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BLOWING-UP PROPERTIES OF THE POSITIVE PRINCIPAL EIGENVALUE FOR INDEFINITE ROBIN-TYPE BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we consider the positive principal eigenvalue for some linear elliptic eigenvalue problem with Robin-type boundary conditions having indefinite coefficients, where its asymptotic behavior for indefinite varying weights is investigated. The aim of this paper is to study necessary and sufficient conditions for the positive principal eigenvalue to blow up to infinity. The analysis is based on variational characterization of the positive principal eigenvalue.

1. Introduction and results Let Ω be a bounded domain of $\mathbf{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. This paper is devoted to the study of the following Robin-type eigenvalue problem with indefinite weights.

$$\begin{cases} -\Delta\varphi = \lambda g(x)\varphi & \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial\mathbf{n}} = \lambda h(x)\varphi & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, $\Delta = \sum_{j=1}^N \partial^2/\partial x_j^2$ is the usual Laplacian in $\mathbf{R}^N$, λ is a real eigenvalue parameter, $g \in L^\infty(\Omega)$, $h \in W^{1-(1/p),\, p}(\partial\Omega)$ for any $p > 1$, and $\mathbf{n}$ is the unit outer normal to $\partial\Omega$. By $L^p(\Omega)$, $1 \leq p \leq \infty$, we denote the usual Lebesgue space with norm $\|\cdot\|_p$, by $W^{m,p}(\Omega)$, $m = 1, 2, 3, \dots, p > 1$, the usual Sobolev space with norm $\|\cdot\|_{m,p}$, and by $W^{1-(1/p),\, p}(\partial\Omega)$, $p > 1$, the set of traces on $\partial\Omega$ of functions in $W^{1,p}(\Omega)$, equipped with norm $\|\cdot\|_{1-(1/p),\, p,\, \partial\Omega}$. It is well known ([**1**, Theorem 7.53]) that the trace operator T defined by $Tu = u|_{\partial\Omega}$ is an isomorphism and a homeomorphism of $W^{1,p}(\Omega)$ onto $W^{1-(1/p),\, p}(\partial\Omega)$

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for each $p > 1$. It should be remarked that functions g and h may both change sign, and that, by Sobolev's imbedding theorem, the hypothesis of h implies that $h \in C^{\theta}(\partial\Omega)$ for any $0 < \theta < 1$.

Throughout this paper, we assume either that

$$g > 0 \text{ on a set of positive measure,} \tag{1.2}$$

or that

$$h(x_0) > 0 \text{ for some } x_0 \in \partial\Omega. \tag{1.3}$$

By an eigenfunction φ for an eigenvalue λ of (1.1), we mean that $\varphi \in W^{2,p}(\Omega)$ for any $p > N$. A principal eigenvalue of (1.1) means an eigenvalue with an eigenfunction which does not change sign in Ω. By the strong maximum principle ([**9**, Theorem 8.19]) and Hopf's boundary point lemma ([**9**, Lemma 3.4]), the nonnegative principal eigenfunction is strictly positive everywhere in $\overline{\Omega}$.

It is clear that $\lambda = 0$ is a principal eigenvalue of (1.1). In addition, we can prove that there exists a positive principal eigenvalue of (1.1) if and only if

$$\int_{\Omega} g\,dx + \int_{\partial\Omega} h\,da < 0, \tag{1.4}$$

and moreover that it is unique, denoted by $\lambda_1(g,h)$, and is characterized by the variational formula

$$\lambda_1(g,h) = \inf\left\{ \frac{\int_{\Omega} |\nabla v|^2\,dx}{\int_{\Omega} gv^2\,dx + \int_{\partial\Omega} hv^2\,da} : v \in W^{1,2}(\Omega), \right. \\ \left. \int_{\Omega} gv^2\,dx + \int_{\partial\Omega} hv^2\,da > 0 \right\}. \tag{1.5}$$

Here, da denotes the surface element of $\partial\Omega$. Indeed, if (1.4) is satisfied, then the infimum (1.5) is positive and is attained by a nonnegative function $\varphi_1 \in W^{1,2}(\Omega)$, and is a weak solution of (1.1) for $\lambda = \lambda_1(g,h)$:

$$\int_{\Omega} \nabla\varphi_1 \nabla w\,dx - \lambda_1(g,h)\left(\int_{\Omega} g\varphi_1 w\,dx + \int_{\partial\Omega} h\varphi_1 w\,da \right) = 0 \\ \text{for all } w \in C^1(\overline{\Omega}).$$

Moreover, by elliptic regularity, we have that $\varphi_1 \in W^{2,p}(\Omega)$ for any $p > N$, meaning that φ_1 is a principal eigenfunction of (1.1). These existence, uniqueness and characterization results for positive principal eigenvalues of (1.1) have been obtained in [**11**] under the stronger assumption that $g \in C^\theta(\overline{\Omega})$ and $h \in C^{1+\theta}(\partial\Omega)$ for some $0 < \theta < 1$. However, the results remain true under our assumptions of g and h, which will be verified in Section 2. It should be mentioned that the case that $h \equiv 0$ is due to Brown and Lin [**4**, Theorem 3.13] (also see Afrouzi and Brown [**2**]) and the case of nonnegative coefficients was considered by Amann [**3**, Theorem 2.2].

In this paper, we study necessary and sufficient conditions on a sequence of weight functions g_j and h_j to give the blowing-up behavior of $\lambda_1(g_j, h_j)$

$$\lim_{j\to\infty} \lambda_1(g_j, h_j) = \infty. \tag{1.6}$$

In the Dirichlet condition case $\varphi|_{\partial\Omega} = 0$, it is well known (see Brown and Lin [**4**]) that there exists a unique positive principal eigenvalue, provided that condition (1.2) is satisfied. Under the condition that

$$\sup_{j\geq 1} \|g_j\|_\infty < \infty, \tag{1.7}$$

Cantrell and Cosner [**5**, Theorem 3.1] proved that the positive principal eigenvalue for g_j goes to infinity as $j \to \infty$ if and only if

$$\limsup_{j\to\infty} \int_\Omega g_j \psi \, dx \leq 0 \tag{1.8}$$

for all $\psi \in L^1(\Omega)$ satisfying that $\psi \geq 0$ almost everywhere in Ω. Meanwhile, in [**12**], the Neumann case $h \equiv 0$ was considered under (1.7), in which condition (1.8) was also verified to be necessary but no longer sufficient in order to have that $\lambda_1(g_j, 0) \to \infty$. The first main purpose of this paper is to consider blowing-up behavior (1.6) in the case $h \not\equiv 0$.

By $L^p(\partial\Omega)$, $1 \leq p \leq \infty$, we denote the set of functions u defined on $\partial\Omega$ whose usual norm $\|u\|_{p,\partial\Omega}$ is finite. Our first main result is on necessary conditions for (1.6). Theorem 1.1 is a generalization of [**12**, Theorem 1.1].

Theorem 1.1. *Assume that g_j and h_j satisfy that*

$$\sup_{j\geq 1}\|g_j\|_\infty < \infty \text{ and } \sup_{j\geq 1}\|h_j\|_{\infty,\partial\Omega} < \infty. \tag{1.9}$$

Then, the following condition is necessary in order to have (1.6):

$$\begin{cases} \limsup_{j\to\infty}\int_\Omega g_j\psi\,dx \leq 0 & \text{for all } \psi\in L^1(\Omega) \text{ such that } \psi\geq 0 \\ & \text{a.e. in } \Omega, \\ \limsup_{j\to\infty}\int_{\partial\Omega} h_j\phi\,da \leq 0 & \text{for all } \phi\in L^1(\partial\Omega) \text{ such that } \phi\geq 0 \\ & \text{a.e. on } \partial\Omega. \end{cases} \tag{1.10}$$

Our second main result is on sufficient conditions for (1.6). Theorem 1.2 is a generalization of [**12**, Theorem 2.2].

Theorem 1.2. *Assume that g_j and h_j satisfy* (1.9)*. Then, the following two assertions hold.*

(i) *If we suppose that*

$$\limsup_{j\to\infty}\left(\int_\Omega g_j\,dx + \int_{\partial\Omega} h_j\,da\right) < 0, \tag{1.11}$$

then condition (1.10) *is sufficient in order to have* (1.6)*.*

(ii) *If we suppose additionally that*

$$\sup_{j\geq 1}\|h_j\|_{1-\frac{1}{p},\,p,\,\partial\Omega} < \infty \text{ for any } p > 1, \tag{1.12}$$

then the following weaker condition is sufficient in order to have (1.6)*.*

$$\limsup_{j\to\infty}\int_\Omega g_j v\,dx \leq 0 \text{ and } \limsup_{j\to\infty}\int_{\partial\Omega} h_j v\,da \leq 0 \tag{1.13}$$

for all $v\in C^1(\overline{\Omega})$ such that $v\geq 0$ in $\overline{\Omega}$.

Remark 1.3. We see from (1.4) and (1.10) that the condition

$$\limsup_{j\to\infty}\left(\int_\Omega g_j\,dx + \int_{\partial\Omega} h_j\,da\right) \leq 0 \tag{1.14}$$

is necessary for (1.6). In the case when the equality holds in (1.14), condition (1.10) is no longer sufficient for (1.6), see [**12**, Example 2.1]. A further discussion of this matter will be given in the latter part of this paper, see Theorems 1.5 and 1.7.

The following result is a direct consequence of Theorems 1.1 and 1.2, which tells us that condition (1.8) due to Cantrell and Cosner is necessary and sufficient in order to have (1.6) for a class of coefficients h.

Corollary 1.4. *Let h be such that $\int_{\partial\Omega} h\,da < 0$ and g_j satisfy* (1.7). *Then, condition* (1.8) *is necessary and sufficient in order to have that $\lambda_1(g_j, h) \to \infty$.*

Next, let us consider the Neumann case $h \equiv 0$ and focus our attention on the case when

$$\lim_{j\to\infty} \int_\Omega g_j\,dx = 0. \tag{1.15}$$

This case is quite delicate and is not discussed by Theorem 1.2. Indeed, the following example satisfies (1.8) but does not give us the blowing-up behavior $\lambda_1(g_j, 0) \to \infty$:

$$g_j(x) = \frac{1}{j}\chi_{(0,\,1-(1/j))}(x) - \chi_{\Omega\setminus(0,\,1-(1/j))}(x), \quad x \in \Omega = (0,1) \subset \mathbf{R},$$

see [**12**, Example 2.1]. Here χ_A is the characteristic function of a subset $A \subset \Omega$. In [**12**, Theorem 2.6], we imposed the following additional condition to (1.8) for ensuring the blowing-up behavior:

$$\lim_{j\to\infty} \frac{\|g_j\|_{q'}^2}{\left|\int_\Omega g_j\,dx\right|} = 0, \tag{1.16}$$

where

$$q' = \begin{cases} \dfrac{2N}{N+2} & N \geq 3, \\ 1 & N = 1, 2. \end{cases}$$

The second main purpose of this paper is to show how condition (1.16) characterizes the blowing-up behavior $\lambda_1(g_j, 0) \to \infty$ in the one-dimensional case $N = 1$. We denote by $g^+ = \max(g, 0)$ the positive part of g and put $g^- = g^+ - g$ as the negative part. Our third main result is the following.

Theorem 1.5. *Let $\Omega \subset \mathbf{R}$ be an open interval. Assume that condition* (1.7) *and the following two conditions hold.*

(1.17)
$$\lim_{j\to\infty} \int_\Omega (g_j)^+ \, dx = 0,$$
(1.18)
$$\gamma_j := \frac{\int_\Omega (g_j)^- \, dx}{\int_\Omega (g_j)^+ \, dx} \geq 1 + c\left(\int_\Omega (g_j)^+ \, dx \right)^\sigma \text{ for all } j \geq 1 \text{ large enough,}$$

with some constants $c > 0$ and $0 < \sigma < 1$. Then, we have that $\lambda_1(g_j, 0) \to \infty$.

Remark 1.6. We observe from (1.18) that if $j \geq 1$ is sufficiently large, then $\gamma_j > 1$, so that $\int_\Omega g_j \, dx < 0$, and $\lambda_1(g_j, 0)$ is well defined. Since we have already seen in Theorem 1.2 (i) that $\lambda_1(g_j, 0) \to \infty$ when $\limsup_{j\to\infty} \int_\Omega g_j \, dx < 0$, we have only to consider case (1.15). Conditions (1.17) and (1.18) tell us that the difference between γ_j and 1 decays *at most* like $(\int_\Omega (g_j)^+ \, dx)^\sigma$ as $j \to \infty$.

Now, we restrict our consideration of g_j to a class of simple functions. Let $\Omega = (\alpha, \beta) \subset \mathbf{R}$, and define

$$g_j(x) = k_j \chi_{(a_j, a_j+T_j)}(x) - m_j \chi_{\Omega \setminus (a_j, a_j+T_j)}(x), \tag{1.19}$$

where k_j, m_j, a_j and T_j are all constants satisfying that

$$k_j > 0, \quad m_j > 0, \quad 0 < T_j < \beta - \alpha \text{ and } \alpha \leq a_j \leq \beta - T_j.$$

Our forth main result tells us that Theorem 1.5 does not remain true for $\sigma = 1$, provided that g_j is given by (1.19).

Theorem 1.7. *Let* Ω *and* g_j *be given by* (1.19) *and satisfy that* $\int_\Omega g_j \, dx < 0$. *Additionally if we assume* (1.17) *together with the condition that there exists a constant* $c_0 > 0$ *such that*

$$\gamma_j \leq 1 + c_0 \int_\Omega (g_j)^+ \, dx \text{ for all } j \geq 1 \text{ large enough}, \tag{1.20}$$

then we have that $\lambda_1(g_j, 0)$ *is bounded above.*

At the end of this section, we explain our motivation for studying blowing-up behavior (1.6) from an ecological point of view. We consider the existence of positive solutions to the following diffusive logistic problem of elliptic type.

$$\begin{cases} -\nabla \cdot (\mu \nabla u) = g(x)u - u^2 & \text{in } \Omega, \\ (\mu \nabla u) \cdot \mathbf{n} = h(x)u & \text{on } \partial\Omega. \end{cases} \tag{1.21}$$

Problem (1.21) denotes the steady state u of the population density of some species inhabiting the region Ω, diffusing at rate $\mu > 0$ and taking into account the crowding effect $-u^2$, cf. [**6**]. The sign-changing functions g and h denote, respectively, the local growth or decay rate of the species in the region Ω and the rate of the population flux on the boundary. It has been proved by the method of super and subsolutions ([**11**, Theorem 2.3]) that problem (1.21) has a unique positive solution $u_\mu \in C^{2+\theta}(\overline{\Omega})$ if $0 < \mu < 1/\lambda_1(g, h)$ and no positive solution otherwise, provided that functions $g \in C^\theta(\overline{\Omega})$ and $h \in C^{1+\theta}(\partial\Omega)$, $0 < \theta < 1$, satisfy (1.4). Thus, the open interval $(0, 1/\lambda_1(g, h))$ of the diffusion rate μ can be regarded as an *interval for survival* for the species. For instance, the shorter the length of this interval is, the more unfavorable for the species the environment is. If $\lambda_1(g_j, h_j)$ goes to infinity, then the length of the interval for survival should shrink to *zero* eventually, and this describes a worst environment for the species.

The rest of this paper is organized as follows. In Section 2, we prove the existence and uniqueness of the positive principal eigenvalue $\lambda_1(g, h)$ for (1.1) with coefficients g and h having the weak regularity assumptions. This is a slight modification of the proofs of [**11**, Theorems 2.1 and 2.2]. In Section 3, we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2. In Section 5, we prove Theorems 1.5 and 1.7. The proofs of Theorems 1.1 and 1.2 are by contradiction arguments, following the line of that of Cantrell and Cosner [**5**, Theorem 3.1]. Meanwhile, the proof of Theorem 1.7 is rather direct, where $\lambda_1(g_j, 0)$ is estimated from above by a concrete test function associated with g_j. All the arguments are based on (1.5).

2. Existence and uniqueness of the positive principal eigenvalue. In this section, we prove the following existence and uniqueness result for positive principal eigenvalues of (1.1), which is an extension of [**11**, Theorem 2.2] to the lower regularity case of g and h.

Theorem 2.1. *Assume that either* (1.2) *or* (1.3) *is satisfied. Then, there exists a positive principal eigenvalue of* (1.1) *if and only if condition* (1.4) *holds. Moreover, it is unique and is characterized by formula* (1.5) *if it exists.*

Proof. To prove this theorem, it is sufficient to verify the following proposition for the corresponding auxiliary eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda g(x)\phi + \mu(\lambda)\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = \lambda h(x)\phi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Proposition 2.2. *The following three assertions hold*:

(1) *For any* $\lambda \in \mathbf{R}$, *there exists a unique principal eigenvalue* $\mu_1(\lambda)$ *of* (2.1), *given by the formula*

$$\mu_1(\lambda) = \inf\left\{ \int_\Omega |\nabla v|^2\,dx - \lambda\left(\int_\Omega gv^2\,dx + \int_{\partial\Omega} hv^2\,da\right) : v \in W^{1,2}(\Omega),\ \int_\Omega v^2\,dx = 1\right\}. \tag{2.2}$$

(2) *Mapping* $\lambda \mapsto \mu_1(\lambda)$ *is concave and satisfies that*

$$\mu_1(\lambda) \longrightarrow -\infty, \quad \lambda \longrightarrow \infty. \tag{2.3}$$

(3) *The principal eigenvalue $\mu_1(\lambda)$ has a unique local maximum (i.e.. global maximum). Moreover, the sign of the global maximum point coincides with that of $-(\int_\Omega g\,dx + \int_{\partial\Omega} h\,da)$.*

It is clear that λ is a principal eigenvalue of (1.1) if and only if $\mu_1(\lambda) = 0$. Since $\mu_1(0) = 0$, we see from assertions (2) and (3) of Proposition 2.2 that, in order to have the existence of a positive principal eigenvalue of (1.1), it is necessary and sufficient that $\int_\Omega g\,dx + \int_{\partial\Omega} h\,da < 0$. Furthermore, the uniqueness is straightforward from the concavity. Finally, formula (1.5) comes from (2.2), just as in the proof of [**11**, Theorem 2.2]. Theorem 2.1 now follows once Proposition 2.2 is proved.

It remains to prove Proposition 2.2. Let S_λ be an energy functional associated with (2.1), defined as

$$S_\lambda(v) = \int_\Omega |\nabla v|^2\,dx - \lambda\bigg(\int_\Omega g v^2\,dx + \int_{\partial\Omega} h v^2\,da\bigg), \quad v \in M,$$

where $M = \{v \in W^{1,2}(\Omega) : \|v\|_2 = 1\}$. Just as in [**11**, Lemma 3.1], we can show that S_λ is bounded below. By the standard argument by Smoller [**10**, Chapter 11], we have the existence of a minimizer $\phi_1 \in M$ for S_λ, nontrivial and nonnegative almost everywhere in Ω:

$$S_\lambda(\phi_1) = \inf_{v \in M} S_\lambda(v).$$

By the Lagrange multiplier rule, we can show that $S_\lambda(\phi_1)$ is a principal eigenvalue of (2.1) and ϕ_1 is the eigenfunction which is strictly positive in $\overline{\Omega}$, and we have

$$\int_\Omega \nabla\phi_1 \nabla w\,dx - \lambda\bigg(\int_\Omega g\phi_1 w\,dx + \int_{\partial\Omega} h\phi_1 w\,da\bigg) = S_\lambda(\phi_1)\int_\Omega \phi_1 w\,dx \quad \text{for all } w \in W^{1,2}(\Omega).$$

If μ is another principal eigenvalue of (2.1) for λ and ϕ_2 is a positive eigenfunction with μ, then we see that

$$\int_\Omega \nabla\phi_2 \nabla\phi_1\,dx - \lambda\bigg(\int_\Omega g\phi_2\phi_1\,dx + \int_{\partial\Omega} h\phi_2\phi_1\,da\bigg) = \mu\int_\Omega \phi_2\phi_1\,dx.$$

It follows that

$$(\mu - S_\lambda(\phi_1))\int_\Omega \phi_2\phi_1\,dx = 0.$$

Hence, $\mu = S_\lambda(\phi_1)$, and the uniqueness of the principal eigenvalue follows. Assertion (1) has been verified.

Next, we verify assertion (2). By $\mu_1(\lambda)$, we denote the principal eigenvalue of (2.1), meaning that $\mu_1(\lambda) = S_\lambda(\phi_1)$. The concavity of mapping $\lambda \mapsto \mu_1(\lambda)$ follows from the fact that mapping $\lambda \mapsto S_\lambda(v)$ is affine. For verifying (2.3), we first consider case (1.2), and let g satisfy that $g > 0$ in a measurable set $A \subset \Omega$ with $|A| > 0$. Since $|A| < \infty$, we can choose a compact subset $E \subset \Omega$ such that $E \subset A$ and $|E| > 0$. It follows that

(2.4) $\quad g > 0$ in a compact set $E \subset \Omega$ of positive measure.

For the measurable set E and a constant $\delta > 0$, there exists an open subset $G \subset \Omega$ such that $G \supset E$ and $|G \setminus E| < \delta$. For a constant $\varepsilon > 0$, let

$$G_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, E) < \varepsilon\}.$$

Here, G_ε is an open subset of Ω. Since the inclusion $E \subset G$ is compact, we can choose a constant $\varepsilon_0 > 0$ so small that $G \supset G_{\varepsilon_0} \supset E$.

Now, we put $u_0 \in C^1(\Omega)$ with compact support in Ω as

$$\begin{cases} 0 \le u_0 \le 1 & \text{in } \Omega, \\ u_0 = 1 & \text{in } G_{\varepsilon_0/2}, \\ \operatorname{supp} u_0 \subset G_{\varepsilon_0}. \end{cases}$$

Then, since $|G_{\varepsilon_0} \setminus E| < \delta$, we have

$$\begin{aligned} \int_\Omega g u_0^2\, dx &= \int_{G_{\varepsilon_0}} g u_0^2\, dx \\ &= \int_E g u_0^2\, dx + \int_{G_{\varepsilon_0} \setminus E} g u_0^2\, dx \\ &\ge \int_E g\, dx - \|g^+\|_\infty |G_{\varepsilon_0} \setminus E| \\ &\ge \int_E g\, dx - \delta \|g^+\|_\infty. \end{aligned}$$

By using (2.4), it follows that, for some $\delta > 0$ small,

$$\int_\Omega g u_0^2\, dx \ge \frac{1}{2} \int_E g\, dx > 0.$$

Meanwhile, since u_0 has a compact support in Ω, we have that $\int_{\partial\Omega} hu_0^2\,da = 0$. By letting $v_0 = u_0/\|u_0\|_2$, formula (2.2) gives us assertion (2.3) as follows.

$$\begin{aligned}\mu_1(\lambda) &\le \int_\Omega |\nabla v_0|^2\,dx - \lambda\int_\Omega gv_0^2\,dx - \lambda\int_{\partial\Omega} hv_0^2\,da\\ &= \|u_0\|_2^{-2}\bigg\{\int_\Omega |\nabla u_0|^2\,dx - \lambda\int_\Omega gu_0^2\,dx - \lambda\int_{\partial\Omega} hu_0^2\,da\bigg\}\\ &\le \|u_0\|_2^{-2}\bigg(\int_\Omega |\nabla u_0|^2\,dx - \frac{\lambda}{2}\int_E g\,dx\bigg) \longrightarrow -\infty, \quad \lambda\to\infty.\end{aligned}$$

Secondly, we consider case (1.3). However, the verification of (2.3) is parallel as in the proof of [**11**, Theorem 2.2], because $h\in C(\partial\Omega)$. Assertion (2) has been verified.

Finally, we verify assertion (3). Let $\phi_1(\lambda)$ be a positive eigenfunction with $\mu_1(\lambda)$, normalized as $\|\phi_1(\lambda)\|_2 = 1$, and we have, for all $w\in W^{1,2}(\Omega)$,

$$\begin{aligned}\int_\Omega \nabla\phi_1(\lambda)\nabla w\,dx - \lambda\bigg(\int_\Omega g\phi_1(\lambda)w\,dx + \int_{\partial\Omega} h\phi_1(\lambda)w\,da\bigg)&\\ = \mu_1(\lambda)\int_\Omega \phi_1(\lambda)w\,dx.&\end{aligned}\tag{2.5}$$

Differentiate both sides with respect to λ, and then

$$\begin{aligned}\int_\Omega \nabla\phi_1'(\lambda)\nabla w\,dx - \bigg(\int_\Omega g\phi_1(\lambda)w\,dx + \int_{\partial\Omega} h\phi_1(\lambda)w\,da\bigg)&\\ - \lambda\bigg(\int_\Omega g\phi_1'(\lambda)w\,dx + \int_{\partial\Omega} h\phi_1'(\lambda)w\,da\bigg)&\\ = \mu_1'(\lambda)\int_\Omega \phi_1(\lambda)w\,dx + \mu_1(\lambda)\int_\Omega \phi_1'(\lambda)w\,dx.&\end{aligned}\tag{2.6}$$

From (2.5) and (2.6), we derive, respectively, the assertions

$$\begin{aligned}&\int_\Omega \nabla\phi_1(\lambda)\nabla\phi_1'(\lambda)\,dx\\ &\qquad - \lambda\bigg(\int_\Omega g\phi_1(\lambda)\phi_1'(\lambda)\,dx + \int_{\partial\Omega} h\phi_1(\lambda)\phi_1'(\lambda)\,da\bigg)\\ &\quad = \mu_1(\lambda)\int_\Omega \phi_1(\lambda)\phi_1'(\lambda)\,dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \nabla\phi_1'(\lambda)\nabla\phi_1(\lambda)\,dx \\ &- \Big(\int_\Omega g\phi_1(\lambda)^2\,dx + \int_{\partial\Omega} h\phi_1(\lambda)^2\,da\Big) \\ &- \lambda\Big(\int_\Omega g\phi_1'(\lambda)\phi_1(\lambda)\,dx + \int_{\partial\Omega} h\phi_1'(\lambda)\phi_1(\lambda)\,da\Big) \\ &= \mu_1'(\lambda)\int_\Omega \phi_1(\lambda)^2\,dx + \mu_1(\lambda)\int_\Omega \phi_1'(\lambda)\phi_1(\lambda)\,dx.\end{aligned}$$

It follows that

$$\mu_1'(\lambda) = -\Big(\int_\Omega g\phi_1(\lambda)^2\,dx + \int_{\partial\Omega} h\phi_1(\lambda)^2\,da\Big). \tag{2.7}$$

From (2.7), λ is a critical point for μ_1, that is, $\mu_1'(\lambda) = 0$, if and only if $\int_\Omega g\phi_1(\lambda)^2\,dx + \int_{\partial\Omega} h\phi_1(\lambda)^2 da = 0$. Moreover, by proving the following lemma, a critical point for μ_1 is unique if exists, and so it is the global maximum point.

Lemma 2.3. *If λ_0 is a critical point for μ_1, then we have*

$$\mu_1(\lambda) < \mu_1(\lambda_0), \quad \lambda \neq \lambda_0.$$

Proof. By the definition of $\mu_1(\lambda)$, it follows that $\mu_1(\lambda) \le S_\lambda(\phi_1(\lambda_0))$. Since $\int_\Omega g\phi_1(\lambda_0)^2\,dx + \int_{\partial\Omega} h\phi_1(\lambda_0)^2\,da = 0$, we have that $S_\lambda(\phi_1(\lambda_0)) = S_{\lambda_0}(\phi_1(\lambda_0)) = \mu_1(\lambda_0)$. Hence, $\mu_1(\lambda) \le \mu_1(\lambda_0)$. If we assume to the contrary that $\mu_1(\lambda) = \mu_1(\lambda_0)$ for some $\lambda \neq \lambda_0$, then $\phi_1(\lambda)$ satisfies (2.5) and attains the infimum $\mu_1(\lambda_0)$, and we have, for all $w \in W^{1,2}(\Omega)$,

$$\begin{aligned}\int_\Omega \nabla\phi_1(\lambda)\nabla w\,dx - \lambda_0\Big(\int_\Omega g\phi_1(\lambda)w\,dx + \int_{\partial\Omega} h\phi_1(\lambda)w\,da\Big) \\ = \mu_1(\lambda_0)\int_\Omega \phi_1(\lambda)w\,dx.\end{aligned}$$

It follows that

$$(\lambda - \lambda_0)\Big(\int_\Omega g\phi_1(\lambda)w\,dx + \int_{\partial\Omega} h\phi_1(\lambda)w\,da\Big) = 0.$$

Since $\phi_1(\lambda)$ is strictly positive in $\overline{\Omega}$, it follows, by the same argument as in the proof of assertion (2), that $g = 0$ almost everywhere in Ω and $h = 0$ everywhere on $\partial\Omega$. This is a contradiction. The proof of Lemma 2.3 is complete. □

Now, we see from (2.7) that

$$\mu_1'(0) = -\frac{\int_\Omega g\,dx + \int_{\partial\Omega} h\,da}{|\Omega|}, \tag{2.8}$$

where we have used that $\phi_1(0) = |\Omega|^{-1/2}$. Since $\mu_1(0) = 0$, assertion (3) follows from combining assertion (2), (2.8), and Lemma 2.3. The proof of Proposition 2.2 is complete. □

The proof of Theorem 2.1 is now complete. □

3. Proof of Theorem 1.1. This section is devoted to the proof of Theorem 1.1. This is by a contradiction argument. If we assume to the contrary that assertion (1.10) does not hold, then we may obtain one of the following possibilities: (i) There exist a constant $\delta_0 > 0$ and $\psi_0 \in L^1(\Omega)$, satisfying that $\psi_0 \geq 0$ almost everywhere in Ω, such that

$$\int_\Omega g_j \psi_0\,dx > \delta_0 \text{ for all } j \geq 1, \tag{3.1}$$

(ii) There exist a constant $\delta_0 > 0$ and $\phi_0 \in L^1(\partial\Omega)$, satisfying that $\phi_0 \geq 0$ almost everywhere on $\partial\Omega$, such that

$$\int_{\partial\Omega} h_j \phi_0\,da > \delta_0 \text{ for all } j \geq 1. \tag{3.2}$$

First, we consider case (3.1). Since $\sqrt{\psi_0} \in L^2(\Omega)$, for any $\varepsilon > 0$ there exists a $v_\varepsilon \in C^1(\Omega)$ with compact support in Ω such that $\|v_\varepsilon - \sqrt{\psi_0}\|_2 < \varepsilon$. It follows that

$$\begin{aligned}\int_\Omega g_j v_\varepsilon^2\,dx + \int_{\partial\Omega} h_j v_\varepsilon^2\,da &= \int_\Omega g_j v_\varepsilon^2\,dx \\ &= \int_\Omega g_j \psi_0\,dx + \int_\Omega g_j (v_\varepsilon^2 - \psi_0)\,dx \\ &> \delta_0 + \int_\Omega g_j (v_\varepsilon^2 - \psi_0)\,dx.\end{aligned}$$

By Schwarz's inequality, we have

$$\left|\int_\Omega g_j(v_\varepsilon^2 - \psi_0)\,dx\right| \le \|g_j\|_\infty \|v_\varepsilon - \sqrt{\psi_0}\|_2(\|v_\varepsilon - \sqrt{\psi_0}\|_2 + 2\|\sqrt{\psi_0}\|_2).$$

Hence, we can choose ε so small that

$$\int_\Omega g_j v_\varepsilon^2\,dx + \int_{\partial\Omega} h_j v_\varepsilon^2\,da > \frac{\delta_0}{2} \text{ for all } j \ge 1.$$

By the definition of $\lambda_1(g_j, h_j)$, it follows that

$$\lambda_1(g_j, h_j) \le \frac{\int_\Omega |\nabla v_\varepsilon|^2\,dx}{\int_\Omega g_j v_\varepsilon^2\,dx + \int_{\partial\Omega} h_j v_\varepsilon^2\,da} < \frac{2\int_\Omega |\nabla v_\varepsilon|^2\,dx}{\delta_0} < \infty.$$

This is contradictory for (1.6).

Next, we consider case (3.2). From the fact that $\sqrt{\phi_0} \in L^2(\partial\Omega)$, it follows similarly that for any $\varepsilon > 0$ there exists a $w_\varepsilon \in C^1(\partial\Omega)$ such that

$$\|w_\varepsilon - \sqrt{\phi_0}\|_{2,\partial\Omega} < \varepsilon, \tag{3.3}$$

see Adams [**1**, Section 7.51]. We define by $\widetilde{w}_\varepsilon \in C^1(\overline{\Omega})$ an extension of w_ε to Ω and by $\eta_{\Omega'} \in C^1(\overline{\Omega})$ a cut-off function satisfying that

$$\eta_{\Omega'} = \begin{cases} 0 & \text{in a compact subset } \Omega' \text{ of } \Omega, \\ 1 & \text{in a tublar neighborhood of } \partial\Omega. \end{cases}$$

If we put $\varphi_{\varepsilon,\Omega'} = \eta_{\Omega'}\widetilde{w}_\varepsilon$, then $\varphi_{\varepsilon,\Omega'} \in C^1(\overline{\Omega})$ and its trace on $\partial\Omega$ coincides with w_ε. Moreover, there exists $\Omega' \Subset \Omega$ such that

$$\left|\int_\Omega g_j \varphi_{\varepsilon,\Omega'}^2\,dx\right| < \frac{\delta_0}{2} \text{ for all } j \ge 1.$$

It follows from (3.2) that

$$\begin{aligned} &\int_\Omega g_j \varphi_{\varepsilon,\Omega'}^2\,dx + \int_{\partial\Omega} h_j \varphi_{\varepsilon,\Omega'}^2\,da \\ &\qquad > -\frac{\delta_0}{2} + \int_{\partial\Omega} h_j \phi_0\,da + \int_{\partial\Omega} h_j(w_\varepsilon^2 - \phi_0)\,da \\ &\qquad > \frac{\delta_0}{2} + \int_{\partial\Omega} h_j(w_\varepsilon^2 - \phi_0)\,da. \end{aligned}$$

Hence, by (3.3), there exists a $\varepsilon > 0$ so small that

$$\int_\Omega g_j \varphi_{\varepsilon,\Omega'}^2 \, dx + \int_{\partial\Omega} h_j \varphi_{\varepsilon,\Omega'}^2 \, da > \frac{\delta_0}{4} \text{ for all } j \geq 1,$$

which leads us to a contradiction, just as in case (3.1). The proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.2. In this section, we prove Theorem 1.2, which is also by a contradiction argument. First, we verify assertion (i). Assume to the contrary that $\lambda_j := \lambda_1(g_j, h_j)$ is bounded above. Let φ_j be a positive eigenfunction associated with λ_j, normalized as $\int_\Omega |\nabla \varphi_j|^2 \, dx = 1$. By definition, we have that

$$1 = \lambda_j \left(\int_\Omega g_j \varphi_j^2 \, dx + \int_{\partial\Omega} h_j \varphi_j^2 \, da \right). \tag{4.1}$$

If we decompose φ_j by the average $t_j = 1/|\Omega| \int_\Omega \varphi_j \, dx$, just as $\varphi_j = t_j + w_j$, then it follows that

$$1 = \lambda_j t_j^2 \left(\int_\Omega g_j \left(1 + \frac{w_j}{t_j}\right)^2 dx + \int_{\partial\Omega} h_j \left(1 + \frac{w_j}{t_j}\right)^2 da \right). \tag{4.2}$$

Since $\int_\Omega w_j \, dx = 0$, the so called Poincaré-Wirtinger inequality ([**8**, Theorem 1, Section 5.8]) shows that $\|w_j\|_{1,2}$ and $\|\nabla w_j\|_2$ are equivalent. Hence, w_j is bounded in $W^{1,2}(\Omega)$, since $\int_\Omega |\nabla w_j|^2 \, dx = 1$. Moreover, $|t_j|$ is bounded. Indeed, since the imbeddings $W^{1,2}(\Omega) \subset L^2(\Omega)$ and $W^{1,2}(\Omega) \subset L^2(\partial\Omega)$ are both continuous, conditions (1.11) and (4.2) allow us to have a constant $\delta > 0$ such that

$$\left\| \frac{w_j}{t_j} \right\|_{1,2} \geq \delta \text{ for all } j \geq 1 \text{ large enough.}$$

Hereby, the function $\varphi_j = t_j + w_j$ is bounded in $W^{1,2}(\Omega)$. By a standard compactness argument, there exist $\varphi_0 \in W^{1,2}(\Omega)$ and a subsequence of φ_j, still denoted by the same notation, such that

$$\varphi_j \longrightarrow \varphi_0 \text{ in } L^2(\Omega) \text{ and } \varphi_j \longrightarrow \varphi_0 \text{ in } L^2(\partial\Omega). \tag{4.3}$$

Hence, it follows from (4.1) that

$$1 = \lambda_j \left(\int_\Omega g_j \varphi_0^2 \, dx + \int_{\partial\Omega} h_j \varphi_0^2 \, da + \int_\Omega g_j (\varphi_j^2 - \varphi_0^2) \, dx + \int_{\partial\Omega} h_j (\varphi_j^2 - \varphi_0^2) \, da \right).$$

Since $\lambda_j > 0$ is bounded above, conditions (1.10) and (4.3) lead us to the assertion

$$\begin{aligned}
&\limsup_{j\to\infty} \lambda_j \int_\Omega g_j \varphi_0^2 \, dx \leq 0, \\
&\limsup_{j\to\infty} \lambda_j \int_{\partial\Omega} h_j \varphi_0^2 \, da \leq 0, \\
&\lim_{j\to\infty} \lambda_j \int_\Omega g_j (\varphi_j^2 - \varphi_0^2) \, dx = 0, \\
&\lim_{j\to\infty} \lambda_j \int_{\partial\Omega} h_j (\varphi_j^2 - \varphi_0^2) \, da = 0.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
1 &= \limsup_{j\to\infty} \lambda_j \left(\int_\Omega g_j \varphi_0^2 \, dx + \int_{\partial\Omega} h_j \varphi_0^2 \, da + \int_\Omega g_j (\varphi_j^2 - \varphi_0^2) \, dx \right. \\
&\qquad \left. + \int_{\partial\Omega} h_j (\varphi_j^2 - \varphi_0^2) \, da \right) \\
&\leq \limsup_{j\to\infty} \lambda_j \int_\Omega g_j \varphi_0^2 \, dx + \limsup_{j\to\infty} \lambda_j \int_{\partial\Omega} h_j \varphi_0^2 \, da \\
&\qquad + \lim_{j\to\infty} \lambda_j \int_\Omega g_j (\varphi_j^2 - \varphi_0^2) \, dx \\
&\qquad + \lim_{j\to\infty} \lambda_j \int_{\partial\Omega} h_j (\varphi_j^2 - \varphi_0^2) \, da \leq 0,
\end{aligned}$$

a contradiction.

Next, we verify assertion (ii). This is by a bootstrap argument relying on elliptic regularity and Sobolev's imbedding theorem. We recall that the mapping

$$\begin{aligned}
W^{2,p}(\Omega) &\longrightarrow L^p(\Omega) \times W^{1-(1/p),p}(\partial\Omega) \\
u &\longmapsto \left((-\Delta + 1)u, \ \frac{\partial u}{\partial \mathbf{n}} \right)
\end{aligned} \tag{4.4}$$

is homeomorphic for each $p > 1$. By the same argument as in assertion (i), we note that φ_j is bounded in $W^{1,2}(\Omega)$ and satisfies (4.3). Then, we show how to obtain that φ_j is bounded in $W^{2,2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$. Indeed, by (1.12), we can choose $H_j \in W^{1,p}(\Omega)$ such that $H_j|_{\partial\Omega} = h_j$ and

$$\sup_{j\geq 1} \|H_j\|_{1,p} < \infty \text{ for any } p > 1.$$

Since λ_j is bounded above, Hölder's inequality shows that

$$\sup_{j\geq 1} \|\lambda_j H_j \varphi_j\|_{1,2-\varepsilon} < \infty \text{ for any } \varepsilon > 0.$$

The trace theorem gives us that

$$\sup_{j\geq 1} \|\lambda_j h_j \varphi_j\|_{1-(1/2-\varepsilon),\, 2-\varepsilon,\, \partial\Omega} < \infty,$$

so that φ_j is bounded in $W^{2,2-\varepsilon}(\Omega)$, by using (4.4).

Now, applying Sobolev's imbedding theorem, we obtain, since ε is arbitrary, that φ_j is bounded in $W^{1,(N(2-\varepsilon)/N-2+\varepsilon)}(\Omega)$, from which φ_j is bounded in $W^{1,(2N/N-1)}(\Omega)$. It can be shown by the same procedure that φ_j is bounded in $W^{2,(2N/N-1)-\varepsilon}(\Omega)$ for any $\varepsilon > 0$. Furthermore, by repeating the same step in finite times, we get that φ_j is bounded in $W^{2,p}(\Omega)$ for each $p > 1$. By Sobolev's imbedding theorem and a compactness argument, a subsequence of φ_j, still denoted by the same notation, converges to some φ_0 in $C^1(\overline{\Omega})$. In particular, $\varphi_0^2 \in C^1(\overline{\Omega})$. Hence, condition (1.13) leads us to a contradiction in the same manner just as in assertion (i). The proof of Theorem 1.2 is complete. □

5. Proof of Theorems 1.5 and 1.7. In this section, we prove Theorems 1.5 and 1.7.

Proof of Theorem 1.5. First, we assert that condition (1.17) implies (1.8). Indeed, assume to the contrary that there exists $\psi \in L^1(\Omega)$, nonnegative almost everywhere in Ω, satisfying that

$$\limsup_{j\to\infty} \int_\Omega g_j \psi \, dx > 0,$$

so that, without loss of generality,

$$\int_\Omega g_j \psi \, dx > \delta \text{ for all } j \geq 1$$

with some constant $\delta > 0$. From (1.17) it follows that some subsequence of $(g_j)^+$, denoted by the same notation, converges to 0 almost everywhere in Ω pointwisely, so that $(g_j)^+\psi \to 0$ almost everywhere in Ω. Since $|(g_j)^+\psi| \leq (\sup_{j\geq 1} \|g_j\|_\infty)\psi$ and the right-hand side is integrable over Ω, we obtain by Lebesgue's convergence theorem that

$$\limsup_{j\to\infty} \int_\Omega g_j \psi \, dx \leq \limsup_{j\to\infty} \int_\Omega (g_j)^+ \psi \, dx = 0,$$

a contradiction.

Now, if we assume that

$$\limsup_{j\to\infty} \int_\Omega g_j \, dx < 0,$$

then Theorem 1.2 applies, thanks to (1.8), and thus $\lambda_1(g_j, 0) \to \infty$. Hereby, we have only to consider case (1.15). By a direct calculation, we see that

$$\frac{\|g_j\|_1^2}{\left|\int_\Omega g_j \, dx\right|} = -\int_\Omega g_j \, dx + 4\left\{1 + (\gamma_j - 1)^{-1}\right\} \int_\Omega (g_j)^+ \, dx,$$

from which we get (1.16), by virtue of (1.17) and (1.18). Hence, as a direct consequence of [**12**, Theorem 2.6], we have that $\lambda_1(g_j, 0) \to \infty$. The proof of Theorem 1.5 is complete. □

Proof of Theorem 1.7. Without loss of generality, we may consider the case when $\Omega = (0, 1)$. The following proposition tells us that Theorem 1.7 is true when $a_j = 0$.

Proposition 5.1. *Under the hypotheses of Theorem* 1.7, *we let*

$$a_j = 0 \text{ for all } j \geq 1.$$

Then, $\lambda_1(g_j, 0)$ *is bounded above.*

Proof. First of all, we note that condition (1.17) means that

$$k_j T_j \longrightarrow 0, \quad j \to \infty, \tag{5.1}$$

and that conditions (1.17) and (1.20) imply that

$$\frac{m_j(1-T_j)}{k_j^2 T_j^2} \leq \frac{1}{k_j T_j} + c_0 \text{ for any } j \geq 1 \text{ large enough.} \tag{5.2}$$

For a fixed constant $0 < \ell < 1/c_0$, where c_0 is a given positive constant by (1.20), we put

$$v_j(x) = -k_j T_j x + \ell.$$

On the one hand, we have

$$\int_0^1 (v_j')^2 \, dx = k_j^2 T_j^2. \tag{5.3}$$

On the other hand, a direct computation gives us that

$$\begin{aligned}
&\int_0^1 g_j v_j^2 \, dx \\
&= \int_0^{T_j} k_j(-k_j T_j x + \ell)^2 \, dx - \int_{T_j}^1 m_j(-k_j T_j x + \ell)^2 \, dx \\
&= \frac{1}{3T_j} \left\{ \ell^3 - (\ell - k_j T_j^2)^3 \right\} \\
&\quad - \frac{m_j}{3k_j T_j} \left\{ (\ell - k_j T_j^2)^3 - (\ell - k_j T_j)^3 \right\} \\
&= \ell^2 k_j T_j - \ell k_j^2 T_j^3 + \frac{k_j^3 T_j^5}{3} \\
&\quad - \frac{m_j(1-T_j)}{3} \left\{ (\ell - k_j T_j^2)^2 + (\ell - k_j T_j^2)(\ell - k_j T_j) + (\ell - k_j T_j)^2 \right\} \\
&= k_j^2 T_j^2 \left\{ \frac{\ell^2}{k_j T_j} - \ell T_j + \frac{k_j T_j^3}{3} \right. \\
&\quad \left. - \frac{m_j(1-T_j)}{3k_j^2 T_j^2} (3\ell^2 - 3\ell k_j T_j - 3\ell k_j T_j^2 + k_j^2 T_j^2 + k_j^2 T_j^3 + k_j^2 T_j^4) \right\} \\
&=: k_j^2 T_j^2 I_j.
\end{aligned}$$

Since we see that

$$\limsup_{j\to\infty} (3\ell^2 - 3\ell k_j T_j - 3\ell k_j T_j^2 + k_j^2 T_j^2 + k_j^2 T_j^3 + k_j^2 T_j^4) \geq 3\ell^2 > 0,$$

we use (5.1) and (5.2) to estimate I_j as follows.

$$\begin{aligned} I_j &\geq \frac{\ell^2}{k_j T_j} - \ell T_j + \frac{k_j T_j^3}{3} \\ &\quad - \frac{1}{3}\left(\frac{1}{k_j T_j} + c_0\right)(3\ell^2 - 3\ell k_j T_j - 3\ell k_j T_j^2 + k_j^2 T_j^2 + k_j^2 T_j^3 + k_j^2 T_j^4) \\ &= \ell(1 - c_0\ell) + \frac{k_j T_j}{3}\left\{(3c_0\ell - 1)(1 + T_j) - c_0 k_j T_j (1 + T_j + T_j^2)\right\} \\ &= \ell(1 - c_0\ell) + \mathrm{o}(1), \quad j \to \infty. \end{aligned}$$

This implies that

$$\int_0^1 g_j v_j^2 \, dx \geq k_j^2 T_j^2 \left\{\frac{\ell(1 - c_0\ell)}{2}\right\} \quad \text{for any } j \geq 1 \text{ large enough.} \tag{5.4}$$

On the basis of (1.5), we derive from (5.3) and (5.4) that

$$\lambda_1(g_j, 0) \leq \frac{\int_0^1 (v_j')^2 \, dx}{\int_0^1 g_j v_j^2 \, dx} \leq \frac{2}{\ell(1 - c_0\ell)} \quad \text{for any } j \geq 1 \text{ large enough.}$$

The proof of Proposition 5.1 is complete. □

Now, we end the proof of Theorem 1.7. Let g_j be given by (1.19) and satisfy that $\int_0^1 g_j \, dx < 0$. The index j is fixed, the notation $\lambda_1(a_j)$ is used as $\lambda_1(g_j, 0)$, by which we understand $\lambda_1(g_j, 0)$ parameterized by a_j. We introduce a new function G_j, defined by $G_j = g_j / m_j$, and we have

$$G_j(x) = \frac{k_j}{m_j} \chi_{(a_j, a_j + T_j)}(x) - \chi_{\Omega \setminus (a_j, a_j + T_j)}(x).$$

Then, $\int_0^1 G_j \, dx < 0$ and the unique positive principal eigenvalue $\lambda_1(G_j, 0)$, where the notation $\Lambda_1(a_j)$ is used as $\lambda_1(G_j, 0)$ in the same sense as $\lambda_1(a_j)$, satisfies that

$$\lambda_1(a_j) = \frac{\Lambda_1(a_j)}{m_j}. \tag{5.5}$$

Due to Cantrell and Cosner [**7**, Section 2], we know that

$$\Lambda_1(0) \leq \Lambda_1(a_j) \leq \Lambda_1\left(\frac{1-T_j}{2}\right), \quad 0 \leq a_j \leq 1-T_j. \tag{5.6}$$

Meanwhile, it is easy to see that

$$\lambda_1\left(\frac{1-T_j}{2}\right) = 4\lambda_1(0), \quad 0 < T_j < 1. \tag{5.7}$$

Hence, condition (5.6) together with (5.5) and (5.7) implies that

$$\lambda_1(0) \leq \lambda_1(a_j) \leq \lambda_1\left(\frac{1-T_j}{2}\right) = 4\lambda_1(0), \quad 0 \leq a_j \leq 1-T_j. \tag{5.8}$$

Combining (5.8) with Proposition 5.1 completes the proof of Theorem 1.7. □

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