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GRAPH DOUGLAS ALGEBRAS

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ABSTRACT. We develop a notion of Douglas algebras for the free semi-groupoid algebras arising from directed graphs. We analyze two extreme examples of the structure of such algebras, the first coming from the graphs with a single vertex, the second coming from cycle graphs. In the first example we demonstrate a lack of algebraic structure while in the second example we completely describe the Douglas algebras.

1. Introduction. The non self-adjoint operator algebras associated to directed graphs are often viewed as noncommutative generalizations of the algebra H^∞. In particular, there are many results which extend the classical results about H^∞ to the directed graph framework, including for example: a Beurling type theorem [**11**], a functional calculus [**10**] and interpolation results [**8**].

This paper is a general discussion of the notion of Douglas algebras for directed graph operator algebras. It takes its shape, primarily, as a presentation of two classes of examples at opposite extremes of results we might hope for from the commutative context. The first class of examples consists of the algebras $\mathcal{L}_n$ arising from the graph with a single vertex and n directed edges. For the second class of examples we discuss the cycle graphs.

The starting points for this article are a pair of results on the spaces of the form $H^\infty + C(\mathbf{T})$ in the context of directed graphs. The first comes from [**4**] where in a discussion after Lemma 1.11 an analogue of the space $H^\infty + C(\mathbf{T})$ in the context of $\mathcal{L}_n$ is shown to be closed, although it is not an algebra. The second is from a paper [**1**] where a slightly different analogue (a distinction we will take up in Section 3) was shown to be a closed algebra in the case that the graph is a cycle graph of length n.

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In the first section of the paper we discuss background results on Douglas algebras to emphasize connections with the classical situation of Douglas algebras. The interested reader is directed to Chapter 6 of [**6**] for a very readable account of the ideas we pursue as well as references for more technical details on the theory of Douglas algebras. In the section following we establish what we feel is the appropriate context for the remaining discussion for the general directed graph operator algebras. We will then prove elementary results for the case of general directed graphs with no sources.

In the first of the two main sections of the paper we present a class of examples which is somewhat surprising in how badly directed graph operator algebras can behave in comparison to standard Douglas algebras. In the second main section we will present a different class of examples to show how nicely certain examples can behave. In a final section we will discuss differences between the two examples.

2. Douglas algebras. In the classical setting a Douglas algebra is any norm closed algebra between H^∞ and $L^\infty(\mathbf{T})$. The standard construction of an arbitrary Douglas algebra is to let Σ be some semi-group of inner functions in H^∞. The Douglas Algebra $\mathfrak{A}_\Sigma$ is the norm closure of the set

$$\{\varphi\overline{\psi} : \varphi \in H^\infty, \psi \in \Sigma\}.$$

That this set forms an algebra follows since L^∞ is commutative and Σ is a semi-group. It was established in a pair of papers [**2, 13**] that every norm closed algebra between H^∞ and L^∞ is of this form.

If one looks at the trivial semi-group containing just the identity, then the Douglas algebra is H^∞. On the other hand, L^∞ is given by the semi-group of all inner functions in H^∞. A more interesting example is provided by the semi-group $\{z^n : n \geq 0\}$. In this case the algebra has the surprising form $H^\infty + C(\mathbf{T})$. That this set is a norm closed algebra was first seen in [**15**]. Perhaps more surprising is the fact that there are no norm closed algebras properly between H^∞ and $H^\infty + C(\mathbf{T})$.

In the rest of this paper we will explore generalizations of these ideas to the operator algebras associated to directed graphs. The motivation for this work is provided by identifying the operator algebra of the graph with a single vertex and a single edge as H^∞. We must be careful however in this identification since the von Neumann algebra generated by the left regular representation of this graph is not L^∞; it is not even commutative. Hence, before discussing algebras between H^∞ and the von Neumann algebra it generates, we must make sure we are in the right context.

3. Getting the setting right and elementary results. By a directed graph Q we mean a pair of sets E and V (whose elements are called edges and vertices, respectively) together with a pair of functions $s : E \to V$ and $r : E \to V$, called the source and range maps, respectively. The set E will be called the edge set of Q, and V will be called the vertex set of Q. We say that a directed graph has no sources if the map r is onto. We say that a finite sequence of edges $e_1 e_2 \cdots e_n$ is a path in Q if $r(e_i) = s(e_{i-1})$ for all $2 \leq i \leq n$. We say that such a path is a cycle if $r(e_1) = s(e_n)$. We let $\mathcal{P}$ denote the set of all paths in Q.

For each path $w \in \mathcal{P}$ denote the basis element in $\ell^2(\mathcal{P})$ which is 1 at w and 0 elsewhere by e_w. Now for each edge f define an operator $L_f : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P})$ by

$$L_f(e_w) = \begin{cases} e_{fw} & \text{if } s(f) = r(w) \\ 0 & \text{else.} \end{cases}$$

Also, for each $v \in V$ define $P_v : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P})$ by

$$P_v(e_w) = \begin{cases} e_w & \text{if } v = r(w) \\ 0 & \text{else.} \end{cases}$$

Denote the norm closed algebra generated by $\{L_e, P_v : e \in E, v \in V\}$ by $A(Q)$, and its WOT-closure by $\mathcal{L}_Q$. We call the second algebra the free semi-groupoid algebra of Q, see [**11**] for more information about these algebras.

We will be analyzing two particular cases in more depth. The first case arises from the graph B_n given by a single vertex and n directed edges. To simplify notation in this particular case, we will denote the algebras by A_n and $\mathcal{L}_n$. Also, when referring to the generators of the algebra which come from the edges, we will put an ordering on the edge set, and denote the ith generator by L_i. The other example will correspond to the cycle graph $\mathcal{C}_n$ with n vertices $\{v_1, v_2, \ldots, v_n\}$ and n edges $\{e_1, e_2, \ldots e_n\}$. In this case the range and source maps will be one-to-one and onto, such that $e_1 e_2 \cdots e_n$ is a cycle.

We start by analyzing the graph B_1 which corresponds to the classical case of H^∞. For the algebra $\mathcal{L}_1$ we have already mentioned that the left regular representation, though isomorphic to H^∞, does not generate a commutative von Neumann algebra. However, the von Neumann algebra generated by the left regular representation of $\mathcal{L}_1$ contains a copy of the compact operators on $\ell^2(\mathcal{P})$. Further, the image of this von Neumann algebra under the Calkin map is isomorphic to L^∞. Another important point here is that the Calkin map is an isometry on $\mathcal{L}_1$ and hence the isometric image of $\mathcal{L}_1$ under the Calkin map is H^∞.

Now notice that A_1, the unital norm closed subalgebra of $\mathcal{L}_1$ generated by the generator L_1, is isomorphic to $A(\mathbf{D})$. Let us denote by $\mathcal{E}_1$ the C^*-algebra generated by A_1 acting as operators on $\ell^2(\mathcal{P})$. It is not hard to see that the image of $\mathcal{E}_1$ under the Calkin map is $C(\mathbf{T})$ which is the C^*-envelope of $A(\mathbf{D})$. These facts are going to help us set the stage when we deal with arbitrary directed graphs. We begin with a result of Katsoulis and Kribs. Recall that a graph has no sources if for every $v \in V(Q)$ there exists an edge $e \in E(Q)$ such that $r(e) = v$.

Proposition 1 [**10**, Proposition 7.3]. *A directed graph Q has no sources if and only if the Calkin map is an isometry on $\mathcal{L}_Q$.*

In view of this our first simplification will be to deal only with graphs with no sources. There is a method of *desingularization*, first seen in [**7**] and used in [**9**], which effectively removes all sources of a directed graph by adding a "tail" at a source, yet still preserves the necessary information about the operator algebras thus generated. We feel however that for the purposes of this paper the extreme generality achieved through this ingenious technique will distract from the main purpose of the paper so we forgo the technicalities. Just to emphasize, we will assume, unless stated otherwise, that from this point on all graphs have no sources. We will try to remind the reader of this fact as appropriate.

Thus for a directed graph Q we can identify $\mathcal{L}_Q$ and $A(Q)$ with their image under the Calkin map unless the actual context is ambiguous. However, for the $*$-algebras, the context is more important so we will denote by $W(Q)$ the von Neumann algebra generated by $\mathcal{L}_Q$ as a subalgebra of $B(\ell^2(\mathcal{P}))$. We will then let $W_e(Q)$ be the image of $W(Q)$

under the Calkin map. In a similar manner, define $C^*(Q)$ and $C_e^*(Q)$ to be the C^*-algebras generated by $A(Q)$ in the appropriate contexts. We are now ready to define our notion of "noncommutative Douglas algebras" which we plan to analyze below.

Definition 1. We say that an algebra $\mathfrak{A}$ is a *graph Douglas algebra* for the graph Q if $\mathfrak{A}$ is a norm closed subalgebra with $\mathcal{L}_Q \subseteq \mathfrak{A} \subseteq W_e(Q)$.

One may be tempted to make the obvious conjecture that $C_e^*(Q)+\mathcal{L}_Q$ is a graph Douglas algebra and that there are no proper subalgebras between $\mathcal{L}_Q$ and $C_e^*(Q)+\mathcal{L}_Q$. This conjecture turns out to be very wrong, as we will see in Sections 4 and 5. However, we do have more to say before we look at our examples.

Proposition 2. *The set $C^*(Q)+\mathcal{L}_Q$ is a norm closed subspace of $W(Q)$ for all directed graphs Q with no sources.*

Proof. We will use Theorem 1.2 of [**4**] which gives conditions under which the sum of two closed subspaces of a Banach space are closed. Notice by [**4**, Lemma 1.1] that there exists a family of maps $\Sigma_k : W^*(Q) \to C^*(Q)$, each contractive such that every element $T \in C^*(Q)$ is the norm limit $\lim \Sigma_k(T)$. Also notice that if $X \in \mathcal{L}_Q$ then $\Sigma_k(X) \in A(Q) \subseteq \mathcal{L}_Q$. Thus, Theorem 1.2 of [**14**] applies to yield the desired conclusion. $\square$

For simplicity our proof follows the technique of [**14**]. The more traditional argument used in [**15**] works equally well in the context of $\mathcal{L}_n$, see the discussion following Lemma 3.11 in [**4**]. There it is shown that $C^*(B_n)+\mathcal{L}_{B_n}$ is the inverse image of a closed set under a quotient map. We suspect a similar technique will work for arbitrary graphs with no sources. We now present an easy corollary to the previous result which puts the result into the setting we wish to discuss. The corollary follows immediately by noting that the Calkin map is an open mapping.

Corollary 1. *The set $C_e^*(Q)+\mathcal{L}_Q$ is a norm closed subspace of $W_e(Q)$ for all directed graphs Q with no sources.*

By breaking the proof into two parts we have simplified the proof but the reader might wonder what is gained by passing to the quotient. To simplify notation we will denote the compact operators in $W(Q)$ as $\mathcal{K}_Q$. For the following we refer the reader to [**10**] where it was first stated as Corollary 7.4. We present only the proof due to its simplicity.

Proposition 3 [**10**, Corollary 7.4]. *The set $\mathcal{L}_Q+\mathcal{K}_Q$ is a norm closed subalgebra of $W(Q)$ for Q a graph with no sources.*

Proof. That the set is norm closed follow by noticing that $\mathcal{L}_Q+\mathcal{K}_Q$ is the inverse image of the closed algebra $\mathcal{L}_Q \subset W_e(Q)$ under the Calkin map. That the set is an algebra follows, since $\mathcal{K}_Q$ is an ideal in $W(Q)$. □

If we now focus on the example of $\mathcal{L}_1$ we notice that the standard result, see Corollary 6.40 in [**6**], on Douglas algebras can be stated as follows:

Proposition 4. *Let $\mathfrak{A}$ be a norm closed subalgebra of $W(B_1)$ such that $\mathcal{L}_1 \subset \mathfrak{A} \subset \mathfrak{L}_1 + C^*(A)$ with both inclusions proper. Then $\mathcal{L}_1 \subseteq \mathfrak{A} \subseteq \mathcal{L}_1 + \mathcal{K}_{B_1}$.*

Proof. Under the Calkin map any such algebra will have image H^∞. Since there are no closed algebras properly between H^∞ and $H^\infty + C(\mathbf{T})$, the result now follows. □

In effect, we only complicate the picture by adding compact operators. We are then left with the question of which algebras lie between $\mathcal{L}_1$ and $\mathcal{L}_1 + \mathcal{K}_{B_1}$. This, however, is not necessarily an easy task. Let K be the compact operator $1 - L_1^2(L_1^*)^2$. Notice that the algebra generated by K inside the compact operators is given by $\{\alpha K : \alpha \in \mathbf{C}\}$ since K is a projection. Now the algebra generated by K and $\mathcal{L}_2$ will not contain all of the compact operators. This can be seen by viewing $\mathcal{L}_2$ as infinite dimensional lower triangular Toeplitz matrices and noting that the algebra generated by K and $\mathcal{L}_2$ will be upper triangular and hence won't contain all the compact operators.

On the other hand, the following proposition shows that when we look at algebras between $\mathcal{L}_1 + \mathcal{K}_{B_1}$ and $W(B_1)$ there are no new algebras that we do not see in the quotient algebra.

Proposition 5. *Every norm closed subalgebra $\mathcal{L}_1 + \mathcal{K}_{B_1} \subseteq \mathfrak{A} \subseteq W(B_1)$ is of the form $A_\Sigma + \mathcal{K}_{B_1}$ where A_Σ is the norm closure of*

$$\{\varphi\psi^* : \varphi \in \mathcal{L}_1, \psi \in \Sigma\}$$

where Σ is a semi-group of isometries in $\mathcal{L}_1$.

Proof. This follows from the previous proposition and by noting that the image of any such algebra under the Calkin map will yield a Douglas subalgebra of L^∞. $\square$

These questions are only made more technical for arbitrary graphs with no sources and so we dispense with the technicalities inherent in adding compact operators. We now present an example of a graph with a source to see why, in that context, we cannot throw away the compact operators so easily. The readers presumably can find more complicated examples to suit their own tastes.

Example. Let Q be the directed graph

$$\bullet \longrightarrow \bullet.$$

In this case it is well known that $\ell^2(\mathcal{P})$ is isomorphic to $\mathbf{C}^3$ and that $A(Q)$ and $\mathcal{L}_Q$ are isomorphic since the WOT and norm topology coincide on finite-dimensional spaces. In fact, $\mathcal{L}_Q$ is the algebra

$$\left\{ \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbf{C} \right\}.$$

Further, notice that every operator on $\mathbf{C}^3$ is compact. In this case one sees easily that $\mathcal{L}_Q + C^*(Q) = W(Q) = M_2(\mathbf{C}) \oplus \mathbf{C}$ and that the only subalgebra properly between $\mathcal{L}_Q$ and $W(Q)$ is of the form $T_2 \oplus \mathbf{C}$ where T_2 denotes the lower triangular 2×2 matrices.

4. The first example. Here we look at the graphs B_n, given by a single vertex with n edges. Notice that B_1 is the classical case and so we will exclude that case in our analysis. Another important fact is that $C_e^*(B_n)$ is isomorphic to the Cuntz algebra, since it is generated by isometries L_i such that $\sum L_i L_i^* = 1$, see [**3**, Corollary V.4.7]. We know from Corollary 1 that the set $\mathcal{L}_n + C_e^*(B_n)$ is norm closed in $W_e(Q)$. We notice first that this subspace is not an algebra.

Proposition 6. *For $n \geq 2$, the space $\mathcal{L}_n + C_e^*(B_n)$ is not an algebra.*

Proof. Let $X \in \mathcal{L}_n$ be chosen such that $X - Y \notin C_e^*(B_n)$ for all $Y \in C_e^*(B_n)$ and X is in the norm closed nonunital subalgebra of $\mathcal{L}_n$ generated by L_1. Notice that the nonunital subalgebra of $\mathcal{L}_n$ generated by L_1 has empty intersection with the ideal in $\mathcal{L}_n$ generated by L_2, call this ideal I. Now, if $\mathcal{L}_n + C_e^*(B_n)$ is an algebra, then $XL_2^* \in \mathcal{L}_n + C_e^*(B_n)$. Notice however that if $A \in C_e^*(B_n)$ and $B \in \mathcal{L}_n$ with $XL_2^* = A + B$ then $X = AL_2 + BL_2$, and hence $AL_2 \in \mathcal{L}_n$. Using elementary calculations for the Cuntz algebra, see [**3**, Lemma V.4.1], we get that $A = YL_2^*$ with $Y \in \mathcal{L}_n$. But then we get $B = (X - Y)L_2^*$ which unless $X = Y$ is impossible since $B \in \mathcal{L}_n$. Of course our choice of X contradicts the possibility of $X = Y$. $\square$

It is interesting to note that the norm closed algebra generated by $\mathcal{L}_n + C_e^*(B_n)$ does have an interesting structure which is reminiscent of Douglas algebras. Let Σ_n be the semigroup of isometries in $\mathcal{L}_n$ generated by $\{L_i : 1 \leq i \leq n\}$.

Proposition 7. *The norm closed algebra generated by $\mathcal{L}_n + C_e^*(B_n)$ is the norm closure of the of the set*

$$A_{\Sigma_n} := \operatorname{span}\{AX^* : A \in \mathcal{L}_n, X \in \Sigma_n\}.$$

Proof. Recalling calculations from the Cuntz algebra, again see [**3**, Lemma V.4.1], we know that an arbitrary element of $C_e^*(B_n)$ can be written as a norm limit of elements of the form

$$\operatorname{span}\{L^\mu (L^*)^\nu\}$$

where μ and ν are multi-indices signifying a sequence of elements in the usual generating set for $\mathcal{L}_n$. The result now follows since $\operatorname{span}\{L^\mu(L^*)^\nu\}$ is a subset of A_{Σ_n}. $\square$

It may be tempting to ask whether there are any closed subalgebras between $\mathcal{L}_n$ and the norm closed algebra generated by $\mathcal{L}_n + C_e^*(B_n)$. And, if so, is there a nice characterization of them. However, by looking at the generators from the previous proposition there is an obvious set of such algebras, each with a nice characterization reminiscent of Douglas algebras.

Proposition 8. *Let Σ be a sub semi-group of Σ_n. Then the norm closure of the set*

$$A_\Sigma := \operatorname{span}\{AX^* : A \in \mathcal{L}_n, X \in \Sigma\}$$

is a graph Douglas algebra. Moreover, if Σ is generated by a proper nonempty subset of $\{L_i\}$, then the algebra A_Σ sits properly between $\mathcal{L}_n$ and the algebra generated by $\mathcal{L}_n + C_e^*(B_n)$.

Proof. A_Σ is an algebra for the same reason that A_{Σ_n} is, it is contained in an algebra, and contains a dense subspace of the same algebra. Only the second part of the statement requires proof. But notice, if the semigroup generated by L_i is not in Σ, then in particular L_i^* is not in A_Σ and hence the inclusion is proper. $\square$

It is however not evident that if Σ is any semi-group of isometries in $\mathcal{L}_n$ then the norm closure of the set $\operatorname{span}\{AX^* : A \in \mathcal{L}_n, X \in \Sigma\}$ will be a graph Douglas algebra. We now present an example of a graph Douglas algebra properly between $\mathcal{L}_n$ and $\mathcal{L}_n + C_e^*(B_n)$ which does not have this form.

Example. Let A be an element of $\mathcal{L}_n \setminus C_e^*(B_n)$, for $n \geq 2$. We claim that there is a norm closed algebra generated by $\mathcal{L}_n$ and elements of the form $AL_1L_2^*$ which is a graph Douglas algebra that can not be written in the special form described in Proposition 8 above.

We look at the set of elements $\mathfrak{A}_{L_1L_2^*} := \{A + BL_1L_2^* : A, B \in \mathcal{L}_n\} \subseteq \mathcal{L}_n + C_e^*(B_n)$. We first show that this set forms an algebra.

To see this, notice that $(L_1L_2^*)^2 = 0$ because $L_2^*L_1$ is zero. Further, $L_1L_2^*L_j$ is equal to L_1 if $j = 2$ and is 0 otherwise. It follows that $\mathfrak{A}_{L_1L_2^*}$ is an algebra containing $L_1L_2^*$ and $\mathcal{L}_n$. That this algebra is norm closed follows by its definition. Notice also that since $L_1L_2^*$ is nilpotent then $L_1L_2^* \notin \mathcal{L}_n$ since $\mathcal{L}_n$ contains no quasinilpotent elements [**4**, Corollary 1.8]. Similarly, if $L_1L_2^* \in (\mathcal{L}_n)^*$ then $L_2L_1^*$ would be a nonzero nilpotent element of $\mathcal{L}_n$. This too is an impossibility and hence $L_1L_2^* \notin \mathcal{L}_n \cup (\mathcal{L}_n)^*$.

Lastly if this algebra were of the form in Proposition 8 then since $L_1L_2^*$ is in the algebra the semi-group Σ would have to contain L_2. But if Σ contains L_2 then $(L_2^*)^2$ would be in the algebra which it clearly is not.

Question. Is it true that if we let Σ be the set of isometries in $\mathcal{L}_n$ then the associated Douglas algebra is $W_e(B_n)$?

5. The second example. We now investigate an example where the graph Douglas algebras can be well understood. For this we let the graph, $\mathcal{C}_n$, be the cycle graph of length n. We begin by noticing that the algebra $\mathcal{L}_{\mathcal{C}_n}$, once we quotient by the compact operators, can be written in the form:

$$\begin{bmatrix} f_{1,1}(z^n) & z^{n-1}f_{1,2}(z^n) & z^{n-2}f_{1,3}(z^n) & \cdots & zf_{1,n}(z^n) \\ zf_{2,1}(z^n) & f_{2,2}(z^n) & z^{n-1}f_{2,3}(z^n) & \cdots & z^2f_{2,n}(z^n) \\ z^2f_{3,1}(z^n) & z^1f_{3,2}(z^n) & f_{3,3}(z^n) & \cdots & z^3f_{3,n}(z^n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z^{n-1}f_{n,1}(z^n) & z^{n-2}f_{n,2}(z^n) & z^{n-3}f_{n,3}(z^n) & \cdots & f_{n,n}(z^n) \end{bmatrix}$$

where $f_{i,j} \in H^\infty$ for all $1 \le i, j \le n$.

Here $W_e(\mathcal{C}_n)$ is given by

$$\begin{bmatrix} f_{1,1}(z^n) & z^{n-1}f_{1,2}(z^n) & z^{n-2}f_{1,3}(z^n) & \cdots & z^1f_{1,n}(z^n) \\ z^1f_{2,1}(z^n) & f_{2,2}(z^n) & z^{n-1}f_{2,3}(z^n) & \cdots & z^2f_{2,n}(z^n) \\ z^2f_{3,1}(z^n) & z^1f_{3,2}(z^n) & f_{3,3}(z^n) & \cdots & z^3f_{3,n}(z^n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z^{n-1}f_{n,1}(z^n) & z^{n-2}f_{n,2}(z^n) & z^{n-3}f_{n,3}(z^n) & \cdots & f_{n,n}(z^n) \end{bmatrix}$$

where $f_{i,j} \in L^\infty$ for all $1 \le i, j \le n$, and $C_e^*(\mathcal{C}_n)$ can be written as

$$\begin{bmatrix} f_{1,1}(z^n) & z^{n-1}f_{1,2}(z^n) & z^{n-2}f_{1,3}(z^n) & \cdots & z^1 f_{1,n}(z^n) \\ z^1 f_{2,1}(z^n) & f_{2,2}(z^n) & z^{n-1}f_{2,3}(z^n) & \cdots & z^2 f_{2,n}(z^n) \\ z^2 f_{3,1}(z^n) & z^1 f_{3,2}(z^n) & f_{3,3}(z^n) & \cdots & z^3 f_{3,n}(z^n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z^{n-1} f_{n,1}(z^n) & z^{n-2} f_{n,2}(z^n) & z^{n-3} f_{n,3}(z^n) & \cdots & f_{n,n}(z^n) \end{bmatrix}$$

where $f_{i,j} \in C(\mathbf{T})$ for all $1 \le i, j \le n$.

It was established in [**1**] that $\mathcal{L}_{\mathcal{C}_n} + C_e^*(\mathcal{C}_n)$ is a norm closed subalgebra of $W_e(\mathcal{C}_n)$. We now take this result and extend it to describe all norm closed subalgebras between $\mathcal{L}_{\mathcal{C}_n}$ and $W_e(\mathcal{C}_n)$. We take advantage of known results about Douglas algebras in our description.

Theorem 1. *Let $\mathcal{L}_{\mathcal{C}_n} \subseteq A \subseteq W_e(\mathcal{C}_n)$. If there exists an i such that the i–i entry of A contains an element $f(z^n)$ such that $f \in L^\infty \setminus H^\infty$, then A can be written as*

$$\begin{bmatrix} f_{1,1}(z^n) & z^{n-1}f_{1,2}(z^n) & z^{n-2}f_{1,3}(z^n) & \cdots & z^1 f_{1,n}(z^n) \\ z^1 f_{2,1}(z^n) & f_{2,2}(z^n) & z^{n-1}f_{2,3}(z^n) & \cdots & z^2 f_{2,n}(z^n) \\ z^2 f_{3,1}(z^n) & z^1 f_{3,2}(z^n) & f_{3,3}(z^n) & \cdots & z^3 f_{3,n}(z^n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z^{n-1} f_{n,1}(z^n) & z^{n-2} f_{n,2}(z^n) & z^{n-3} f_{n,3}(z^n) & \cdots & f_{n,n}(z^n) \end{bmatrix}$$

where $f_{i,j} \in A_\Sigma$ for all $1 \le i, j \le n$, and A_Σ is a Douglas algebra containing $H^\infty + C(\mathbf{T})$.

Proof. Notice that, since A is an algebra, the restriction of A to the i–j position is a subspace of A, call it $A_{i,j}$. Next notice that $A_{i,i}$ is a subalgebra of A for all $1 \le i \le n$. Notice that this subalgebra will be of the form $\{f(z^n) : f \text{ in a Douglas subalgebra of } L^\infty\}$. Now if there exists some function $f(z^n)$ in $A_{i,i}$, such that $f \in L^\infty \setminus H^\infty$, then $A_{i,i}$ contains $\{f(z^n) : f \in H^\infty + C(\mathbf{T})\}$.

We will show that this implies that $A_{j,k} = z^{k-j} A_{i,i}$ for $j \ge k$ and $A_{j,k} = z^{k-j+n} A_{i,i}$ for $j \ge i$. We first notice that if $j \ge k$ then $z^{k-j} A_{j,j} \subseteq A_{j,k}$ and if $k \ge j$ then $z^{k-j+n} A_{j,j} \subseteq A_{j,k}$. Similarly, if $j \ge k$ then $z^{k-j} A_{k,k} \subseteq A_{j,k}$ and if $k \ge j$ then $z^{k-j+n} A_{k,k} \subseteq A_{j,k}$.

To see this in the case of $j = 1, k = 2$, look at

$$\begin{bmatrix} A_{1,1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & z^{n-1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & z^{n-1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & A_{2,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and notice that these are subsets of A. The cases for general j and k follow in the same manner.

Next notice in a similar manner that for all $j \geq k$ we have $n^{n-k-j}A_{j,k} \subseteq A_{j,j}$ and $z^{n-(k-j)}A_{j,k} \subseteq A_{k,k}$, and a similar pair of inclusions is true when $k \geq j$. Now since, $\bar{z}^2 n \in A_{i,i}$ it follows that for $j \neq i$, $z\bar{z}^2 n \subseteq A_{i,j}$ and hence $\bar{z}^n \in A_{j,j}$ for all j. Now, for each i, there exists an $f(z^n)$ such that $f \in L^\infty \setminus H^\infty$ and hence each $A_{j,j}$ contains $H^\infty + C(\mathbf{T})$.

Without loss of generality, assume that $j \geq k$. We now show that $z^{k-j}A_{i,i} = A_{j,k}$. This actually is not difficult, since we can easily see that $z(A_\Sigma) = A_\Sigma$, for all Douglas algebras $A_\Sigma \supseteq H^\infty + C(\mathbf{T})$. Hence, $A_{j,k} \subseteq z^{k-j}A_{j,j}$ and $A_{j,k} \subseteq z^{k-j}A_{k,k}$ for all j and k, and hence $z^{k-j}A_{j,j} = A_{j,k} = z^{k-j}A_{k,k}$ for all $j \geq k$. The result now follows. □

Notice that each of these algebras contains $C_e^*(\mathcal{C}_n)$. We now describe those subalgebras which lie between $\mathcal{L}_{\mathcal{C}_n}$ and $\mathcal{L}_{\mathcal{C}_n} + C_e^*(\mathcal{C}_n)$. We denote, for $1 \leq i \leq n-1$ by Z_i the partial isometry on $\mathcal{L}_{\mathcal{C}_n}$ with a z in the i–$(i+1)$ position and 0 elsewhere. Let Z_n be the partial isometry with a z in the 1–n position. Notice that the set $S = \{Z_i : 1 \leq i \leq n\}$ together with the identity generates a semi-group of partial isometries. We will denote this semi-group by Σ_S.

Proposition 9. *If A is the closure of the set*

$$\{XY^* : X \in \mathcal{L}_{\mathcal{C}_n}, Y \in \Sigma_S\},$$

then $A = \mathcal{L}_{\mathcal{C}_n} + C_e^*(\mathcal{C}_n)$.

Proof. Clearly we have that $A \subseteq \mathcal{L}_{\mathcal{C}_n} + C_e^*(\mathcal{C}_n)$. But notice that from Theorem 1 we need only show that some diagonal element of A contains an element in $L^\infty \setminus H^\infty$. But we have $\bar{z}^n$ is in the 1-1 diagonal since it is equal to $(Z_1 Z_2 Z_3 \cdots Z_n)^*$. The result now follows. $\square$

For Λ a proper sub semi-group of Σ_S, notice that if we order S with the usual ordering then a monomial in Λ is nonzero if and only if the monomial is given by consecutive generators in the cyclic ordering inherited from S_0. For example, if $n = 5$ and Λ is generated by $\{Z_1, Z_2, Z_3, Z_5\}$, then

$$\Lambda = \{Z_1, Z_2, Z_3, Z_5, Z_1Z_2, Z_2Z_3, Z_5Z_1, Z_1Z_2Z_3, Z_5Z_1Z_2, Z_5Z_1Z_2Z_3\}.$$

On the other hand, if Λ were generated by the set $\{Z_5Z_1Z_2, Z_1, Z_3\}$, then Λ would equal

$$\{Z_1, Z_3, Z_5Z_1Z_1, Z_5Z_1Z_2Z_3\}.$$

Theorem 2. *Let A be a graph Douglas algebra contained properly in $\mathcal{L}_{\mathcal{C}_n} + C_e^*(\mathcal{C}_n)$. Then there exists a unique proper sub semi-group of Σ_S, denoted Λ, such that A is the norm closed subspace*

$$\mathcal{L}_{\mathcal{C}_n} + \sum_{t \in \Lambda} \{\alpha_t t^* : \alpha_t \in \mathbf{C}\}.$$

Proof. We see from the proof of Theorem 1 that, if a diagonal subalgebra of A contains an element $f(z^n)$ such that $f \in L^\infty \setminus H^\infty$, then A is not a proper subset of $\mathcal{L}_{\mathcal{C}_n} + C_e^*(\mathcal{C}_n)$. Hence, we need only look at what is required to prevent that from happening.

We also know that the i–j entry of an arbitrary element of A is of the form $z^{i-j} f(z^n)$ if $j \leq i$ and $z^{i-j+n} f(z^n)$ if $i \leq j$. Here we assume that $f \in L^\infty$. Now if $f(z^n)$ is in $L^\infty \setminus H^\infty$, then $\bar{z}^n f(z^n)$ is contained in a diagonal subalgebra of A. It follows that the only way that this is not the case is if $f(z^n)$ is of the form $\alpha \bar{z}^n + g(z)$ where $g(z) \in H^\infty$. In other words, there is an element t in the semi-group Σ_S such that $\alpha_t t^* \in A$, with $\alpha_t \in \mathbf{C}$. Now, if $s, t \in \Sigma_S$ with $\alpha_s s^* \in A$ and $\alpha_t t^* \in A$

for some $\alpha_s, \alpha_t \in \mathbf{C}$, then t^*s^* and s^*t^* are both in A. The result now follows. $\square$

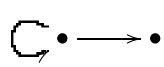
6. Concluding remarks. After having seen the previous examples an obvious question arises. Are there noncycle graphs Q with no sources such that $\mathcal{L}_Q + C_e^*(Q)$ are norm closed algebras? and if so, can we classify them?

A first partial result uses the idea of partly free algebras from [**12**] to exclude a large collection of graphs. Recall that a free semi-groupoid algebra is *partly free* if there is an injection $\iota : W^*(B_2) \to W^*(Q)$ such that the restriction of ι to $\mathcal{L}_2$ is completely isometric. We say that a cycle $w = e_1 e_2 \cdots e_m$ is *supported on* v if $r(e_i) = v$ implies $i = 1$. We call an infinite sequence of edges in Q a *proper infinite path* if any finite segment of the sequence is a path in Q and no two edges are repeated. Lastly, say that a graph satisfies the *aperiodic path property* if either there is a vertex v such that two distinct cycles are supported on v, or Q contains a proper infinite path. It is shown in [**12**, Theorem 2.5] that, for a graph Q, with no sources, $\mathcal{L}_Q$ is partly free if and only if Q has the aperiodic path property.

Proposition 10. *If $\mathcal{L}_Q$ is a partly free graph algebra, then $\mathcal{L}_Q + C_e^*(Q)$ is not an algebra.*

Proof. Since $\mathcal{L}_Q$ is partly free there is a completely isometric map $i : \mathcal{L}_2 \to \mathcal{L}_Q$ which is the restriction of an injection $\iota : W_e^*(B_2) \to W_e^*(Q)$. Now using the element $XL_2^* \in W_e^*(B_2) \setminus (\mathcal{L}_2 + C_e^*(Q))$, we notice that $\iota(XL_2^*) \in W_e^*(Q) \setminus (\mathcal{L}_Q + C_e^*(Q))$. By our choice of X these properties will be preserved by the Calkin map and the result is proved. $\square$

There are, of course, graphs which do not give rise to partly free algebras which are not cycle graphs. A simple example is the graph



Here the algebra $\mathcal{L}_Q$ can be written as

$$\begin{bmatrix} f(z) & 0 \\ g(z) & \lambda \end{bmatrix}$$

such that $f, g \in H^\infty$, $g(0) = 0$ and $\lambda \in \mathbf{C}$.

The algebra $C_e^*(Q)$ can then be written as

$$\begin{bmatrix} f_{1,1}(z) & f_{1,2}(z) \\ f_{2,1}(z) & f_{2,2}(z^2) \end{bmatrix}$$

such that $f_{i,j} \in C(\mathbf{T})$ for all i, j and $f_{1,2}(0) = f_{2,1}(0) = 0$.

This particular representation makes it clear that the set $\mathcal{L}_Q + C_e^*(Q)$ is not an algebra. An alternate approach mimics the proof of Proposition 6 to yield the following proposition.

Proposition 11. *Let Q be a graph with no sources. Assume that there exists a vertex v and a primitive cycle $w = e_1 e_2 \cdots e_n$ with $s(e_n) = v$. If there exists an edge e not in the cycle with $s(e) = v$, then $\mathcal{L}_Q + C_e^*(Q)$ is not an algebra.*

Proof. Let L_w denote the partial isometry given by $L_{e_1} L_{e_2} \cdots L_{e_n}$, and let X in the subalgebra generated by L_w be chosen so that $X \notin C_e^*(Q)$. Then $X L_e^*$ will not be in $\mathcal{L}_Q + C_e^*(Q)$. □

Recalling some examples from [**12**], we see that there are partly free graphs with no cycles and no sources, so Propositions 10 and 11 speak to different classes of graphs. However, using the description of partly free algebras from [**12**, Theorem 2.5] we can see that these two propositions, when put together, completely describe when $\mathcal{L}_Q + C_e^*(Q)$ is an algebra using simple graph theory.

Theorem 3. *Let Q be a graph with no sources. Then $\mathcal{L}_Q + C_e^*(Q)$ is an algebra if and only if Q is a cycle graph.*

Proof. Based on the preceding two propositions, we need only show that either a graph is a cycle graph (it gives rise to a partly free algebra) or there exists a cycle supported on a vertex v and an edge e not in the cycle such that $r(e) = v$.

Assume that the graph is not a cycle graph. Choose a vertex v in Q. Let $\mathfrak{P}_v$ denote the set of all finite, or infinite paths which end at v. Notice $\mathfrak{P}_v$ is nonempty since Q contains no sources. If there is any cycle which is a proper subpath of a path in $\mathfrak{P}_v$, then we are in the case of Proposition 11, so assume that every path in $\mathfrak{P}_v$ does not contain a loop as a proper subpath. In this case, since Q has no sources, every path in $\mathfrak{P}_v$ is aperiodic and hence $\mathcal{L}_Q$ is partly free. $\square$

This raises the more interesting question:

Question. What property of the cycle graph algebras allows us to describe the graph Douglas algebras so succinctly?

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