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RINGS OVER WHICH ALL MODULES ARE STRONGLY GORENSTEIN PROJECTIVE

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ABSTRACT. One of the main results of this paper is the characterization of the rings over which all modules are strongly Gorenstein projective. We show that these kinds of rings are very particular cases of the well known quasi-Frobenius rings. We give examples of rings over which all modules are Gorenstein projective but not necessarily strongly Gorenstein projective.

1. Introduction. Throughout this paper all rings are commutative with identity element and all modules are unital. It is convenient to use "m-local" to refer to (not necessarily Noetherian) rings with a unique maximal ideal m.

For background on the following definitions, we refer the reader to [**3, 5**–**7**].

Definition 1. A module M is said to be *Gorenstein projective* if there exists an exact sequence of projective modules

$$\mathbf{P} = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

such that $M \cong \operatorname{Im}(P_0 \to P^0)$ and such that $\operatorname{Hom}(-,Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective module.

The exact sequence $\mathbf{P}$ is called a *complete projective* resolution.

The *Gorenstein injective* modules are defined dually.

Recently in [**3**], the authors studied a simple particular case of Gorenstein projective and injective modules, which are defined, respectively, as follows:

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Definition 1.2 [**3**]. A module M is said to be *strongly Gorenstein projective*, if there exists a complete projective resolution of the form

$$\mathbf{P} = \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that $M \cong \mathrm{Im}\,(f)$.

The exact sequence $\mathbf{P}$ is called a *strongly complete projective* resolution.

The *strongly Gorenstein injective* modules are defined dually.

The principal role of the strongly Gorenstein projective and injective modules is to give a simple characterization of Gorenstein projective and injective modules, respectively, as follows:

Theorem 1.3 [**3**, Theorem 2.7]. *A module is Gorenstein projective, respectively injective, if and only if it is a direct summand of a strongly Gorenstein projective, respectively injective, module.*

The importance of this last result manifests in showing that the strongly Gorenstein projective and injective modules have simpler characterizations than their Gorenstein correspondent modules. For instance:

Proposition 1.4 [**3**, Proposition 2.9]. *A module M is strongly Gorenstein projective if and only if there exists a short exact sequence of modules* $0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$, *where P is projective, and* $\mathrm{Ext}\,(M, Q) = 0$ *for any projective module Q.*

The aim of this paper is to investigate the two following classes of rings:

1. The rings over which all modules are Gorenstein projective, respectively injective, which are called *G-semisimple* rings (please see Proposition 2.1).

2. The rings over which all modules are strongly Gorenstein projective, respectively injective, which are called *SG-semisimple* rings (please see Proposition 3.1).

In Section 2, we show that the G-semisimple rings are just the well-known quasi-Frobenius rings, i.e., Noetherian and self-injective rings. The SG-semisimple rings are then particular cases of the quasi-Frobenius rings. In Section 3, we characterize the SG-semisimple rings. Namely, we show that an m-local ring is SG-semisimple if and only if it has at most one proper nonzero ideal; in general, a ring is SG-semisimple if and only if it is a finite direct product of local SG-semisimple rings.

Before starting, we need to recall some useful results about quasi-Frobenius rings (for more details about these kinds of rings, see for example [**1, 8**]). The quotient ring R/I, where R is a principal ideal domain and I is any nonzero ideal of R, is a classical example of a quasi-Frobenius ring [**9**, Exercise 9.24]. The quasi-Frobenius rings have several characterizations. Here, we need the following:

Theorem 1.5 [**8**, Theorems 1.50, 7.55, and 7.56]. *For a ring R, the following are equivalent*:

1. *R is quasi-Frobenius*;

2. *R is Artinian and self-injective*;

3. *Every projective R-module is injective*;

4. *Every injective R-module is projective;*

5. *R is Noetherian and, for every ideal I, $\mathrm{Ann}\,(\mathrm{Ann}\,(I)) = I$, where $\mathrm{Ann}\,(I)$ denotes the annihilator of I.*

Quasi-Frobenius rings are particular cases of the perfect rings, i.e., the rings over which all flat modules are projective. Namely, a ring is quasi-Frobenius if and only if it is perfect and self-injective [**8**, Theorem 6.39]. The perfect rings are introduced by Bass in [**2**]. They have the following characterizations (needed later):

Theorem 1.6 [**2**, Theorem P and Example 6, page 476]. *For a ring R, the following are equivalent*:

1. *R is perfect*;

2. *Every direct limit* (*with directed index set*) *of projective R-modules is projective*;

3. *R is a finite direct product of local rings, each with T-nilpotent maximal ideal (i.e., if we pick a sequence* $a_1, a_2, \dots$ *of elements in the maximal ideal, then for some index* j, $a_1 a_2 \dots a_j = 0$*).*

From Theorems 1.5 and 1.6 above and [**8**, Lemma 5.64], we may give the following structural characterization of quasi-Frobenius rings, which will be used later:

Proposition 1.7. *A ring R is quasi-Frobenius if and only if* $R = R_1 \times \cdots \times R_n$*, where each* R_i *is a local quasi-Frobenius ring.*

2. ***G*-semisimple rings.** In this section we investigate the G-semisimple rings, i.e., the rings that satisfy each of the following equivalent conditions:

Proposition 2.1. *Let R be a ring. The following are equivalent*:

1. *Every R-module is Gorenstein projective*;
2. *Every R-module is Gorenstein injective.*

Proof. We prove the implication (1) $\Rightarrow$ (2), and the proof of the converse implication is analogous.

Assume that every module is Gorenstein projective. Then, any injective module is projective (since, as a Gorenstein projective module, it embeds in a projective module). This is equivalent, by Theorem 1.5, to saying that every projective module is injective. Then, every complete projective resolution is also a complete injective resolution, and therefore, every R-module is Gorenstein injective. □

Note that the equivalence in Proposition 2.1 is already known when R is Noetherian, and that each of the conditions (1) and (2) is equivalent to the ring being quasi-Frobenius (see for example [**6**, Theorem 12.3.1]). Next, we show how Proposition 2.1 and its proof show that a G-semisimple ring is the same as a quasi-Frobenius ring.

Theorem 2.2. *For any ring R, the following are equivalent*:

1. *R is G-semisimple*;

2. *Every Gorenstein injective R-module is Gorenstein projective*;

3. *Every strongly Gorenstein injective R-module is strongly Gorenstein projective*;

4. *Every Gorenstein projective R-module is Gorenstein injective*;

5. *Every strongly Gorenstein injective R-module is strongly Gorenstein projective*;

6. *R is quasi-Frobenius.*

Proof. First note that a G-semisimple ring is Noetherian. Indeed, from the proof of Proposition 2.1, we have that if R is a G-semisimple ring, then every projective R-module is injective. This means from Theorem 1.5 that R is quasi-Frobenius and so is Noetherian. This gives a proof of the implication (1) $\Rightarrow$ (6). For the proof of the remaining implications use also Proposition 2.1 and its proof. □

We have the following relationship between semisimple rings and G-semisimple rings; compare to [**9**, Exercise 9.2].

Proposition 2.3. *A G-semisimple ring is semisimple if and only if it has finite global dimension.*

Proof. Follows from the fact that a Gorenstein projective module is projective if and only if it has finite projective dimension [**7**, Proposition 2.27]. □

Finally, it is important to say that numerous examples exist of G-semisimple rings which are not semisimple, for instance $\mathbf{Z}/4\mathbf{Z}$.

3. ***SG*-semisimple rings.** We investigate, in this section, SG-semisimple rings, i.e., rings that satisfy each of the following equivalent conditions.

Proposition 3.1. *Let R be a ring. The following are equivalent*:

1. *Every R-module is strongly Gorenstein projective*;
2. *Every R-module is strongly Gorenstein injective.*

Proof. It suffices to prove the implication (1) $\Rightarrow$ (2), and the proof of the converse implication is analogous.

Assume that every module is strongly Gorenstein projective. Then, by Theorem 2.2, R is G-semisimple (i.e., quasi-Frobenius). Thus, we can show that a strongly complete projective resolution is also a strongly complete injective resolution. □

Naturally, an SG-semisimple ring is G-semisimple (i.e., quasi-Frobenius). Later, we give examples of SG-semisimple rings and other examples of G-semisimple rings which are not SG-semisimple (see Corollaries 3.9 and 3.10). Before that, we give a characterization of SG-semisimple rings. We begin by a structure theorem. For that, we need the following lemma.

Lemma 3.2. *Let $R = R_1 \times \cdots \times R_n$ be a finite direct product of rings R_i. An R-module M is* (*strongly*) *Gorenstein projective if and only if $M = M_1 \oplus \cdots \oplus M_n$, where each M_i is a* (*strongly*) *Gorenstein projective R_i-module.*

Proof. This follows from the structure of (projective) modules and homomorphisms over a finite direct product of rings (see for example [**4**, subsection 2.6]). □

Theorem 3.3. *A ring R is SG-semisimple if and only if $R = R_1 \times \cdots \times R_n$, where each R_i is a local SG-semisimple ring.*

Proof. The result is a consequence of Proposition 1.7 and Lemma 3.2 above. □

Theorem 3.3 leads us to restrict the study of the SG-semisimple rings to the local SG-semisimple rings.

Lemma 3.4. *Let R be an m-local ring, and let $x \neq 0$ be a zero-divisor element of R. If the ideal xR is strongly Gorenstein projective, then* $\operatorname{Ann}(xR) \cong xR$ *and therefore* $\operatorname{Ann}(\operatorname{Ann}(xR)) = \operatorname{Ann}(xR)$.

Particularly, if $xR = m$, we get $\operatorname{Ann}(m) = m$.

Proof. Since xR is strongly Gorenstein projective, there exists, by [**3**, Proposition 2.9], a short exact sequence of R-modules

$$(\star) \qquad 0 \longrightarrow xR \longrightarrow P \longrightarrow xR \longrightarrow 0,$$

where P is projective, then free (since R is m-local). In the sequence $(\star)$ xR is finitely generated, then so is the free R-module P. Thus, there exists a nonzero positive integer n such that $P \cong R^n$. Hence, we get the following exact sequence:

$$(\star\star) \qquad 0 \longrightarrow xR \longrightarrow R^n \longrightarrow xR \longrightarrow 0.$$

Consider also the following canonical short exact sequence of R-modules: $0 \to \operatorname{Ann}(xR) \to R \to xR \to 0$. From Schanuel's lemma [**9**, Theorem 3.62], we have:

$$\operatorname{Ann}(xR) \oplus R^n \cong R \oplus (xR).$$

Then, since R is m-local, the minimal generating sets of both $\operatorname{Ann}(xR) \oplus R^n$ and $R \oplus (xR)$ have the same numbers of elements which is necessarily 2. On the other hand, since x is a zero-divisor element of R, $\operatorname{Ann}(xR) \neq 0$. Thus, $\operatorname{Ann}(xR)$ is generated by at least one element, and so $\operatorname{Ann}(xR) \oplus R^n$ is generated by at least $n+1$ elements. Then, by the reason above, n must equal 1. So the sequence $(\star\star)$ becomes: $0 \to xR \to R \xrightarrow{f} xR \to 0$. Now, let $\alpha \in R$ with $f(1) = \alpha x$. Since f is surjective, there exists $\beta \in R$ such that $f(\beta) = x$. So, $x = \beta\alpha x$, and then $(1-\beta\alpha)x = 0$, which means that $(1-\beta\alpha) \in \operatorname{Ann}(xR) \subseteq m$. Then, $\beta\alpha$ is invertible and so is α. This implies that:

$$\operatorname{Ker} f = \{y \in R \mid 0 = yf(1) = y\alpha x\} = \operatorname{Ann}(xR).$$

Consequently, $xR \cong \operatorname{Ker} f = \operatorname{Ann}(xR)$. Therefore, $\operatorname{Ann}(\operatorname{Ann}(xR)) = \operatorname{Ann}(xR)$, as desired.

Now, if $m = xR$, then $m = xR \subseteq \operatorname{Ann}(\operatorname{Ann}(xR)) = \operatorname{Ann}(xR) \subseteq m$. □

Lemma 3.5. *Let R be an m-local ring, and let I be a nonzero proper ideal of R. If R/I is a strongly Gorenstein projective R-module, then I is a cyclic strongly Gorenstein projective ideal generated by a zero-divisor element of R.*

Proof. Since R is an m-local ring and similarly to the first part of the proof of Lemma 3.4 above, we get a short exact sequence of R-modules:

$$(*) \qquad 0 \longrightarrow R/I \longrightarrow R^n \longrightarrow R/I \longrightarrow 0,$$

where n is a nonzero positive integer.

And also, the same argument as in the proof of Lemma 3.4 above, and using the short exact sequence of R-modules:

$$(**) \qquad 0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

we get $n = 1$ and $I = xR$ for some zero-divisor element x of R.

Now, to show that I is strongly Gorenstein projective, note at first that it is Gorenstein projective (by the sequence $(**)$ and from [**7**, Theorem 2.5]). Then, $\operatorname{Ext}(I, P) = 0$ for every projective R-module P (by [**7**, Proposition 2.3]). On the other hand, the two sequences $(*)$ and $(**)$ with the Horseshoe lemma [**9**, Lemma 6.20] give the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & Q & \longrightarrow & I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & R \oplus R & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R/I & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that Q is a projective R-module. Therefore, by the top horizontal sequence and Proposition 1.4, I is a strongly Gorenstein projective ideal. $\square$

Lemma 3.6. *If R is a local G-semisimple ring, then every R-module M is of the form: $M = R^{(I)} \oplus N$, where I is an index set and N is an R-module with* $\mathrm{Ann}\,(x) \neq 0$ *for every element x of N.*

Proof. We may assume that M admits an element x such that $rx \neq 0$ for all $0 \neq r \in R$. Consider the set E of all free submodules of M. The set E is not empty, since xR is a free submodule of M. On the other hand, since R is a local G-semisimple ring and from Theorem 1.6, a direct limit of free R-modules is a free R-module. Then, for every subchain E_i of E, $\cup E_i$ is a free submodule of M. Then, by Zorn's lemma, E admits a maximal element F. We may set $F = R^{(I)}$ which is injective (since R is G-semisimple). Then, F is a direct summand of M and so $M = F \oplus N$ for some R-module N. If there exists $x \in N$ such that $rx \neq 0$ for all $r \in R$, then $xR \cong R$ is injective and then a direct summand of N. Hence, there exists an R-module N' such that $N = xR \oplus N'$, and so $M = F \oplus N = F \oplus xR \oplus N'$. But, the free submodule $F \oplus xR$ of M contradicts the maximality of F. $\square$

The main result in this section is the following characterization of local SG-semisimple rings.

Theorem 3.7. *Let R be an m-local ring. The following are equivalent*:

1. *R is SG-semisimple*;
2. *R/m is a strongly Gorenstein projective R-module*;
3. *R has a most one nonzero proper ideal* (*which is necessarily m*).

Proof. (1) $\Rightarrow$ (2). By definition.

(2) $\Rightarrow$ (3). From Lemma 3.5, $m = xR$ is a cyclic strongly Gorenstein projective ideal and x is zero-divisor. Then, by Lemma 3.4, $m^2 = 0$. Therefore, a standard argument shows that either $m = 0$ or m is the unique nonzero proper ideal of R.

(3) $\Rightarrow$ (1). We may assume that R is not a field. Clearly $m = xR$ (for some $0 \neq x \in R$) and $m^2 = 0$. Then, from Theorem 1.5, R is G-semisimple (i.e., quasi-Frobenius), and so m is a Gorenstein projective ideal of R. Hence, by [**7**, Proposition 2.3], $\mathrm{Ext}\,(m, Q) = 0$ for every projective R-module Q. Then, by the short exact sequence

$$0 \longrightarrow \mathrm{Ann}\,(m) = m \longrightarrow R \longrightarrow m \longrightarrow 0,$$

and from Proposition 1.4, m is a strongly Gorenstein projective R-module.

Now, consider an arbitrary R-module M. By Lemma 3.6, there exists an index set I such that $M \cong R^{(I)} \oplus N$, where N is an R-module with $\mathrm{Ann}\,(y) \neq 0$ for every nonzero element $y \in N$. Then, necessarily $xN = 0$, and so $N \cong (R/m)^{(J)}$ for some index set J. Since $R/m \cong m$ is a strongly Gorenstein projective R-module and, by [**3**, Proposition 2.2], N is a strongly Gorenstein projective R-module. Therefore, M is a strongly Gorenstein projective R-module. □

Corollary 3.8. *A ring R is SG-semisimple if and only if $R = R_1 \times \cdots \times R_n$, where each R_i is a ring with at most one nonzero proper ideal.*

Proof. Combine Theorems 3.3 and 3.7. □

We end with some examples of G-semisimple and SG-semisimple rings.

Corollary 3.9. *For every principal ideal domain R and every nonzero prime ideal p of R, the ring R/p^2 is local SG-semisimple.*

The following result shows how to construct G-semisimple rings which are not SG-semisimple.

Corollary 3.10. *For every principal ideal domain R and every nonzero prime ideal p of R, the ring R/p^n, where $n \geq 3$, is a local G-semisimple, but it is not SG-semisimple.*

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