times H^n} i\Big(\sum_{j=1}^{n} (h_j, w(s_j))^{\sim} \Big) \exp\left\{ i \sum_{j=1}^{n} (h_j, y(s_j))^{\sim} - \frac{i}{2q_2} V(\vec{s}, \vec{h}) \right\} d\mu(\vec{s}, \vec{h}),$$

and hence the proof is complete. □

From now on, we examine some relationships involving all of the concepts of the Fourier-Feynman transform, the convolution product and the first variation of functionals in $\mathcal{S}''_{n,B}$.

Theorem 4.4. *Let F and G be given as in Theorem* 3.4 *with corresponding measures μ and η, where μ and η satisfy the conditions $\int_{\Delta_n \times H^n} \sqrt{s_j} |h_j| \, d|\mu|(\vec{s}, \vec{h}) < \infty$ for all $j = 1, \dots, n$ and $\int_{\Delta_m \times H^m} \sqrt{t_j} |k_j| \, d|\eta|(\vec{t}, \vec{k}) < \infty$ for all $j = 1, \dots, m$. Let p, q_1 and q_2 be given as in Theorem* 4.3, *and let q be any nonzero real number. Then for s-a.e. $y, w \in C_0(B)$,*

$$\delta(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y|w) = \int_{\Delta_n \times H^n} \sum_{m_1 + \cdots + m_{n+1} = m} \int_{\Delta_{m;m_1, \dots, m_{n+1}} \times H^m} \frac{i}{\sqrt{2}} W(w; \vec{s}, \vec{h}; \vec{t}, \vec{k}) \exp\left\{ \frac{i}{\sqrt{2}} W(y; \vec{s}, \vec{h}; \vec{t}, \vec{k}) - \frac{i}{2q_1} V(\vec{s}, \vec{h}) - \frac{i}{2q_2} V(\vec{t}, \vec{k}) - \frac{i}{4q} R(\vec{s}, \vec{h}; \vec{t}, \vec{k}) \right\} d\eta(\vec{t}, \vec{k}) \, d\mu(\vec{s}, \vec{h}). \tag{4.4}$$

Proof. The proof is similar to that of Theorem 4.1. A key point is to justify interchanging the differentiation and the integral sign. But this can be done because

$$\begin{aligned}\int_{\Delta_n\times H^n}\int_{\Delta_m\times H^m}&|W(w;\vec{s},\vec{h};\vec{t},\vec{k})|\,d|\eta|(\vec{t},\vec{k})\,d|\mu|(\vec{s},\vec{h})\\&\le\|\eta\|\sum_{j=1}^{n}\int_{\Delta_n\times H^n}\sqrt{s_j}|h_j|\,d|\mu|(\vec{s},\vec{h})\\&\quad+\|\mu\|\sum_{j=1}^{m}\int_{\Delta_m\times H^m}\sqrt{t_j}|k_j|\,d|\eta|(\vec{t},\vec{k})<\infty\end{aligned}$$

where $\|\mu\|$ and $\|\eta\|$ denote the total variation of μ and η, respectively. □

In the next two theorems, we obtain the convolution product with respect to the first and second argument of the first variation. Theorem 4.5 is easily obtained by applying Theorem 3.4 to expression (4.1) for F and G.

Theorem 4.5. *Let F, G, p, q_1, q_2 and q be given as in Theorem 4.4. Then for s-a.e. $w_1, w_2\in C_0(B)$,*

(4.5)

$$\begin{aligned}&(\delta T_{q_1}^{(p)}(F)(\cdot|w_1)*\delta T_{q_2}^{(p)}(G)(\cdot|w_2))_q(y)\\&=-\int_{\Delta_n\times H^n}\sum_{m_1+\cdots+m_{n+1}=m}\int_{\Delta_{m;m_1,\dots,m_{n+1}}\times H^m}\\&\Big[\sum_{j=1}^{n}(h_j,w(s_j))^{\sim}\Big]\Big[\sum_{j=1}^{m}(k_j,w(t_j))^{\sim}\Big]\exp\Big\{\frac{i}{\sqrt{2}}W(y;\vec{s},\vec{h};\vec{t},\vec{k})\\&\quad-\frac{i}{2q_1}V(\vec{s},\vec{h})-\frac{i}{2q_2}V(\vec{t},\vec{k})-\frac{i}{4q}R(\vec{s},\vec{h};\vec{t},\vec{k})\Big\}\,d\eta(\vec{t},\vec{k})\,d\mu(\vec{s},\vec{h})\end{aligned}$$

for s-a.e. $y\in C_0(B)$.

Theorem 4.6. *Let F, G, p, q_1, q_2 and q be given as in Theorem* 4.4. *Then for s-a.e. $y \in C_0(B)$,*

$$(4.6)\quad (\delta T_{q_1}^{(p)}(F)(y|\cdot) * \delta T_{q_2}^{(p)}(G)(y|\cdot))_q(w) = -\frac{1}{2}\int_{\Delta_n\times H^n}\int_{\Delta_m\times H^m}\sum_{j=1}^{n}\sum_{l=1}^{m}[(h_j, w(s_j))^{\sim}(k_l, w(t_l))^{\sim} - \frac{i}{q}\min\{s_j, t_l\}\langle h_j, k_l\rangle] \exp\left\{iW(y;\vec{s},\vec{h};\vec{t},\vec{k}) - \frac{i}{2q}V(\vec{s},\vec{h}) - \frac{i}{2q}V(\vec{t},\vec{k})\right\} d\eta(\vec{t},\vec{k})\, d\mu(\vec{s},\vec{h})$$

for s-a.e. $w \in C_0(B)$.

Proof. Using Theorem 4.1, we have for $z > 0$ and for s-a.e. $y \in C_0(B)$,

$$(\delta T_{q_1}^{(p)}(F)(y|\cdot) * \delta T_{q_2}^{(p)}(G)(y|\cdot))_z(w) = -\frac{1}{2}\int_{C_0(B)}\int_{\Delta_n\times H^n}\int_{\Delta_m\times H^m} \left(\sum_{j=1}^{n}(h_j, w(s_j) + z^{-1/2}x(s_j))^{\sim}\right)\left(\sum_{j=1}^{m}(k_j, w(t_j) - z^{-1/2}x(s_j))^{\sim}\right) \exp\left\{iW(y;\vec{s},\vec{h};\vec{t},\vec{k}) - \frac{i}{2q}V(\vec{s},\vec{h}) - \frac{i}{2q}V(\vec{t},\vec{k})\right\} d\eta(\vec{t},\vec{k})\, d\mu(\vec{s},\vec{h})\, dm_B(x)$$

for s-a.e. $w \in C_0(B)$. Evaluating the above Wiener integral using the facts that

$$\int_{C_0(B)}(h_j, x(s_j))^{\sim}\, dm_B(x) = 0, \qquad \int_{C_0(B)}(k_j, x(t_j))^{\sim}\, dm_B(x) = 0$$

and

$$\int_{C_0(B)}(h_j, x(s_j))^{\sim}(k_l, x(t_l))^{\sim}\, dm_B(x) = \min\{s_j, t_l\}\langle h_j, k_l\rangle,$$

we obtain the expressions in the righthand side of (4.6). Since it is independent of the variable z, it has an analytic extension to $z \in \mathbf{C}_+$, and, letting $z \to -iq$, we complete the proof. □

In the following two theorems, we obtain the Fourier-Feynman transform of $\delta(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q$ with respect to the first and second argument of the variation.

Theorem 4.7. *Let F, G, p, q_1, q_2 and q be given as in Theorem 4.4. Then for s-a.e. $y \in C_0(B)$,*

$$(4.7)\quad \begin{aligned} T_q^{(p)}(\delta(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(\cdot|w))(y) \\ &= \delta T_q^{(p)}((T_{q_1}^{(p)}(F)) * T_{q_2}^{(p)}(G))_q)(y|w) \\ &= \delta T_{q_1'}^{(p)}(F)\left(\frac{y}{\sqrt{2}}\middle|\frac{w}{\sqrt{2}}\right) T_{q_2'}^{(p)}(G)\left(\frac{y}{\sqrt{2}}\right) \\ &\quad + T_{q_1'}^{(p)}(F)\left(\frac{y}{\sqrt{2}}\right) \delta T_{q_2'}^{(p)}(G)\left(\frac{y}{\sqrt{2}}\middle|\frac{w}{\sqrt{2}}\right) \end{aligned}$$

for s-a.e. $y \in C_0(B)$, where q_j' are nonzero extended real numbers satisfying $(1/q) + (1/q_j) = (1/q_j')$ for $j = 1, 2$.

Proof. Using Theorem 3.5, we obtain the second equality. Taking the Fourier-Feynman transform, with respect to the first argument of the variation, of the expression in equation (4.4) shows that the first expression in (4.7) is given by (the proof is similar to that of Theorem 3.5)

$$\int_{\Delta_n \times H^n} \sum_{m_1+\cdots+m_{n+1}=m} \int_{\Delta_{m;m_1,\dots,m_{n+1}} \times H^m} \frac{i}{\sqrt{2}} W(w;\vec{s},\vec{h};\vec{t},\vec{k})$$
$$\exp\left\{\frac{i}{\sqrt{2}} W(y;\vec{s},\vec{h};\vec{t},\vec{k}) - \frac{i}{2q_1'} V(\vec{s},\vec{h}) - \frac{i}{2q_2'} V(\vec{t},\vec{k})\right\} d\eta(\vec{t},\vec{k})\, d\mu(\vec{s},\vec{h}).$$

Now, by Theorems 3.1 and 4.1, the last expression is equal to the third expression in (4.7). □

Theorem 4.8. *Let F, G, p, q_1, q_2 and q be given as in Theorem 4.4. Then for s-a.e. $y \in C_0(B)$*

$$(4.8)\quad T_q^{(p)}(\delta(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y|\cdot))(w) = \delta(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y|w)$$

for s-a.e. $w \in C_0(B)$.

Proof. By the same methods as in the proof of Theorem 4.3, (4.8) is obtained. □

Theorem 4.9. *Let F, G, p, q_1, q_2 and q be given as in Theorem 4.4. Then for s-a.e. $w_1, w_2 \in C_0(B)$*

$$(4.9)\quad \begin{aligned}(T_q^{(p)}(\delta T_{q_1}^{(p)}(F)(\cdot|w_1) * \delta T_{q_2}^{(p)}(G)(\cdot|w_2))_q)(y) &= T_q^{(p)}(\delta T_{q_1}^{(p)}(F)(\cdot|w_1))\left(\frac{y}{\sqrt{2}}\right) T_q^{(p)}(\delta T_{q_2}^{(p)}(G)(\cdot|w_2))\left(\frac{y}{\sqrt{2}}\right)\\ &= \delta T_{q_1'}^{(p)}(F)\left(\frac{y}{\sqrt{2}}\middle| w_1\right) \delta T_{q_2'}^{(p)}(G)\left(\frac{y}{\sqrt{2}}\middle| w_2\right)\end{aligned}$$

for s-a.e. $y \in C_0(B)$, where q_j' are nonzero extended real numbers satisfying $(1/q) + (1/q_j) = (1/q_j')$ for $j = 1, 2$.

Proof. The first equality is obtained by Theorem 3.6 and the second equality is obtained by Theorem 4.2. □

Next we extend the results in Theorems 4.1 through 4.9 to the functionals in $\mathcal{S}_B''$. We give four series expressions ((4.11), (4.15), (4.16), (4.17)) and five relationships ((4.12), (4.13) and the three relationships in Theorem 4.13). Again, each of the four series expressions can be expressed explicitly using the expressions (4.1), (4.4), (4.5) and (4.6), respectively.

Theorem 4.10. *Let $F \in \mathcal{S}_B''$ be given by (2.15) with*

$$(4.10)\qquad \sum_{n=1}^{\infty} \int_{\Delta_n \times H^n} \left(\sum_{j=1}^{n} \sqrt{s_j}|h_j| \right) d|\mu_n|(\vec{s}, \vec{h}) < \infty.$$

Let $1 \le p < \infty$, and let q be a nonzero extended real number. Then for s-a.e. $w \in C_0(B)$ the first variation $\delta F(y|w)$ exists, belongs to $\mathcal{S}_B''$ as a function of y and is given by the formula

$$(4.11)\qquad \delta T_q^{(p)}(F)(y|w) = \sum_{n=1}^{\infty} \delta T_q^{(p)}(F_n)(y|w)$$

for s-a.e. $y \in C_0(B)$.

Proof. Note that

$$\delta T_q^{(p)}(F)(y|w) = \frac{\partial}{\partial t}\bigg(\sum_{n=1}^{\infty}(T_q^{(p)}(F_n))(y|w)\bigg)\bigg|_{t=0}.$$

Hence we obtain (4.11) provided we pass the differentiation under the infinite summation. But this can be justified by (4.10). □

The proofs of Theorems 4.11 through 4.13 are similar to that of Theorem 4.10 and so we will not give those here. For example, to prove Theorems 4.11 and 4.13, first justify to pass the integration or the differentiation under the infinite summation, and in the next step use the relationships (4.2), (4.3), (4.7), (4.8) and (4.9).

Theorem 4.11. *Let F and p be given as in Theorem 4.10, and let q_1 and q_2 be nonzero extended real numbers. Then, for s-a.e. $y, w \in C_0(B)$,*

(4.12)
$$T_{q_1}^{(p)}(\delta T_{q_2}^{(p)}(F)(\cdot|w))(y) = \delta T_{q_1}^{(p)}(T_{q_2}^{(p)}(F))(y|w) = \delta T_q^{(p)}(F)(y|w),$$

$$T_{q_1}^{(p)}(\delta T_{q_2}^{(p)}(F)(y|\cdot))(w) = \delta T_{q_2}^{(p)}(F)(y|w) \tag{4.13}$$

where q is a nonzero extended real number satisfying $(1/q_1)+(1/q_2) = (1/q)$.

Theorem 4.12. *Let F and G be given as in Theorem 3.4 with corresponding measures μ_n in $\mathcal{M}_n''$ and η_m in $\mathcal{M}_m''$ where μ_n satisfies (4.10) and η_m satisfies*

$$\sum_{m=1}^{\infty}\int_{\Delta_m\times H^m}\bigg(\sum_{j=1}^{m}\sqrt{t_j}|k_j|\bigg)\,d|\eta_m|(\vec{t},\vec{k}) < \infty. \tag{4.14}$$

Let p, q_1 and q_2 be given as in Theorem 4.11, and let q be any nonzero real number. Then for s-a.e. $y, w, w_1, w_2 \in C_0(B)$,

$$\begin{aligned}\delta(T_{q_1}^{(p)}(F) * T_{q_2}^{(p)}(G))_q(y|w) \\ = \sum_{r=0}^{\infty}\sum_{n+m=r}\delta(T_{q_1}^{(p)}(F_n) * T_{q_2}^{(p)}(G_m))_q(y|w),\end{aligned} \tag{4.15}$$

$$(4.16)\quad (\delta T_{q_1}^{(p)}(F)(\cdot|w_1) * \delta T_{q_2}^{(p)}(G)(\cdot|w_2))_q(y) = \sum_{r=0}^{\infty} \sum_{n+m=r} (\delta T_{q_1}^{(p)}(F_n)(\cdot|w_1) * \delta T_{q_2}^{(p)}(G_m)(\cdot|w_2))_q(y),$$

and

$$(4.17)\quad (\delta T_{q_1}^{(p)}(F)(y|\cdot) * \delta T_{q_2}^{(p)}(G)(y|\cdot))_q(w) = \sum_{r=0}^{\infty} \sum_{n+m=r} (\delta T_{q_1}^{(p)}(F)(y|\cdot) * \delta T_{q_2}^{(p)}(G)(y|\cdot))_q(w).$$

Theorem 4.13. *Let F, G, p, q_1, q_2 and q be given as in Theorem* 4.11. *Then for s-a.e. $y, w, w_1, w_2 \in C_0(B)$, relationships* (4.7), (4.8) *and* (4.9) *hold.*

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