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AN EXPLICIT EXAMPLE CONCERNING THE INVARIANT SUBSPACE PROBLEM FOR BANACH SPACES

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ABSTRACT. We simplify the negative solution to the invariant subspace problem for Banach spaces. Developing the ideas of Read, we give an explicit example of a continuous linear operator on the Banach space l_1 without nontrivial closed invariant subspaces.

1. Introduction. By a *space* we mean a linear manifold over the field $\mathbf{K}$ of real or complex numbers and by an *operator* we understand a linear mapping. Let T be an operator on a space E. A subspace M of E is a nontrivial invariant subspace of T if $\{0\} \neq M \neq E$ and $T(M) \subset M$. One of the most famous problems of operator theory is the invariant subspace problem for Hilbert spaces. It asks whether every continuous operator on an infinite-dimensional (i.d.) separable Hilbert space has a nontrivial closed invariant subspace. This problem is still open. A vast literature exists dedicated to the invariant subspace problem for various important classes of Banach spaces and continuous operators. The invariant subspace problem for complex Banach spaces solved negatively Enflo [**2**] and Read [**5, 6, 7**]. Enflo constructed an i.d. separable Banach space X and a continuous operator on X without nontrivial closed invariant subspaces. The paper containing this very difficult example was submitted for publication in the Acta Mathematica in 1981. It was accepted in 1985 and it appeared in 1987. In the meantime, Read also constructed a counterexample and submitted it for publication in the Bulletin of the London Mathematical Society in 1983. The paper appeared in 1984 [**3**]. A shorter version of this proof was published by Read in 1986 [**5**]. He also constructed continuous operators on Banach spaces l_1 and c_0 without nontrivial

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closed invariant subspaces [**4, 6**]. This l_1-example was simplified by Davie and can be found in Beauzamy's book [**1**, Chapter 14].

Note that it is not known whether there exists a continuous operator without a nontrivial closed invariant subspace on an i.d. reflexive Banach space.

Let $P(\mathbf{d})$ be a proposition depending on a strictly increasing sequence $\mathbf{d} = (d_n) \subset \mathbf{N}$. We say that $P(\mathbf{d})$ is true for all $\mathbf{d}$ increasing sufficiently rapidly if there is a constant $c \in \mathbf{N}$ and functions $f_i : \mathbf{N}^i \to \mathbf{N}$, $i \in \mathbf{N}$, with the following property: Whenever the strictly increasing sequence $\mathbf{d} = (d_i)$ satisfies $d_1 > c$ and $d_{r+1} > f_r(d_1, \dots, d_r)$ for all $r \in \mathbf{N}$, then the proposition $P(\mathbf{d})$ is true [**3**, Definition 1.1]. Clearly, $P(\mathbf{d})$ is true for some strictly increasing sequences $\mathbf{d}$ if it is true for all $\mathbf{d}$ increasing sufficiently rapidly.

All known examples of a continuous operator T on a Banach space E without nontrivial closed invariant subspaces depend on some strictly increasing sequence $\mathbf{d} = (d_i) \subset \mathbf{N}$. It is proved that T is continuous and has no nontrivial closed invariant subspace for all $\mathbf{d}$ increasing sufficiently rapidly. However, any concrete strictly increasing sequence $\mathbf{d}$ such that T is continuous and has no nontrivial closed invariant subspace cannot be indicated by means of the proof.

In our proof we do not use the notion of sequences increasing sufficiently rapidly. We indicate concrete increasing sequences $\mathbf{d} = (d_i) \subset \mathbf{N}$ such that the operator T is continuous and has no nontrivial closed invariant subspace. Note also that Enflo, Read and Davie used the local compactness of $\mathbf{K}$ and we do not. The heart of our innovation is Lemma 6. That is the way we can obtain an explicit growth condition.

2. Results. Let $\alpha \geq 8$. Let $d_0 = 2$ and $(d_n) \subset \mathbf{N}$ with $d_n \geq \alpha^{2d_{n-1}}$ for $n \in \mathbf{N}$. It is easy to see that $d_{n-1} \geq 2n$ for $n \in \mathbf{N}$. Hence we have $d_n \geq \alpha^{4n}$, $n \in \mathbf{N}$. Since the function $f(x) = x^{8/\sqrt{x}}$ is decreasing in the interval (e^2, ∞) we get $d_n^{8/\sqrt{d_n}} \leq \alpha^{32n/\alpha^{2n}} \leq \alpha^{1/(2n)}$ for $n \in \mathbf{N}$. Thus, $d_n^{4n} \leq \alpha^{\sqrt{d_n}/4} < d_{n+1}$ for $n \in \mathbf{N}$.

Put $v_0 = 0$, $a_n = d_{2n-1}$, $b_n = d_{2n}$ and $v_n = (n-1)(a_n+b_n)$ for $n \in \mathbf{N}$. Then $4n < \alpha^{4n} \leq a_n$, $a_n^{4n} < \alpha^{\sqrt{a_n}/4} < b_n$ and $b_n^{4n} < \alpha^{\sqrt{b_n}/4} < a_{n+1}$ for every $n \in \mathbf{N}$. Hence, we get $8na_n < b_n$, $8nb_n < a_{n+1}$, $8(v_{n-1}+1) < a_n$, $\alpha a_n^2 < \alpha^{\sqrt{a_n}/4}$, $\alpha b_n^2 < \alpha^{\sqrt{b_n}/4}$ and $n^2 a_n^2 < b_n$ for every $n \in \mathbf{N}$.

For $a, b \in \mathbf{Z}$ we denote the set $\{k \in \mathbf{Z} : a < k \leq b\}$ by $(a, b]$; similarly we define $[a, b)$, $[a, b]$ and (a, b).

For nonempty sets $A, B \subset \mathbf{N}$ we write $A < B$ if $1 + \max A = \min B$.

For $n, r \in \mathbf{N}$ with $n > r$ we put:

$$\begin{aligned} J_{n,r} &= ((r-1)a_n + v_{n-r}, ra_n), \\ I_{n,r} &= [ra_n, ra_n + v_{n-r-1}], \\ L_{n,r} &= ((n-1)a_n + (r-1)b_n, r(a_n + b_n)) \end{aligned}$$

and

$$K_{n,r} = [r(a_n + b_n), (n-1)a_n + rb_n].$$

These sets are nonempty and $J_{n,r} < I_{n,r} < J_{n,r+1}$, $L_{n,r} < K_{n,r} < L_{n,r+1}$ for $n, r \in \mathbf{N}$ with $n > r + 1$ and $J_{n,n-1} < I_{n,n-1}$, $L_{n,n-1} < K_{n,n-1}$ for $n \geq 2$. Moreover, $X_n := \cup_{r=1}^{n-1}(J_{n,r} \cup I_{n,r}) = (v_{n-1}, (n-1)a_n]$, $Y_n := \cup_{r=1}^{n-1}(L_{n,r} \cup K_{n,r}) = ((n-1)a_n, v_n]$ for $n \geq 2$; so $X_n < Y_n < X_{n+1}$ for $n \geq 2$ and $\cup_{n=2}^{\infty}(X_n \cup Y_n) = \mathbf{N}$.

Let $F = \mathbf{K}[x]$ and $F_n = \{f \in F : \deg(f) \leq n\}$ for $n \in \mathbf{N}_0$, where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. F is a linear algebra over $\mathbf{K}$ and F_n is a subspace of F for every $n \in \mathbf{N}_0$. Put

$$f_i = \begin{cases} x^i & \text{if } i = 0; \\ \alpha^{[(2r-1)a_n - 2i]/\sqrt{4a_n}} x^i & \text{if } i \in J_{n,r} \text{ and } n, r \in \mathbf{N} \text{ with } n > r; \\ a_{n-r}(x^i - x^{i-a_n}) & \text{if } i \in I_{n,r} \text{ and } n, r \in \mathbf{N} \text{ with } n > r; \\ \alpha^{[(2r-1)b_n - 2i]/\sqrt{4b_n}} x^i & \text{if } i \in L_{n,r} \text{ and } n, r \in \mathbf{N} \text{ with } n > r; \\ x^i - b_n x^{i-b_n} & \text{if } i \in K_{n,r} \text{ and } n, r \in \mathbf{N} \text{ with } n > r. \end{cases}$$

Clearly, $\operatorname{lin}\{f_i : 0 \leq i \leq n\} = F_n$ for $n \in \mathbf{N}_0$. Thus, $(f_i)_{i=0}^{\infty}$ is a Hamel base in F. For $f \in F$ of the form $f = \sum_{i=0}^{m} c_i f_i$ we put $\|f\| = \sum_{i=0}^{m} |c_i|$; $\|\cdot\|$ is a norm on F.

Denote by $E = (E, \|\cdot\|)$ the completion of the normed space $(F, \|\cdot\|)$. The Banach space E is linearly isometric to l_1.

Put $A_m = \alpha^{\sqrt{a_m}}$ and $B_m = \alpha^{\sqrt{b_m}}$ for $m \in \mathbf{N}$. We have the following

Lemma 1. (a) *Let* $m \in \mathbf{N}$ *with* $m \geq 2$. *Then* $\max_{0 \leq i \leq (m-1)a_m} \|x^i\| \leq A_m$ *and* $\max_{0 \leq i \leq v_m} \|x^i\| \leq B_m$.

(b) *Let* $n, r \in \mathbf{N}$ *with* $n > r$. *Then* $\|x^i - x^{i-ra_n}\| \le 2a_{n-r}^{-1}$ *for* $i \in I_{n,r}$ *and* $\|x^i - b_n^r x^{i-rb_n}\| \le 2b_n^{r-1}$ *for* $i \in K_{n,r}$.

Proof. For $m \in \mathbf{N}$ we have $[0, ma_{m+1}] = [0, v_m] \cup \cup_{r=1}^m (J_{m+1,r} \cup I_{m+1,r})$ and $[0, v_{m+1}] = [0, ma_{m+1}] \cup \cup_{r=1}^m (L_{m+1,r} \cup K_{m+1,r})$.

Clearly $\|x^0\| = 1$. Let $n, r \in \mathbf{N}$ with $n > r$. It is easy to check that $\|x^i\| < \alpha^{\sqrt{a_n}}$ for $i \in J_{n,r}$ and $\|x^i\| < \alpha^{\sqrt{b_n}}$ for $i \in L_{n,r}$.

Let $i \in I_{n,r}$. For $j \in [0, r)$ we have $i - ja_n \in I_{n,r-j}$, so $a_{n-r+j}^{-1} f_{i-ja_n} = x^{i-ja_n} - x^{i-(j+1)a_n}$. Hence, $\sum_{j=0}^{r-1} a_{n-r+j}^{-1} f_{i-ja_n} = x^i - x^{i-ra_n}$, so $\|x^i - x^{i-ra_n}\| = \sum_{j=0}^{r-1} a_{n-r+j}^{-1} \le 2a_{n-r}^{-1}$. Then $\|x^i\| \le 2a_{n-r}^{-1} + \|x^{i-ra_n}\|$ and $i - ra_n \in [0, v_{n-2}]$.

Let $i \in K_{n,r}$. For $j \in [0, r)$ we have $i - jb_n \in K_{n,r-j}$, so $x^{i-jb_n} = f_{i-jb_n} + b_n x^{i-(j+1)b_n}$. Hence, $x^i = \sum_{j=0}^{r-1} b_n^j f_{i-jb_n} + b_n^r x^{i-rb_n}$, so $\|x^i - b_n^r x^{i-rb_n}\| = \sum_{j=0}^{r-1} b_n^j \le 2b_n^{r-1}$. Then $\|x^i\| \le 2b_n^{r-1} + b_n^r \|x^{i-rb_n}\|$ and $i - rb_n \in [a_n, (n-1)a_n]$.

Hence, by induction, we get $\max_{0 \le i \le (m-1)a_m} \|x^i\| \le \alpha^{\sqrt{a_m}}$ and $\max_{0 \le i \le v_m} \|x^i\| \le \alpha^{\sqrt{b_m}}$ for $m \ge 2$. □

Lemma 2. *The operator* $T : (F, \|\cdot\|) \to (F, \|\cdot\|)$, $Tf = xf$ *is continuous and* $\|T\| \le \alpha$.

Proof. It is enough to show that $\|Tf_i\| \le \alpha$ for any $i \in \mathbf{N}_0$. Since $1 \in J_{2,1}$ we have $f_1 = \alpha^{[a_2-2]/\sqrt{4a_2}} x$. Thus, $Tf_0 = x = \alpha^{-[a_2-2]/\sqrt{4a_2}} f_1$; hence, $\|Tf_0\| \le 1$.

Let $i \in \mathbf{N}$. For some $n, r \in \mathbf{N}$ with $n > r$, we have $i \in J_{n,r} \cup I_{n,r} \cup L_{n,r} \cup K_{n,r}$. Consider four cases (and many subcases).

Case 1. $i \in J_{n,r}$. Then $Tf_i = \alpha^{[(2r-1)a_n - 2i]/\sqrt{4a_n}} x^{i+1}$.

1.1°. $i < ra_n - 1$. Then $i + 1 \in J_{n,r}$, so $f_{i+1} = \alpha^{[(2r-1)a_n - 2(i+1)]/\sqrt{4a_n}} x^{i+1}$. Thus, $Tf_i = \alpha^{2/\sqrt{4a_n}} f_{i+1}$. Hence, $\|Tf_i\| = \alpha^{2/\sqrt{4a_n}} \le \alpha$.

1.2°. $i = ra_n - 1$. Then $Tf_i = \alpha^{[2-a_n]/\sqrt{4a_n}} x^{ra_n}$. By Lemma 1 we have $\|x^{ra_n} - x^0\| \le 2a_{n-r}^{-1}$. Thus, $\|x^{ra_n}\| \le 2a_{n-r}^{-1} + 1 \le 2$ so $\|Tf_i\| \le \alpha^{-\sqrt{a_n}/4} \le a_n^{-1} < 1$.

Case 2. $i \in I_{n,r}$. Then $Tf_i = a_{n-r}(x^{i+1} - x^{i+1-a_n})$.

2.1°. $i < ra_n + v_{n-r-1}$. Then $i+1 \in I_{n,r}$, so $f_{i+1} = a_{n-r}(x^{i+1} - x^{i+1-a_n})$. Thus, $\|Tf_i\| = \|f_{i+1}\| = 1$.

2.2°. $i = ra_n + v_{n-r-1}$. Then $4(i+1) < (4r+1)a_n < b_n$. Put $j = i - a_n + 1$.

If $r < n-1$, then $i+1 \in J_{n,r+1}$; so $f_{i+1} = \alpha^{[(2r+1)a_n - 2(i+1)]/\sqrt{4a_n}} x^{i+1}$. Hence, $\|x^{i+1}\| = \alpha^{-[(2r+1)a_n-2(i+1)]/\sqrt{4a_n}} \leq \alpha^{-\sqrt{a_n}/4}$.

If $r = n-1$, then $i+1 \in L_{n,1}$; so $f_{i+1} = \alpha^{[b_n - 2(i+1)]/\sqrt{4b_n}} x^{i+1}$. Hence, $\|x^{i+1}\| = \alpha^{-[b_n-2(i+1)]/\sqrt{4b_n}} \leq \alpha^{-\sqrt{b_n}/4}$.

If $r = 1$ and $n = 2$, then $j = 1 \in J_{2,1}$ and $\|x^j\| = \alpha^{-[a_2-2]/\sqrt{4a_2}} \|f_j\| \leq \alpha^{-\sqrt{a_2}/4}$.

If $r = 1$ and $n > 2$, then $j \in J_{n-1,1}$; so $f_j = \alpha^{[a_{n-1}-2j]/\sqrt{4a_{n-1}}} x^j$. Hence, $\|x^j\| = \alpha^{-[a_{n-1}-2(v_{n-2}+1)]/\sqrt{4a_{n-1}}} \leq \alpha^{-\sqrt{a_{n-1}}/4}$.

If $1 < r < n-1$, then using Lemma 1 we get $\|x^j - x^{v_{n-r-1}+1}\| \leq 2a_{n-r+1}^{-1}$. Moreover, $v_{n-r-1}+1 \in J_{n-r,1}$, so

$$f_{v_{n-r-1}+1} = \alpha^{[a_{n-r} - 2(v_{n-r-1}+1)]/\sqrt{4a_{n-r}}} x^{v_{n-r-1}+1}.$$

Hence, $\|x^{v_{n-r-1}+1}\| \leq \alpha^{-\sqrt{a_{n-r}}/4}$. Thus, $\|x^j\| \leq 2a_{n-r+1}^{-1} + \alpha^{-\sqrt{a_{n-r}}/4}$.

If $r = n-1$, then $j \in J_{n,n-1}$; so $f_j = \alpha^{[(2n-3)a_n-2j]/\sqrt{4a_n}} x^j$. Hence, $\|x^j\| = \alpha^{-[a_n-2]/\sqrt{4a_n}} \leq \alpha^{-\sqrt{a_n}/4}$.

It follows that $\|Tf_i\| \leq a_{n-r}(\|x^{i+1}\| + \|x^j\|) \leq 2a_{n-r}(a_{n-r+1}^{-1} + \alpha^{-\sqrt{a_{n-r}}/4}) \leq a_{n-r}^{-1}$.

Case 3. $i \in L_{n,r}$. Then $Tf_i = \alpha^{[(2r-1)b_n-2i]/\sqrt{4b_n}} x^{i+1}$.

3.1°. $i < r(a_n + b_n) - 1$. Then $i+1 \in L_{n,r}$, so $f_{i+1} = \alpha^{([(2r-1)b_n - 2(i+1)]/\sqrt{4b_n})} x^{i+1}$. Thus, $Tf_i = \alpha^{2/\sqrt{4b_n}} f_{i+1}$. Hence, $\|Tf_i\| = \alpha^{1/\sqrt{b_n}} < \alpha$.

3.2°. $i = r(a_n + b_n) - 1$. Then $Tf_i = \alpha^{[-b_n-2ra_n+2]/\sqrt{4b_n}} x^j$, where $j = r(a_n + b_n)$. By Lemma 1 we have $\|x^j - b_n^r x^{ra_n}\| \leq 2b_n^{r-1}$. In Case 1 we have shown that $\|x^{ra_n}\| \leq 2$. Hence, $\|x^j\| \leq 3b_n^{n-1}$. Thus, $\|Tf_i\| \leq 3\alpha^{-\sqrt{b_n}/2} b_n^{n-1} \leq b_n^{-1}$.

Case 4. $i \in K_{n,r}$. Then $Tf_i = x^{i+1} - b_n x^{i+1-b_n}$.

4.1°. $i < (n-1)a_n + rb_n$. Then $i+1 \in K_{n,r}$, so $f_{i+1} = x^{i+1} - b_n x^{i+1-b_n} = Tf_i$. Hence, $\|Tf_i\| = 1$.

4.2°. $i = (n-1)a_n + rb_n$. Put $j = i+1-b_n$. Then $j \in L_{n,r}$, so $f_j = \alpha^{[(2r-1)b_n - 2j]/\sqrt{4b_n}} x^j$. Hence, $\|x^j\| = \alpha^{-[(2r+1)b_n - 2(i+1)]/\sqrt{4b_n}} \le \alpha^{-\sqrt{b_n}/4}$.

If $r < n-1$, then $i+1 \in L_{n,r+1}$; so $f_{i+1} = \alpha^{[(2r+1)b_n - 2(i+1)]/\sqrt{4b_n}} x^{i+1}$. Hence, $\|x^{i+1}\| = \|x^j\| \le \alpha^{-\sqrt{b_n}/4}$.

If $r = n-1$, then $i+1 \in J_{n+1,1}$; so $f_{i+1} = \alpha^{[a_{n+1} - 2(i+1)]/\sqrt{4a_{n+1}}} x^{i+1}$. Hence, $\|x^{i+1}\| \le \alpha^{-\sqrt{a_{n+1}}/4} \le \alpha^{-\sqrt{b_n}/4}$. Thus, we have $\|Tf_i\| \le \|x^{i+1}\| + b_n \|x^j\| \le 2b_n \alpha^{-\sqrt{b_n}/4} \le 1$. □

From now on, by T we will denote the continuous operator on E such that $Tf = xf$ for all $f \in F$; clearly $\|T\| \le |\alpha|$.

By the proof of Lemma 2 we get the following

Remark 3. If $n, r \in \mathbf{N}$ with $n > r$, then $\|Tf_{ra_n - 1}\| \le a_n^{-1}$, $\|Tf_{ra_n + v_{n-r-1}}\| \le a_{n-r}^{-1}$ and $\|Tf_{r(a_n+b_n)-1}\| \le b_n^{-1}$.

Let $m \in \mathbf{N}$ with $m > 2$. Put $S_m = \cup_{n=m+1}^{\infty} I_{n,n-m}$. Let $Q_m : F \to F_{(m-1)a_m}$ be an operator such that

$$Q_m f_i = \begin{cases} f_i & \text{if } i \in [0, (m-1)a_m] \\ -amx^{i-(n-m)a_n} & \text{if } i \in I_{n.n-m} \text{ and } n > m \\ 0 & \text{if } i \in (\mathbf{N} \setminus S_m) \text{ with } i > (m-1)a_m. \end{cases}$$

Clearly, $\|Q_m f_i\| = 1$ for $0 \le i \le (m-1)a_m$, and $\|Q_m f_i\| \le a_m B_{m-1} < \sqrt{A_m}$ for $i \in I_{n,n-m}$, $n > m$. Thus, $\sup_{i \in \mathbf{N}_0} \|Q_m f_i\| < \sqrt{A_m}$; so the operator $Q_m : (F, \|\cdot\|) \to (F, \|\cdot\|)$ is continuous and $\|Q_m\| < \sqrt{A_m}$. From now on, by Q_m we will denote its continuous extension on $(E, \|\cdot\|)$.

We have the following lemma.

Lemma 4. *Let $m \in \mathbf{N}$ with $m > 2$, and let $s \in K_{m,1}$. Then $\|T^s - T^s Q_m\| \le \alpha$.*

Proof. It is enough to show that $\|T^s f_i - T^s Q_m f_i\| \le \alpha$ for every $i \in \mathbf{N}_0$. If $i \in [0, (m-1)a_m]$, then $T^s f_i - T^s Q_m f_i = 0$.

Let $i > (m-1)a_m$. For some $n, r \in \mathbf{N}$ with $n > r$ we have $i \in J_{n,r} \cup I_{n,r} \cup L_{n,r} \cup K_{n,r}$. If $i \in J_{n,r} \cup L_{n,r} \cup K_{n,r}$, then $i \notin S_m$; so $Q_m f_i = 0$ and $T^s f_i - T^s Q_m f_i = T^s f_i$.

Consider four cases.

Case 1. $i \in J_{n,r}$. Then $T^s f_i = \alpha^{[(2r-1)a_n - 2i]/\sqrt{4a_n}} x^{i+s}$. We have $m < n$, since $(m-1)a_m < i < (n-1)a_n$. Thus, $s < 2b_m < \sqrt{a_n}$.

1.1°. $i < ra_n - s$. Then $i+s \in J_{n,r}$, so $f_{i+s} = \alpha^{[(2r-1)a_n - 2(i+s)]/\sqrt{4a_n}} \times x^{i+s}$. Hence, $T^s f_i = \alpha^{2s/\sqrt{4a_n}} f_{i+s}$. Thus, $\|T^s f_i\| \le \alpha$.

1.2°. $i \ge ra_n - s$. Then $T^{i+s-ra_n+1} f_{ra_n - 1} = \alpha^{[(2r-1)a_n - 2(ra_n - 1)]/\sqrt{4a_n}} x^{i+s}$, since $ra_n - 1 \in J_{n,r}$. It follows that $T^s f_i = \alpha^{[2ra_n - 2i - 2]/\sqrt{4a_n}} T^{i+s-ra_n} T f_{ra_n - 1}$. Using Remark 3 we get $\|T^s f_i\| \le \alpha \alpha^{i+s-ra_n} a_n^{-1} \le \alpha^s a_n^{-1} \le \alpha^{2b_m} a_{m+1}^{-1} \le 1$.

Case 2. $i \in I_{n,r}$. Then $T^s f_i = a_{n-r}(x^{i+s} - x^{i+s-a_n})$. We have $n > m$, since $(m-1)a_m < i \le ra_n + v_{n-r-1} \le (n-1)a_n$. Thus, $4(i+s) < (4r+1)a_n < b_n$.

2.1°. $r = n - m$. Then $i = ra_n + j$ for some $j \in [0, v_{m-1}]$ and $T^s Q_m f_i = -a_m x^{j+s}$.We have $i + s \in J_{n,r+1}$, so $f_{i+s} = \alpha^{[(2r+1)a_n - 2(i+s)]/\sqrt{4a_n}} x^{i+s}$. Hence, $\|x^{i+s}\| \le \alpha^{-\sqrt{a_n/4}} \le a_n^{-1}$. Using Lemma 1 we get $\|x^{(r-1)a_n + j + s} - x^{j+s}\| \le 2a_{m+1}^{-1}$. Thus, $\|T^s f_i - T^s Q_m f_i\| = a_m \|x^{i+s} - (x^{i-a_n+s} - x^{j+s})\| < 3a_m a_{m+1}^{-1} < 1$.

2.2°. $r \ne n - m$. Then $i \notin S_m$, so $Q_m f_i = 0$ and $T^s f_i - T^s Q_m f_i = T^s f_i$.

(a) $r > n - m$. Then for $j = i + s$ we have $j - a_n \in J_{n,r}$, so $f_{j-a_n} = \alpha^{[(2r-1)a_n - 2(j-a_n)]/\sqrt{4a_n}} x^{j-a_n}$. Thus, $\|x^{j-a_n}\| \le \alpha^{-\sqrt{a_n}/4}$.

If $r+1 < n$, then $j \in J_{n,r+1}$; so $f_j = \alpha^{[(2r+1)a_n - 2j]/\sqrt{4a_n}} x^j$. Hence, $\|x^j\| = \|x^{j-a_n}\| \le \alpha^{-\sqrt{a_n}/4}$.

If $r + 1 = n$, then $j \in L_{n,1}$; so $f_j = \alpha^{[b_n - 2j]/\sqrt{4b_n}} x^j$. Hence, $\|x^j\| \le \alpha^{-\sqrt{b_n}/4}$.

It follows that $\|T^s f_i\| \le a_{n-r}(\|x^j\| + \|x^{j-a_n}\|) \le a_{n-1}(\alpha^{-\sqrt{a_n}/4} + \alpha^{-\sqrt{a_n}/4}) < 1$.

(b) $r < n - m$. Then $i = ra_n + j$ for some $j \in [0, v_{n-r-1}]$.

If $j + s \le v_{n-r-1}$, then $i + s \in I_{n,r}$; so $f_{i+s} = a_{n-r}(x^{i+s} - x^{i+s-a_n}) = T^s f_i$. Thus, $\|T^s f_i\| = 1$.

If $j + s > v_{n-r-1}$, then we have $T^s f_i = a_{n-r}(x^{i+s} - x^{i+s-a_n}) = T^{s+j-v_{n-r-1}}[a_{n-r}(x^{ra_n+v_{n-r-1}} - x^{(r-1)a_n+v_{n-r-1}})] = T^{s+j-v_{n-r-1}-1} T f_{ra_n+v_{n-r-1}}$. Using Remark 3 we get $\|T^s f_i\| \le \alpha^{s+j-v_{n-r-1}-1} a_{n-r}^{-1} \le \alpha^s a_{n-r}^{-1} \le \alpha^{2b_m} a_{m+1}^{-1} \le 1$.

Case 3. $i \in L_{n,r}$. Then $T^s f_i = \alpha^{[(2r-1)b_n-2i]/\sqrt{4b_n}} x^j$, where $j = i + s$. We have $n \ge m$, since $v_{m-1} < a_m < i < r(a_n + b_n) \le v_n$. Thus, $4j < a_{n+1}$.

3.1°. $n = m$ and $j > v_n$. Then $j \in J_{n+1,1}$; so $f_j = \alpha^{[a_{n+1}-2j]/\sqrt{4a_n+1}} x^j$ and $\|x^j\| = \alpha^{-[a_{n+1}-2j]/\sqrt{4a_{n+1}}} \le \alpha^{-\sqrt{a_{n+1}/4}}$. Thus, $\|T^s f_i\| \le \alpha^{\sqrt{b_n}/2}\|x^j\| \le 1$.

3.2°. $n = m$ and $j \le v_n$. Then $j > na_n + rb_n$ and $r < n - 1$.

(a) If $j < (r+1)(a_n+b_n)$, then $j \in L_{n,r+1}$; so $f_j = \alpha^{[(2r+1)b_n-2j]/\sqrt{4b_n}} x^j$. Thus, $\|T^s f_i\| = \alpha^{[-2b_n+2s]/\sqrt{4b_n}} \le \alpha$, since $s - b_n < \sqrt{b_n}$.

(b) If $(r+1)(a_n+b_n) \le j \le (n-1)a_n + (r+1)b_n$, then using Lemma 1 we get $\|x^j - b_n^{r+1} x^{j-(r+1)b_n}\| \le 2b_n^r$ and $\|x^{j-(r+1)b_n}\| \le A_n \le b_n$. Thus, $\|x^j\| \le 2b_n^r + b_n^{r+2} \le 3b_n^n < \alpha^{\sqrt{b_n}/4}$. Moreover, we have $\|T^s f_i\| = \alpha^{[(2r-1)b_n-2i]/\sqrt{4b_n}}\|x^j\| \le \alpha^{-\sqrt{b_n}/4}\|x^j\|$, since $4i = 4(j-s) \ge (4r-1)b_n$. It follows that $\|T^s f_i\| \le 1$.

(c) If $j > (n-1)a_n + (r+1)b_n$, then $r < n - 2$ and $j \in L_{n,r+2}$; so $f_j = \alpha^{[(2r+3)b_n-2j]/\sqrt{4b_n}} x^j$. Thus, $\|T^s f_i\| = \alpha^{[2s-4b_n]/\sqrt{4b_n}} \le \alpha$.

3.3°. $n > m$ and $j < r(a_n + b_n)$. Then $j \in L_{n,r}$, so $f_j = \alpha^{[(2r-1)b_n-2j]/\sqrt{4b_n}} x^j$. Thus, $\|T^s f_i\| = \alpha^{2s/\sqrt{4b_n}} \le \alpha$, since $s < 2b_m < \sqrt{b_n}$.

3.4°. $n > m$ and $j \ge r(a_n + b_n)$. Put $k = r(a_n + b_n)$. Clearly, $k - 1 \in L_{n,r}$, so $f_{k-1} = \alpha^{[(2r-1)b_n-2(k-1)]/\sqrt{4b_n}} x^{k-1}$. Hence, $T^{j-k} T f_{k-1} = \alpha^{[(2r-1)b_n-2(k-1)]/\sqrt{4b_n}} x^j$. Thus, $\|x^j\| \le \alpha^{-[(2r-1)b_n-2(k-1)]/\sqrt{4b_n}} \|T\|^{s-1} \|T f_{k-1}\|$. By Remark 3 we get $\|T f_{k-1}\| \le b_n^{-1}$. It follows that $\|T^s f_i\| \le \alpha^{[2(k-i-1)]/\sqrt{4b_n}} \alpha^{s-1} b_n^{-1} \le \alpha^s b_n^{-1} < \alpha^{2b_m} b_{m+1}^{-1} < 1$, since $k - i - 1 < s < 2b_m < \sqrt{b_n}$.

Case 4. $i \in K_{n,r}$. Then $T^s f_i = x^j - b_n x^{j-b_n}$, where $j = i + s$. We have $n \geq m$, since $(m-1)a_m < i \leq (n-1)a_n + rb_n \leq v_n$.

4.1°. $n = m$ and $j \leq (n-1)a_n + (r+1)b_n$. Then $j \geq (r+1)(a_n + b_n)$, so $r + 1 \leq n - 1$ and $j \in K_{n,r+1}$. Thus, $f_j = x^j - b_n x^{j-b_n} = T^s f_i$, so $\|T^s f_i\| = 1$.

4.2°. $n = m$ and $j > (n-1)a_n + (r+1)b_n$. Then $4j < (4r+5)b_n < a_{n+1}$.

If $r < n - 2$, then $j \in L_{n,r+2}$ and $f_j = \alpha^{[(2r+3)b_n - 2j]/\sqrt{4b_n}} x^j$; so $\|x^j\| \leq 1$.

If $r \geq n - 2$, then $j \in J_{n+1,1}$ and $f_j = \alpha^{[a_{n+1} - 2j]/\sqrt{4a_{n+1}}} x^j$; so $\|x^j\| \leq 1$.

If $r < n - 1$, then $j - b_n \in L_{n,r+1}$ and

$$f_{j-b_n} = \alpha^{[(2r+1)b_n - 2(j-b_n)]/\sqrt{4b_n}} x^{j-b_n};$$

so $\|x^{j-b_n}\| \leq \alpha^{-\sqrt{b_n}/4} \leq b_n^{-1}$.

If $r = n - 1$, then $j - b_n \in J_{n+1,1}$ and

$$f_{j-b_n} = \alpha^{[a_{n+1} - 2(j-b_n)]/\sqrt{4a_{n+1}}} x^{j-b_n};$$

so $\|x^{j-b_n}\| \leq |\alpha|^{-\sqrt{a_{n+1}}/4} \leq \alpha_{n+1}^{-1}$.

It follows that $\|T^s f_i\| \leq \|x^j\| + b_n \|x^{j-b_n}\| \leq 2 < \alpha$.

4.3°. $n > m$ and $j \leq (n-1)a_n + rb_n$. Then $j \in K_{n,r}$ and $f_j = x^j - b_n x^{j-b_n} = T^s f_i$; so $\|T^s f_i\| = 1$.

4.4°. $n > m$ and $j > (n-1)a_n + rb_n$. Then $4j < (4r+1)b_n < a_{n+1}$.

If $r < n - 1$, then $j \in L_{n,r+1}$ and $f_j = \alpha^{[(2r+1)b_n - 2j]/\sqrt{4b_n}} x^j$; so $\|x^j\| \leq 1$.

If $r = n - 1$, then $j \in J_{n+1,1}$ and $f_j = \alpha^{[a_{n+1} - 2j]/\sqrt{4a_{n+1}}} x^j$; so $\|x^j\| \leq 1$.

Moreover, we have $j - b_n \in L_{n,r}$ and

$$f_{j-b_n} = \alpha^{[(2r-1)b_n - 2(j-b_n)]/\sqrt{4b_n}} x^{j-b_n};$$

so $\|x^{j-b_n}\| \leq \alpha^{-\sqrt{b_n}/4} \leq b_n^{-1}$. It follows that $\|T^s f_i\| \leq \|x^j\| + b_n \|x^{j-b_n}\| \leq 2 < \alpha$. □

For $f \in F$ of the form $f = \sum_{i=0}^{m} c_i x^i$ we put $|f| = \sum_{i=0}^{m} |ci|$. The functional $|\cdot| : F \to [0, \infty)$, $f \to |f|$ is a submultiplicative norm on F.

It is easy to check that for $m \in \mathbf{N}$ with $m > 2$ and $y \in F_{(m-1)a_m}$ we have $\|y\| \leq A_m |y|$ and $|y| \leq \max_{0 \leq i \leq (m-1)a_m} |f_i| \|y\| \leq \sqrt{A_m} \|y\|$.

For $n \in \mathbf{N}_0$ we denote by P_n the linear projection from F onto F_n such that $P_n(x^i) = 0$ for $i > n$. We have $x(P_n v) = P_{n+1}(xv)$ for $n \in \mathbf{N}$ and $v \in F$.

We need two other lemmas to prove our theorem.

Lemma 5. *Let $e \in E$ with $e \neq 0$ and $k \in \mathbf{N}$ with $k > 2$. Then there exists an $m \in \mathbf{N}$ with $m > k$ such that $|P_{(m-k)a_m}(Q_m e)| \geq a_m^{-1}$.*

Proof. Suppose by contradiction that for every $m \in \mathbf{N}$ with $m > k$ we have

$$|P_{(m-k)a_m}(Q_m e)| < a_m^{-1}. \tag{1}$$

For some $(e_j) \in l_1(\mathbf{N}_0)$ we have $e = \sum_{j=0}^{\infty} e_j f_j$. Then $\|e\| = \sum_{j=0}^{\infty} |e_j| > 0$. Put $c_n = (n-1)a_n$ for $n \in \mathbf{N}$. For $n \in \mathbf{N}$ we have $\sum_{j=0}^{c_n} e_j f_j = \sum_{j=0}^{c_n} y_{n,j} x^j$ for some $(y_{n,j})_{j=0}^{c_n} \subset \mathbf{K}$. For $n > 2$ we obtain $Q_n(\sum_{j=c_n+1}^{\infty} e_j f_j) = \sum_{i=0}^{v_{n-1}} z_{n,i} x^i$, where

$$z_{n,i} = -a_n \sum_{m=n+1}^{\infty} e_{i+(m-n)a_m}. \tag{2}$$

So we get

$$Q_n e = \sum_{j=0}^{c_n} y_{n,j} x^j + \sum_{j=0}^{v_{n-1}} z_{n,j} x^j. \tag{3}$$

From (1) and (3) we obtain for $m > k$

$$\sum_{j=v_{m-1}+1}^{(m-k)a_m} |y_{m,j}| < a_m^{-1}. \tag{4}$$

Let $m > n > k$ and $M_{m,n} = ((m-n)a_m + v_{n-2}, (m-n)a_m + v_{n-1}]$. Clearly, $M_{m,n} \subset [a_m, c_m] = \cup_{s=1}^{m-1} I_{m,s} \cup \cup_{s=2}^{m-1} J_{m,s}$ and $f_i = a_{m-s}(x^i -$

$x^{i-a_m})$ for $i \in I_{m,s}$, $s \in [1,m)$ and $f_i = \alpha^{[(2s-1)a_m-2i]/\sqrt{4a_m}}x^i$ for $i \in J_{m,s}$, $s \in [1,m)$. If $j \in M_{m,n}$, then $j \in I_{m,m-n}$; if $i \in [a_m, c_m]$ and $i - a_m \in M_{m,n}$, then $i \in J_{m,m-n+2}$. Thus, $y_{m,j} = a_n e_j$ for $j \in M_{m,n}$. Clearly, $M_{m,n} \subset (v_{m-1}, (m-k)a_m]$. Using (2) and (4) we obtain for $n > k$,

$$(5) \qquad \sum_{j=v_{n-2}+1}^{v_{n-1}} |z_{n,j}| \le \sum_{m=n+1}^{\infty} \sum_{j=v_{n-2}+1}^{v_{n-1}} a_n |e_{j+(m-n)a_m}| = \sum_{m=n+1}^{\infty} \sum_{j \in M_{m,n}} |y_{m,j}| \le a_n^{-1}.$$

From (3) and (1) we get for $n > k$

$$(6) \qquad \sum_{j=0}^{v_{n-1}} |y_{n,j} + z_{n,j}| \le |P_{a_n}(Q_n e)| < a_n^{-1}.$$

Hence, by (5), we have for $n > k$,

$$(7) \qquad \sum_{j=v_{n-2}+1}^{v_{n-1}} |y_{n,j}| < a_n^{-1} + a_n^{-1} = 2a_n^{-1}.$$

Let $n > k$. Put $M_n = \sum_{j=v_{n-1}+1}^{c_n} e_j f_j - \sum_{j=v_{n-1}+1}^{c_n} y_{n,j} x^j$.

Clearly, $(v_{n-1}, c_n] = \cup_{s=1}^{n-1} (J_{n,s} \cup I_{n,s})$. If $i \in \cup_{s=1}^{n-1} J_{n,s} \cup \cup_{s=2}^{n-1} I_{n,s}$, then $P_{v_{n-1}} f_i = 0$; if $i \in I_{n,1}$, then $P_{v_{n-1}}(f_i) \in F_{v_{n-2}}$. Thus, $P_{v_{n-1}}(M_n) \in F_{v_{n-2}}$; but $M_n = \sum_{j=0}^{v_{n-1}} y_{n,j} x^j - \sum_{j=0}^{v_{n-1}} e_j f_j \in F_{v_{n-1}}$, so $M_n \in F_{v_{n-2}}$. Hence, $\sum_{j=v_{n-2}+1}^{v_{n-1}} e_j f_j - \sum_{j=v_{n-2}+1}^{v_{n-1}} y_{n,j} x^j = \sum_{j=0}^{v_{n-2}} t_{n,j} f_j$ for some $(t_{n,j})_{j=0}^{v_{n-2}} \subset \mathbf{K}$. By (7) and Lemma 1, we get

$$\sum_{j=v_{n-2}+1}^{v_{n-1}} |e_j| \le \left\| \sum_{j=v_{n-2}+1}^{v_{n-1}} e_j f_j - \sum_{j=0}^{v_{n-2}} t_{n,j} f_j \right\| \le \sum_{j=v_{n-2}+1}^{v_{n-1}} |y_{n,j}| B_{n-1} \le \frac{2B_{n-1}}{a_n} < \frac{1}{a_n^{1/2}}.$$

Hence, for $m \ge k$, we have $\sum_{j=v_{m-1}+1}^{c_m} |e_j| \le \sum_{j=v_{m-1}+1}^{v_m} |e_j| < a_{m+1}^{-1/2} < a_m^{-1}$. Using (2) we get for $n \ge k$: $\sum_{j=0}^{v_{n-1}} |z_{n,j}| \le$

$a_n \sum_{m=n+1}^{\infty} \sum_{j=0}^{v_{n-1}} |e_{j+(m-n)a_m}| \leq a_n \sum_{m=n+1}^{\infty} \sum_{j=v_{m-1}+1}^{v_m} |e_j| \leq a_n \sum_{m=n+1}^{\infty} a_{m+1}^{-1/2} \leq 2a_n a_{n+2}^{-1/2} \leq a_{n+1}^{-1}$.

Let $n > k$. Applying (6), we obtain

$$\left| P_{v_{n-1}}\left(\sum_{j=0}^{c_n} e_j f_j \right) \right| = \left| P_{v_{n-1}}\left(\sum_{j=0}^{c_n} y_{n,j} x^j \right) \right| = \sum_{j=0}^{v_{n-1}} |y_{n,j}| < a_n^{-1} + a_{n+1}^{-1} < 2a_n^{-1}.$$

Moreover, we have

$$\left| P_{v_{n-1}}\left(\sum_{j=v_{n-1}+1}^{c_n} e_j f_j \right) \right| \leq \sum_{j=v_{n-1}+1}^{c_n} |e_j| \max_{j \in (v_{n-1}, c_n]} |P_{v_{n-1}}(f_j)| \leq a_n^{-1} a_{n-1}.$$

Thus, $|\sum_{j=0}^{v_{n-1}} e_j f_j| = |P_{v_{n-1}}(\sum_{j=0}^{c_n} e_j f_j) - P_{v_{n-1}}(\sum_{j=v_{n-1}+1}^{c_n} e_j f_j)| < 2a_n^{-1} a_{n-1}$. For some $(s_{n,j})_{j=0}^{v_{n-1}} \subset \mathbf{K}$ we have $\sum_{j=0}^{v_{n-1}} e_j f_j = \sum_{j=0}^{v_{n-1}} s_{n,j} x^j$. Hence,

$$\sum_{j=0}^{v_{n-1}} |e_j| = \left| \sum_{j=0}^{v_{n-1}} s_{n,j} x^j \right| \leq \sum_{j=0}^{v_{n-1}} |s_{n,j}| B_{n-1} \leq \left| \sum_{j=0}^{v_{n-1}} s_{n,j} x^j \right| B_{n-1} \leq \frac{2a_{n-1}}{a_n} B_{n-1} < a_{n-1}^{-1}$$

for every $n > k$. It follows that $\sum_{j=0}^{\infty} |e_j| = 0$, so $e = 0$; a contradiction. □

Lemma 6. *Let $n \in \mathbf{N}$. Let $m \in [0, n]$, $0 < \varepsilon < 2^{-1}$ and $2 < M < \varepsilon^{-1}$. Assume that $y \in F_n$ with $|y| \leq M$ and $|P_m(y)| \geq M\varepsilon$. Then there exists a $q \in F_n$ with $|q| \leq \varepsilon^{-(2n)!}$ such that $|P_n(qy) - x^m| < \varepsilon$.*

Proof. Clearly, $y = \sum_{i=0}^{n} y_i x^i$ for some $(y_i)_{i=0}^{n} \subset \mathbf{K}$. By assumption, we have $\sum_{i=0}^{n} |y_i| \leq M$ and $\sum_{i=0}^{m} |y_i| \geq M\varepsilon$.

If $n = 1$ and $m = 0$, then $q = y_0^{-1} x^0 - y_0^{-2} y_1 x^1$ satisfies our claim.

If $n = 1$ and $m = 1$, then we can take $q = y_1^{-1}x^0$ if $|y_0| < 2^{-1}\varepsilon^2 M$, and $q = y_0^{-1}x^1$ if $|y_0| \geq 2^{-1}\varepsilon^2 M$.

Suppose that our claim is true for $n = k \geq 1$. We shall prove that it is true for $n = k+1$.

Let $m = 0$, $0 < \varepsilon < 2^{-1}$ and $2 < M < \varepsilon^{-1}$. Then we put $q_0 = y_0^{-1}$ and $q_{i+1} = -y_0^{-1}\sum_{j=1}^{i+1} y_j q_{i+1-j}$ for $0 \leq i \leq k$ and $q = \sum_{i=0}^{k+1} q_i x^i$. It is easy to see that $P_{k+1}(qy) - x^0 = 0$ and

$$|q| = \sum_{i=0}^{k+1} |qi| \leq \sum_{i=0}^{k+1} \varepsilon^{-i}|y_0|^{-1} \leq \frac{\varepsilon^{-(k+2)} - 1}{M\varepsilon(\varepsilon^{-1} - 1)} \leq \varepsilon^{-(k+2)} \leq \varepsilon^{-(2k+2)!}.$$

Let $1 \leq m \leq k+1$, $0 < \varepsilon < 2^{-1}$ and $2 < M < \varepsilon^{-1}$. Consider two cases.

Case 1. $|y_0| < (\varepsilon/2)^{(2k)!+1}$. Then $\sum_{i=1}^{k+1} |y_i| \leq M$ and $\sum_{i=1}^{m} |y_i| \geq M(\varepsilon/2)$. By the inductive assumption for $\bar{y} = \sum_{i=1}^{k+1} y_i x^{i-1}$, there exists a $q = \sum_{i=0}^{k} q_i x^i$ with $|q| \leq (\varepsilon/2)^{-(2k)!}$ and $|P_k(q\bar{y}) - x^{m-1}| < (\varepsilon/2)$. Then we have $|q| \leq \varepsilon^{-(2k+2)!}$ and

$$\begin{aligned} |P_{k+1}(qy) - x^m| &= |P_{k+1}(q(x\bar{y} + y_0 x^0)) - x x^{m-1}| \\ &= |x(P_k(q\bar{y}) - x^{m-1}) + y_0 q| < \varepsilon. \end{aligned}$$

Case 2. $|y_0| \geq (\varepsilon/2)^{(2k)!+1}$. Then we put $q_i = 0$ for $0 \leq i < m$, $q_m = y_0^{-1}$ and $q_{m+j} = -y_0^{-1}\sum_{i=1}^{j} y_i q_{m+j-i}$ for $1 \leq j \leq k+1-m$. For $q = \sum_{i=0}^{k+1} q_i x^i$ it is easy to check that $P_{k+1}(qy) - x^m = 0$ and

$$\begin{aligned} |q| &= \sum_{j=0}^{k+1-m} |q_{m+j}| \leq \frac{1}{|y_0|}\sum_{j=0}^{k+1-m}\left(\frac{M}{|y_0|}\right)^j \\ &\leq \frac{1}{|y_0|}\frac{(M/|y_0|)^{k+1} - 1}{(M/|y_0|) - 1} \leq \varepsilon^{-(2k+2)!}. \qquad \square \end{aligned}$$

Remark. It is easy to see that the assumption $y \in F_n$ in Lemma 6 can be replaced by the assumption $y \in F$.

Now we are ready to show our main result.

Theorem 7. *Assume that* $d_1 \geq \alpha^4$ *and* $d_{n+1} \geq \alpha^{(nd_n)!}$ *for every* $n \in \mathbf{N}$. *Then the linear continuous operator* T *on* E *has no nontrivial closed invariant subspace.*

Proof. Let M be a closed subspace of E with $M \neq \{0\}$ such that $T(M) \subset M$. Then $g(T)(M) \subset M$ for every $g \in F$. Let $e \in M$ with $0 < \|e\| \leq 1$. We shall prove that for every $\delta > 0$ there exists an $f \in F$ such that $\|f(T)e - x^0\| < \delta$. Let $\delta > 0$. Let $k > 2$ with $a_{k-1} > 6\delta^{-1}$. By Lemma 5 we have $|P_{(m-k)a_m}(Q_m e)| \geq a_m^{-1}$ for some $m > k$. For $R_m = (a_m A_m)^{[2(m-2)a_m]!}$ we have $\alpha a_m A_m R_m < b_m$, since $a_m A_m < \alpha^{a_m}$ and $\alpha^{[(2m-1)a_m]!} < b_m$. Put $y = Q_m e$. Then $|y| \leq \sqrt{A_m}\|Q_m\|\|e\| \leq A_m$.

By Lemma 6 there exists a $q \in F_{(m-2)a_m}$ with $|q| \leq R_m$ such that

$$|P_{(m-2)a_m}(qy) - x^{(m-k)a_m}| < (a_m A_m)^{-1}. \tag{8}$$

(S1) Put $f = b_m^{-1} x^{a_m + b_m} q$ and $S = K_{m,1}$. Then $f = \sum_{s\in S} t_s x^s$ for some $(t_s)_{s\in S} \subset \mathbf{K}$. Let $z = fy$. Using Lemma 4 we get $\|f(T)e - z\| = \|\sum_{s\in S} t_s(T^s - T^s Q_m)e\| \leq \sum_{s\in S} |t_s|\alpha = |f|\alpha = |q|\alpha b_m^{-1} < R_m \alpha b_m^{-1} < a_m^{-1}$.

(S2) Let $a = (m-1)a_m + b_m$ and $b = 2(m-1)a_m + b_m$. Clearly, $z \in F_b$; so $z = \sum_{j=0}^{b} s_j x^j$ for some $(s_j)_{j=0}^{b} \subset \mathbf{K}$. Then $\|z - P_a z\| = \|\sum_{j=a+1}^{b} s_j x^j\| \leq \sum_{j=a+1}^{b} |s_j| \max_{a<j\leq b} \|x^j\| \leq |z|$, since $\|x^j\| = \alpha^{-[3b_m - 2j]/\sqrt{4b_m}} \leq \alpha^{-[b_m - 4(m-1)a_m]/\sqrt{4b_m}} \leq 1$ for $j \in (a, b] \subset L_{m,2}$. We have $|z| \leq |f||y| = b_m^{-1}|q||y| \leq b_m^{-1} R_m A_m < a_m^{-1}$. Thus, $\|z - P_a z\| < a_m^{-1}$.

(S3) Let $t = x^{a_m} qy$ and $c = (m-1)a_m$. Clearly $t \in F_{2c}$; so $t = \sum_{j=a_m}^{2c} \gamma_j x^j$ for some $(\gamma_j)_{j=a_m}^{2c} \subset \mathbf{K}$. For $j \in [a_m, c]$, we have $j + b_m \in K_{m,1}$. Thus, $\|b_m^{-1} x^{j+b_m} - x^j\| = b_m^{-1}\|f_{j+b_m}\| = b_m^{-1}$. Hence, we get $\|P_a z - P_c t\| = \|P_a(b_m^{-1} x^{b_m} t) - P_c t\| = \|P_a(\sum_{j=a_m}^{2c} \gamma_j b_m^{-1} x^{j+b_m}) - P_c(\sum_{j=a_m}^{2c} \gamma_j x^j)\| = \|\sum_{j=a_m}^{c} \gamma_j b_m^{-1} x^{j+b_m} - \sum_{j=a_m}^{c} \gamma_j x^j\| \leq \sum_{j=a_m}^{c} |\gamma_j| \, \|b_m^{-1} x^{j+b_m} - x^j\| \leq |t| b_m^{-1}$. Thus, $\|P_a z - P_c t\| \leq b_m^{-1} R_m A_m < a_m^{-1}$, since $|t| \leq |q||y| \leq R_m A_m$.

(S4) Using (8) we get $|P_c t - x^{(m-k+1)a_m}| = |x^{a_m}(P_{c-a_m}(qy) - x^{(m-k)a_m})| = |P_{(m-2)a_m}(qy) - x^{(m-k)a_m}| < (a_m A_m)^{-1}$. Hence, $\|P_c t - x^{(m-k+1)a_m}\| \leq a_m^{-1}$.

(S5) By Lemma 1 we have $\|x^{(m-k+1)a_m} - x^0\| \leq 2a_{k-1}^{-1}$. Since $f(T)e - x^0 = (f(T)e - z) + (z - P_a z) + (P_a z - P_c t) + (P_c t - x^{(m-k+1)a_m}) + (x^{(m-k+1)a_m} - x^0)$, we obtain $\|f(T)e - x^0\| \leq 6a_{k-1}^{-1} < \delta$.

We have shown that for every $\delta > 0$ there exists an $f \in F$ such that $\|f(T)e - x^0\| < \delta$. It follows that $x^0 \in M$. Hence, $x^n = T^n x^0 \in M$ for all $n \in \mathbf{N}_0$. Thus, $F \subset M$, so $M = E$. ◻

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