ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 1, 2010

FINITE SYMMETRIC TRILINEAR INTEGRAL TRANSFORM OF DISTRIBUTIONS. PART III

G.L. WAGHMARE

ABSTRACT. In this paper we extend the finite symmetric trilinear integral transform to distributions and establish an inversion formula using Parseval's identity. The operational calculus generated is applied to find the temperature inside hexagonal prism of semi-infinite length.

1. Introduction. Sen [**5**] with the help of trilinear coordinates has solved different types of boundary value problems relating to boundaries in the form of an equilateral triangle. Any plane in the space is described by the set

$$E = \{x = (x_1, x_2, x_3)/x_1 + x_2 + x_3 = p,\ x_i \in R,\ i = 1, 2, 3\}$$

where x_1, x_2 and x_3 are the trilinear coordinates of a point and p is height of an equilateral triangle. If $a = (k/q)p$, where k, q are integers and $k \leq q$, then the subset of E,

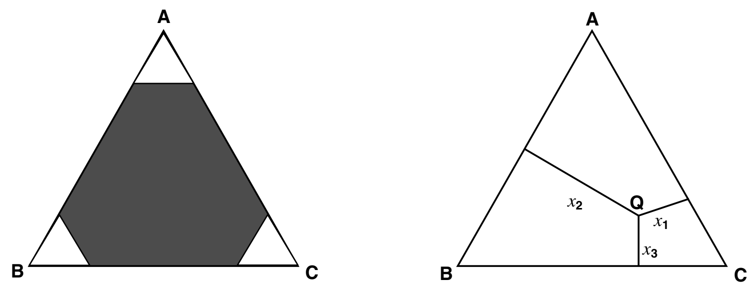
$$Hq = \{x \in E/0 < x_i < a,\ i = 1, 2, 3\}$$

describes a hexagonal region (Figure 1) if $a < p$ and an equilateral triangular region (Figure 2) if $k = q = 1$.

Sen [**5**] has also expressed two-dimensional Laplace operators in trilinear coordinates as

$$\begin{aligned} \nabla_1^2 &\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ &\equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_2 \partial x_3} - \frac{\partial^2}{\partial x_1 \partial x_3} \\ &\equiv -\frac{\partial}{\partial x_1}\frac{\partial}{\partial \eta_1} - \frac{\partial}{\partial x_2}\frac{\partial}{\partial \eta_2} - \frac{\partial}{\partial x_3}\frac{\partial}{\partial \eta_3} \\ &\equiv L \text{ (say)} \end{aligned} \tag{1.1}$$

2010 AMS *Mathematics subject classification.* Primary 46F12, 46F05.
Keywords and phrases. Symmetric trilinear integral transform, distribution.
The author is thankful to UGC, Western Regional office, Pune (India) for the award of minor research project UGC F.No. 47-210/05 dated 23 February, 2006.
Received by the editors on December 8, 2006, and in revised form on July 20, 2007.
DOI:10.1216/RMJ-2010-40-1-339



FIGURES 1 AND 2.

where

$$\begin{aligned}\frac{\partial}{\partial \eta_1} &= -\frac{\partial}{\partial x_1} + \frac{1}{2}\frac{\partial}{\partial x_2} + \frac{1}{2}\frac{\partial}{\partial x_3}\\ \frac{\partial}{\partial \eta_2} &= -\frac{\partial}{\partial x_2} + \frac{1}{2}\frac{\partial}{\partial x_3} + \frac{1}{2}\frac{\partial}{\partial x_1}\\ \frac{\partial}{\partial \eta_3} &= -\frac{\partial}{\partial x_3} + \frac{1}{2}\frac{\partial}{\partial x_1} + \frac{1}{2}\frac{\partial}{\partial x_2}\end{aligned} \tag{1.2}$$

are the derivatives along the outward normal at $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, respectively.

Later Patil [**3**] developed the symmetric integral transform of function of trilinear coordinates, which is defined on Hq as

(1.3)

$$\aleph(f)(n) = F(n) = \int_0^a \int_0^a \int_0^a f(x_1, x_2, x_3)\varphi_{n,q}(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3,$$

where $\varphi_{n,q}(x_1, x_2, x_3) = \cos\lambda_{n,q}x_1 + \cos\lambda_{n,q}x_2 + \cos\lambda_{n,q}x_3$, are eigenfunctions corresponding to the eigenvalues $\lambda_{n,q} = (q2n\pi)/p$, $n = 1, 2, 3, \ldots$, in an eigenvalue problem

$$L\varphi + \lambda^2\varphi = 0 \tag{1.4}$$

subjected to the Neumann type of boundary conditions

$$\frac{\partial\varphi}{\partial\eta} = 0$$

at

$$x_1 = 0, \quad x_1 = a$$
$$x_2 = 0, \quad x_2 = a$$
$$x_3 = 0, \quad x_3 = a.$$

If $f(x_1, x_2, x_3)$ is continuous, has piecewise continuous first and second order partial derivatives on Hq and satisfies the above Neumann type of boundary conditions, then the inverse transform of (1.3) is given by

$$f(x_1, x_2, x_3) = \sum_{n=1}^{\infty} F(n) c_{n,q} \varphi_{n,q}(x_1, x_2, x_3) \tag{1.5}$$

where

$$\frac{1}{c_{n,q}} = \int_0^a \int_0^a \int_0^a \varphi_{n,q}^2 \, dx_1 \, dx_2 \, dx_3 = \frac{3a^3}{2} = \frac{3}{2}\left(\frac{kp}{q}\right)^3,$$

[**3**, page 129].

In [**6, 7**], we consider the testing functions as smooth complex-valued functions ϕ defined on T such that

$$\sup_{x \in T} |L^m \phi(x)| < \infty \text{for each } m = 0, 1, 2, \dots ,$$

where T is an equilateral triangular region described by the set

$$\{x = (x_1, x_2, x_3)/x_1 + x_2 + x_3 = p,\ 0 < x_i < p,\ x_i \in R,\ i = 1, 2, 3\}.$$

In [**6**] the finite symmetric trilinear integral transform is extended to distributions by using L_1 convergence. In [**7**] the testing functions (together with all their images under Laplace operator of any order) further satisfy Dirichlet-type boundary conditions, and the finite symmetric trilinear integral transform is extended to distributions analogous to the method employed in [**1**].

In this paper we consider the testing functions as smooth complex-valued functions ϕ defined on Hq which together with all their images under Laplace operator of any order satisfy von Neumann-type boundary conditions. We extend the finite symmetric trilinear integral transform to distributions analogous to the method employed in [**7**]. At the end we find the temperature inside a hexagonal prism of semi-infinite length.

2. The testing function space Eq. Let Eq denote the set of all infinitely differentiable complex-valued functions ϕ defined on Hq which satisfy the following two conditions:

(i) $L^m\phi$ satisfy von Neumann type boundary conditions on Hq for each $m = 0, 1, 2, \dots$.

(2.1) (ii) $\alpha_{m,q}(\phi) = \sup_{x\in H_q} |L^m\phi(x)| < \infty$ for each $m = 0, 1, 2, \dots$.

We note that Eq is nonempty and for each $n \in N$, eigenfunction $\varphi_{n,q}(x)$ is in Eq. Eq is a linear space. The topology of Eq is that generated by the countable multi-norm $\{\alpha_{m,q}\}_{m=0}^{\infty}$.

Theorem 2.1. *Eq is complete and therefore a Fréchet space.*

The proof of this theorem is similar to the proof of Theorem 3.1 in [**6**].

For every $\phi \in Eq$, the finite symmetric trilinear integral transform

$$\aleph(\phi)(n) = \int_0^a \int_0^a \int_0^a \phi(x)\varphi_{n,q}(x)\,dx_1\,dx_2\,dx_3 \tag{2.2}$$

exists and by (1.5), one has

$$\phi(x) = \sum_{n=1}^{\infty} c_{n,q}\aleph(\phi)(n)\varphi_{n,q}(x). \tag{2.3}$$

We call the sequence $(\aleph(\phi)(n))_{n\in N}$ the finite symmetric trilinear integral transform $\aleph(\phi)$ of ϕ. Therefore,

$$\aleph(\phi) = (\aleph(\phi)(n))_{n\in N}. \tag{2.4}$$

The expression (2.3) can be seen as an inversion formula for the said transform. The map $\phi \to \aleph(\phi)$ is a continuous linear transformation from Eq into l^∞.

Let $L_2(H_q)$ denote the set of complex-valued functions ϕ defined on Hq such that

$$\|\phi\| = \left[\int_0^a \int_0^a \int_0^a |\phi(x)|^2\,dx_1\,dx_2\,dx_3\right]^{1/2} < \infty. \tag{2.5}$$

An inner product in $L_2(H_q)$ is defined by

$$(2.6)\qquad (\phi,\psi)=\int_0^a\int_0^a\int_0^a \phi(x)\overline{\psi(x)}\,dx_1\,dx_2\,dx_3,\quad \phi,\psi\in L_2(H_q)$$

where $\overline{\psi(x)}$ denotes the complex conjugate of $\psi(x)$.

Proposition 2.2. *If $\phi \in Eq$, then*

$$(2.7)\qquad \begin{gathered}\aleph(L^m\phi)(n)=(-1)^m\lambda_{n,q}^{2m}\aleph(\phi)(n),\\ \textit{for every } n\in N,\quad m=0,1,2,\dots .\end{gathered}$$

Proof. From (2.2), we have

$$\begin{aligned}\aleph(L\phi)(n)&=\int_0^a\int_0^a\int_0^a (L\phi)(x)\varphi_{n,q}(x)\,dx_1\,dx_2\,dx_3\\
&=-\int_0^a\int_0^a\int_0^a\left(\frac{\partial}{\partial x_1}\frac{\partial\phi(x)}{\partial\eta_1}+\frac{\partial}{\partial x_2}\frac{\partial\phi(x)}{\partial\eta_2}+\frac{\partial}{\partial x_3}\frac{\partial\phi(x)}{\partial\eta_3}\right)\\
&\qquad\qquad\times\phi_{n,q}(x)\,dx_1\,dx_2\,dx_3,\quad \text{by (1.1)},\\
&=-\bigg\{\int_0^a\int_0^a\int_0^a\frac{\partial}{\partial x_1}\frac{\partial\phi(x)}{\partial\eta_1}\varphi_{n,q}(x)\,dx_1\,dx_2\,dx_3\\
&\quad+\int_0^a\int_0^a\int_0^a\frac{\partial}{\partial x_2}\frac{\partial\phi(x)}{\partial\eta_2}\varphi_{n,q}(x)\,dx_1\,dx_2\,dx_3\\
&\quad+\int_0^a\int_0^a\int_0^a\frac{\partial}{\partial x_3}\frac{\partial\phi(x)}{\partial\eta_3}\varphi_{n,q}(x)\,dx_1\,dx_2\,dx_3\bigg\}.\end{aligned}$$

Integrating the first integral by parts, we have

$$(2.8)\qquad \begin{aligned}&\int_0^a\int_0^a\int_0^a\frac{\partial}{\partial x_1}\frac{\partial\phi(x)}{\partial\eta_1}\varphi_{n,q}(x)\,dx_1\,dx_2\,dx_3\\
&=\int_0^a\int_0^a\left(\frac{\partial\phi(x)}{\partial\eta_1}\varphi_{n,q}(x)\right]_0^a dx_2\,dx_3\\
&\quad-\int_0^a\int_0^a\int_0^a\frac{\partial\phi(x)}{\partial\eta_1}\frac{\partial\varphi_{n,q}(x)}{\partial x_1}\,dx_1\,dx_2\,dx_3\end{aligned}$$

$$= -\int_0^a \int_0^a \int_0^a \frac{\partial \phi(x)}{\partial \eta_1} \frac{\partial \varphi_{n,q}(x)}{\partial x_1}\, dx_1\, dx_2\, dx_3,$$

(by using boundary conditions)

$$= -\int_0^a \int_0^a \int_0^a \left(-\frac{\partial}{\partial x_1} + \frac{1}{2}\frac{\partial}{\partial x_2} + \frac{1}{2}\frac{\partial}{\partial x_3} \right) \phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial x_1}\, dx_1\, dx_2\, dx_3,$$

(by using (1.2)),

$$\begin{aligned}
&= \int_0^a \int_0^a \int_0^a \frac{\partial \phi(x)}{\partial x_1} \frac{\partial \varphi_{n,q}(x)}{\partial x_1}\, dx_1\, dx_2\, dx_3 \\
&\quad - \frac{1}{2}\int_0^a \int_0^a \int_0^a \frac{\partial \phi(x)}{\partial x_2} \frac{\partial \varphi_{n,q}(x)}{\partial x_1}\, dx_1\, dx_2\, dx_3 \\
&\quad - \frac{1}{2}\int_0^a \int_0^a \int_0^a \frac{\partial \phi(x)}{\partial x_3} \frac{\partial \varphi_{n,q}(x)}{\partial x_1}\, dx_1\, dx_2\, dx_3 \\
&= \int_0^a \int_0^a \left(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial x_1} \right]_0^a dx_2\, dx_3 \\
&\quad - \int_0^a \int_0^a \int_0^a \phi(x) \frac{\partial^2 \varphi_{n,q}(x)}{\partial x_1^2}\, dx_1\, dx_2\, dx_3 \\
&\quad - \frac{1}{2}\int_0^a \int_0^a \left(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial x_1} \right]_0^a dx_1\, dx_3 \\
&\quad + \frac{1}{2}\int_0^a \int_0^a \int_0^a \phi(x) \frac{\partial^2 \varphi_{n,q}(x)}{\partial x_2 \partial x_1}\, dx_1\, dx_2\, dx_3 \\
&\quad - \frac{1}{2}\int_0^a \int_0^a \left(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial x_1} \right]_0^a dx_1\, dx_2 \\
&\quad + \frac{1}{2}\int_0^a \int_0^a \int_0^a \phi(x) \frac{\partial^2 \varphi_{n,q}(x)}{\partial x_3 \partial x_1}\, dx_1\, dx_2\, dx_3.
\end{aligned}$$

Similarly, we have

(2.9)
$$\begin{aligned}
&\int_0^a \int_0^a \int_0^a \frac{\partial}{\partial x_2} \frac{\partial \phi(x)}{\partial \eta_2} \varphi_{n,q}(x)\, dx_1\, dx_2\, dx_3 \\
&\qquad = \int_0^a \int_0^a \left(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial x_2} \right]_0^a dx_1\, dx_3
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^a \int_0^a \int_0^a \phi(x) \frac{\partial^2 \varphi_{n,q}(x)}{\partial x_2^2}\, dx_1\, dx_2\, dx_3 \\
&- \frac{1}{2} \int_0^a \int_0^a \left(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial x_2} \right]_0^a dx_1\, dx_2 \\
&+ \frac{1}{2} \int_0^a \int_0^a \int_0^a \phi(x) \frac{\partial^2 \varphi_{n,q}(x)}{\partial x_3 \partial x_2}\, dx_1\, dx_2\, dx_3 \\
&- \frac{1}{2} \int_0^a \int_0^a \left(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial x_2} \right]_0^a dx_2\, dx_3 \\
&+ \frac{1}{2} \int_0^a \int_0^a \int_0^a \phi(x) \frac{\partial^2 \varphi_{n,q}(x)}{\partial x_1 \partial x_2}\, dx_1\, dx_2\, dx_3
\end{aligned}
$$

(2.10)

$$
\begin{aligned}
\int_0^a \int_0^a \int_0^a & \frac{\partial}{\partial x_3} \frac{\partial \phi(x)}{\partial \eta_3} \varphi_{n,q}(x)\, dx_1\, dx_2\, dx_3 \\
&= \int_0^a \int_0^a \left(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial x_3} \right]_0^a dx_1\, dx_2 \\
&\quad - \int_0^a \int_0^a \int_0^a \phi(x) \frac{\partial^2 \varphi_{n,q}(x)}{\partial x_3^2}\, dx_1\, dx_2\, dx_3 \\
&\quad - \frac{1}{2} \int_0^a \int_0^a \left(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial x_3} \right]_0^a dx_2\, dx_3 \\
&\quad + \frac{1}{2} \int_0^a \int_0^a \int_0^a \phi(x) \frac{\partial^2 \varphi_{n,q}(x)}{\partial x_1 \partial x_3}\, dx_1\, dx_2\, dx_3 \\
&\quad - \frac{1}{2} \int_0^a \int_0^a \left(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial x_3} \right]_0^a dx_1\, dx_3 \\
&\quad + \frac{1}{2} \int_0^a \int_0^a \int_0^a \phi(x) \frac{\partial^2 \varphi_{n,q}(x)}{\partial x_2 \partial x_3}\, dx_1\, dx_2\, dx_3.
\end{aligned}
$$

Using (2.8), (2.9) and (2.10) we have

$$
\begin{aligned}
\aleph(L\phi)(n) = -\Bigg\{ & \int_0^a \int_0^a \left(\phi(x) \left(\frac{\partial}{\partial x_1} - \frac{1}{2} \frac{\partial}{\partial x_2} - \frac{1}{2} \frac{\partial}{\partial x_3} \right) \varphi_{n,q}(x) \right]_0^a dx_2\, dx_3 \\
&+ \int_0^a \int_0^a \left(\phi(x) \left(\frac{\partial}{\partial x_2} - \frac{1}{2} \frac{\partial}{\partial x_1} - \frac{1}{2} \frac{\partial}{\partial x_3} \right) \varphi_{n,q}(x) \right]_0^a dx_1\, dx_3 \\
&+ \int_0^a \int_0^a \left(\phi(x) \left(\frac{\partial}{\partial x_3} - \frac{1}{2} \frac{\partial}{\partial x_1} - \frac{1}{2} \frac{\partial}{\partial x_2} \right) \varphi_{n,q}(x) \right]_0^a dx_1\, dx_2
\end{aligned}
$$

$$- \int_0^a \int_0^a \int_0^a \phi(x) \bigg(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_1 \partial x_3} - \frac{\partial^2}{\partial x_2 \partial x_3} \bigg) \varphi_{n,q}(x)\, dx_1\, dx_2\, dx_3 \bigg\}$$

$$= \int_0^a \int_0^a \bigg(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial \eta_1} \bigg]_0^a dx_2\, dx_3 + \int_0^a \int_0^a \bigg(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial \eta_2} \bigg]_0^a dx_1\, dx_3 + \int_0^a \int_0^a \bigg(\phi(x) \frac{\partial \varphi_{n,q}(x)}{\partial \eta_3} \bigg]_0^a dx_1\, dx_2 + \int_0^a \int_0^a \int_0^a \phi(x) L\varphi_{n,q}(x)\, dx_1\, dx_2\, dx_3,$$

(using (1.1) and (1.2)).

Using boundary conditions and (1.4), we get

(2.11) $$\aleph(L\phi)(n) = (-1)\lambda_{n,q}^2 \aleph(\phi)(n), \text{ for every } n \in N.$$

Using (2.11), it is quite simple to obtain (2.7). □

Proposition 2.3. *Let $\phi \in Eq$. Then the series*

(2.12) $$\sum_{n=1}^{\infty} c_{n,q} |\aleph(L^m\phi)(n)|^2$$

converges and Bessel's inequality

(2.13) $$\sum_{n=1}^{\infty} c_{n,q} |\aleph(L^m\phi)(n)|^2 \le \|L^m\phi\|^2 < \infty$$

holds for each $m = 0, 1, 2, 3, \dots$.

The proof is similar to that of [**7**, Proposition 2.4].

Proposition 2.4. *If $\phi \in Eq$, then the series*

(2.14) $$\sum_{n=1}^{\infty} (-1)^m c_{n,q} \lambda_{n,q}^{2m} \aleph(\phi)(n) \varphi_{n,q}(x), \quad m = 0, 1, 2, \dots$$

converges absolutely and uniformly over Hq.

Proof. The proof can be given by using Proposition 2.2 and following the pattern of proof of [**7**, Proposition 2.5].

Proposition 2.5. *If $\phi \in Eq$, then*

$$L^m\phi(x) = \sum_{n=1}^{\infty}(-1)^m c_{n,q}\lambda_{n,q}^{2m}\aleph(\phi)(n)\varphi_{n,q}(x), \quad m = 0, 1, 2, \dots , \tag{2.15}$$

and the series converges uniformly over Hq.

Proof. Employing (1.5), we arrive at

$$L^m\phi(x) = \sum_{n=1}^{\infty} c_{n,q}\aleph(L^m\phi)(n)\varphi_{n,q}(x).$$

Upon using Proposition 2.2, we obtain (2.15).

From (2.15) and Proposition 2.4,

$$\sum_{r=1}^{n}(-1)^m c_{r,q}\lambda_{r,q}^{2m}\aleph(\phi)(r)\varphi_{r,q} \longrightarrow L^m\phi$$

uniformly on Hq can be easily proved. □

The following is an immediate consequence of Proposition 2.5.

Corollary 2.6. *For all $\phi \in Eq$, $\phi_n \to \phi$ in Eq, where $\phi_n(x) = \sum_{k=1}^{n} c_{k,q}\aleph(\phi)(k)\varphi_{k,q}(x)$.*

Theorem 2.7. *For every $\phi \in Eq$, Parseval's identity holds, that is,*

$$\sum_{n=1}^{\infty} c_{n,q}|\aleph(\phi)(n)|^2 = \|\phi\|^2. \tag{2.16}$$

Equivalently,

$$(\aleph(\phi_1), \aleph(\phi_2)) = \sum_{n=1}^{\infty} c_{n,q}\aleph(\phi_1)(n)\overline{\aleph(\phi_2)(n)} = (\phi_1, \phi_2). \tag{2.17}$$

Proof. Expanding the inner product and using the fact that $\{\varphi_{n,q}\}$ is an orthogonal set, one has

$$\|\phi - \phi_n\|^2 = \|\phi\|^2 - \sum_{k=1}^{n} c_{k,q} |\aleph(\phi)(k)|^2,$$

where

$$\phi_n(x) = \sum_{k=1}^{n} c_{k,q} \aleph(\phi)(k) \varphi_{k,q}(x).$$

But

$$\|\phi - \phi_n\|^2 = \int_0^a \int_0^a \int_0^a |\phi(x) - \phi_n(x)|^2 dx_1\, dx_2\, dx_3 \leq [\alpha_{0,q}(\phi - \phi_n)]^2 a^3.$$

Hence,

$$0 \leq \|\phi\|^2 - \sum_{k=1}^{n} c_{k,q} |\aleph(\phi)(k)|^2 \leq [\alpha_{0,q}(\phi - \phi_n)]^2 a^3.$$

Taking the limit as $n \to \infty$ and using Corollary 2.6, we get (2.16).

By using polarization identity we get (2.17). □

3. The space of rapidly decreasing sequences. Let Bq be the set of all complex sequences $(a_n)_{n \in N}$ satisfying

$$\sum_{n=1}^{\infty} c_{n,q} \lambda_{n,q}^{2m} |a_n| < \infty, \text{ for all } m = 0, 1, 2, \dots . \tag{3.1}$$

Bq is a linear space and

$$\beta_{m,q}((a_n)_{n \in N}) = \sum_{n=1}^{\infty} c_{n,q} \lambda_{n,q}^{2m} |a_n|, \quad m = 0, 1, 2, \dots , \tag{3.2}$$

defines a countable multi-norm on Bq. Bq is complete and therefore a Fréchet space.

Theorem 3.1. *For each continuous linear functional f defined on Bq, there exist a positive constant C and a nonnegative integer r such that for every $(a_n)_{n\in N} \in Bq$,*

$$|\langle f, (a_n)_{n\in N}\rangle| \leq C \max_{0\leq k\leq r} \beta_{k,q}((a_n)_{n\in N}). \tag{3.3}$$

Proof. The proof is similar to that of [**8**, Theorem 1.1.8]. Let $\rho_{\gamma,q} = \max.\{\beta_{0,q}, \beta_{1,q}, \dots , \beta_{\gamma,q}\}$. Then $\{\rho_{\gamma,q}\}_{\gamma=0}^{\infty}$ is a countable multinorm on B_q.

Assume that there are no values of C and r for which the inequality (3.3) holds for all $(a_n)_{n\in N} \in Bq$.This means that for each positive integer ν, there exists a sequence a^ν, where $a^\nu = (a_n^{(r)})_{n\in N} = (a_1^{(\nu)}, a_2^{(\nu)}, \dots)$, in Bq such that

$$|\langle f, a^\nu\rangle| > \gamma\rho_{\gamma,q}(a^\nu). \tag{3.4}$$

Since $\rho_{\gamma,q}$ is a norm, $\rho_{\gamma,q}(a^\nu) > 0$. a^ν cannot be the zero element in Bq (otherwise we would have equality in (3.4), since both sides there would be zero).

Set

$$d^\nu = \left(\frac{a^\nu}{\gamma\rho_{\gamma,q}(a^\nu)}\right) \in Bq.$$

With k being an arbitrary but fixed nonnegative integer, we have for $\gamma > k$,

$$\rho_{k,q}(d^\nu) \leq \rho_{\gamma,q}(d^\nu) = \frac{\rho_{\gamma,q}(a^\nu)}{\gamma\rho_{\gamma,q}(a^\nu)} = \frac{1}{\gamma} \longrightarrow 0 \text{ as } \gamma \to \infty.$$

Since the topology generated by $(\rho_{\gamma,q})$ is equal to that generated by $(\beta_{\gamma,q})$, it follows that $d^\nu \to 0$ in Bq. Consequently, $\langle f, d^\nu\rangle \to 0$ because f is a continuous linear functional on Bq. But from (3.4) we have

$$|\langle f, d^\nu\rangle| = \frac{|\langle f, a^\nu\rangle|}{\gamma\rho_{\gamma,q}(a^\nu)} > 1.$$

This contradiction proves the theorem. □

Theorem 3.2. *Let $f \in B_q'$. Then there exists a sequence $(\alpha_n)_{n\in N}$ of slow growth such that*

$$\langle f, (a_n)_{n\in N}\rangle = \sum_{n=1}^{\infty} c_{n,q}\alpha_n a_n, \text{ for every } (a_n)_{n\in N} \in Bq.$$

Proof. By Theorem 3.1, there exist a positive constant C and a nonnegative integer r such that, for every $(a_n)_{n\in N} \in Bq$,

$$|\langle f, (a_n)_{n\in N}\rangle| \leq C \max_{0\leq k\leq r} \beta_{k,q}((a_n)_{n\in N}) = C\sum_{n=1}^{\infty} c_{n,q}\lambda_{n,q}^{2r}|a_n| < \infty.$$

Take $(\alpha_n)_{n\in N}$ such that

$$|\alpha_n| \leq C\lambda_{n,q}^{2r} = C\left(\frac{q2\pi}{p}\right)^{2r} n^{2r}, \text{ for every } n \in N. \qquad \square$$

Theorem 3.3. *The finite symmetric trilinear integral transform $\aleph$ is a homeomorphism from Eq onto the space Bq.*

Proof. For any $\phi \in Eq$, by Proposition 2.4,

$$\sum_{n=1}^{\infty} c_{n,q}\lambda_{n,q}^{2m}|\aleph(\phi)(n)| < \infty \text{ for every } m = 0, 1, 2, \dots .$$

Define the mapping $\aleph : Eq \to Bq$ by $\aleph(\phi) = (\aleph(\phi)(n))_{n\in N}$. It follows from (2.3) that $\aleph$ is one-to-one.

Let $a = (a_n)_{n\in N}$ be an arbitrary member of Bq. Take

$$\phi(x) = \sum_{n=1}^{\infty} c_{n,q}a_n\varphi_{n,q}(x), \quad x \in H_q. \tag{3.5}$$

For each $m = 0, 1, 2, \dots,$

$$\sum_{n=1}^{\infty}(-1)^m\lambda_{n,q}^{2m}c_{n,q}a_n\varphi_{n,q}(x) \leq 3\sum_{n=1}^{\infty} c_{n,q}\lambda_{n,q}^{2m}|a_n| < \infty. \tag{3.6}$$

By the Weierstrass M-test, the series on the lefthand side of inequality (3.6) converges absolutely and uniformly over H_q.

ϕ (infinitely differentiable) together with all its images under L of any order satisfies the Neumann-type boundary conditions. Further,

$$\alpha_{m,q}(\phi) = \sup_{x \in H_q} |L^m \phi(x)| = \sup_{x \in H_q} \left| \sum_{n=1}^{\infty} (-1)^m \lambda_{n,q}^{2m} c_{n,q} a_n \varphi_{n,q}(x) \right| < \infty$$

for each $m = 0, 1, 2, \dots$.

This implies $\phi(x) \in Eq$. Multiplying (3.5) by $\varphi_{k,q}(x)$ and integrating, we have

$$\int_0^a \int_0^a \int_0^a \phi(x) \varphi_{k,q}(x) \, dx_1 \, dx_2 \, dx_3 = \sum_{n=1}^{\infty} c_{n,q} a_n \int_0^a \int_0^a \int_0^a \varphi_{n,q}(x) \varphi_{k,q}(x) \, dx_1 \, dx_2 \, dx_3 = a_k$$

$$\aleph(\phi)(k) = a_k \text{ for all } k \in N$$
$$\aleph(\phi) = (a_n)_{n \in N}.$$

Hence, $\aleph$ is onto. $\aleph^{-1} : B_q \to E_q$ exists and is given by

$$\aleph^{-1}(a)(x) = \aleph^{-1}((a_n)_{n \in N})(x) = \sum_{n=1}^{\infty} c_{n,q} a_n \varphi_{n,q}(x), \quad x \in H_q,$$

for each $a = (a_n)_{n \in N} \in Bq$.

Assume that $(\phi_k)_{k \in N} \to \phi$ in Eq as $k \to \infty$. By Proposition 2.2,

$$\begin{aligned} \lambda_{n,q}^{2m} |\aleph(\phi_k)(n) - \aleph(\phi)(n)| &= |\aleph(L^m(\phi_k - \phi))(n)| \\ &\leq \int_0^a \int_0^a \int_0^a |L^m(\phi_k - \phi)(x)| |\varphi_{n,q}(x)| \, dx_1 dx_2 dx_3 \\ &\leq 3a^3 \alpha_{m,q}(\phi_k - \phi) \longrightarrow 0 \text{ as } k \to \infty, \end{aligned}$$

for all $n \in N$ and $m = 0, 1, 2, \dots$.

$$\beta_{m,q}(\aleph(\phi_k)-\aleph(\phi)) = \sum_{n=1}^{\infty} c_{n,q}\lambda_{n,q}^{2m}|\aleph(\phi_k)(n)-\aleph(\phi)(n)| \longrightarrow 0 \text{ as } k\to\infty,$$

for each $m = 0, 1, 2, \dots$. This proves $\aleph$ is continuous.

Let $a^k \to a$ in Bq as $k \to \infty$, where $a^k = (a_n^{(k)})_{n\in N}$, $a = (a_n)_{n\in N}$.

$$\aleph^{-1}(a^k - a)(x) = \sum_{n=1}^{\infty} c_{n,q}(a_n^{(k)} - a_n)\varphi_{n,q}(x), \quad x \in H_q.$$

$$\begin{aligned}\alpha_{m,q}(\aleph^{-1}(a^k - a)) &= \sup_{x\in H_q}\left|\sum_{n=1}^{\infty}(-1)^m\lambda_{n,q}^{2m}c_{n,q}(a_n^{(k)} - a_n)\varphi_{n,q}(x)\right| \\ &\le 3\sum_{n=1}^{\infty} c_{n,q}\lambda_{n,q}^{2m}|a_n^{(k)} - a_n| \\ &= 3\beta_{m,q}((a_n^{(k)} - a)_{n\in N}) \longrightarrow 0\end{aligned}$$

as $k \to \infty$ for each $m = 0, 1, 2, \dots$. Thus, $\aleph^{-1}$ is continuous and the proof is complete. □

4. Distribution space. In this section we will introduce the space of distributions.

Definition 4.1. A linear functional U on a Fréchet space Eq, $U : Eq \to C$ is called a *distribution* if there exists a sequence $(\zeta_n)_{n\in N}$ in Eq such that

$$\langle U, \phi\rangle = \lim_{n\to\infty}\int_0^a\int_0^a\int_0^a \zeta_n(x)\phi(x)\,dx_1\,dx_2\,dx_3 \tag{4.1}$$

exists for each $\phi \in Eq$.

The set of all distributions is a complex linear space, and it will be denoted by E_q'. The map $\phi \to \xi_\phi(f) = |\langle f, \phi\rangle|$ is a semi-norm on E_q'. The family of semi-norms $\{\xi_\phi\}_{\phi\in E_q}$ is separating and generates a topology on E_q'. It is clear that Eq is a subspace of E_q' and the topology of Eq is stronger than that induced on it by E_q'.

Define

$$d(\phi, \psi) = \sum_{m=0}^{\infty}\frac{1}{2^m}\frac{\alpha_{m,q}(\phi - \psi)}{1 + \alpha_{m,q}(\phi - \psi)}.$$

Then d is a compatible translation invariant metric on Eq [**4**, page 27]. Furthermore, (Eq, d) is a complete metric space.

Theorem 4.2. *Every distribution is a continuous linear functional on Eq.*

The proof is similar to the proof of Theorem 3.143 [**2**].

Proposition 4.3. *E'_q is the dual of Eq, that is, E'_q is precisely the collection of all continuous linear functionals from Eq into C.*

Proof. Let $f : E_q \to C$ be a continuous linear functional. For each $\phi \in E_q$, by Corollary 2.6, $\phi_n \to \phi$ in E_q, where $\phi_n(x) = \sum_{k=1}^n c_{k,q} \aleph(\phi)(k)\varphi_{k,q}(x)$.

We have

$$\begin{aligned}
\langle f, \phi\rangle &= \lim_{n\to\infty} \langle f, \phi_n\rangle \\
&= \lim_{n\to\infty} \sum_{k=1}^n c_{k,q} \aleph(\phi)(k)\langle f, \varphi_{k,q}\rangle \\
&= \lim_{n\to\infty} \sum_{k=1}^n c_{k,q} \bigg\{ \int_0^a \int_0^a \int_0^a \phi(x)\varphi_{k,q}(x)\, dx_1\, dx_2\, dx_3 \bigg\} \langle f, \varphi_{k,q}\rangle \\
&= \lim_{n\to\infty} \int_0^a \int_0^a \int_0^a \phi(x) \bigg[\sum_{k=1}^n c_{k,q} \langle f, \varphi_{k,q}\rangle \varphi_{k,q}(x) \bigg] dx_1\, dx_2\, dx_3 \\
&= \lim_{n\to\infty} \int_0^a \int_0^a \int_0^a \phi(x)\chi_n(x)\, dx_1\, dx_2\, dx_3.
\end{aligned}$$

Thus, the condition in the definition of distribution is satisfied with

$$\chi_n(x) = \sum_{k=1}^n c_{k,q} \langle f, \varphi_{k,q}\rangle \varphi_{k,q}(x). \qquad \square$$

Let $\mathcal{D}(Hq)$ denote the space of all complex-valued smooth functions with compact support in Hq provided with the topology induced by the family of semi-norms

$$\gamma_k(\phi) = \sup_{x \in H_q} |D^k \phi(x)|, \quad \phi \in \mathcal{D}(Hq),$$

where

$$D^k \equiv \frac{\partial^{k_1+k_2+k_3}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}.$$

$\mathcal{D}(Hq)$ is a subspace of Eq [**6**, page 146]. The topology of the space $\mathcal{D}(Hq)$ is stronger than the induced topology on it by Eq and the restriction of any $f \in E_q'$ to $\mathcal{D}(Hq)$ is in $\mathcal{D}'(Hq)$, the space of Schwartz distributions.

Let A_0 denote the set of all functions $f(x)$ which are continuous, has piecewise continuous first and second order partial derivatives on Hq and satisfies the Neumann type of boundary conditions on Hq.

Proposition 4.4. *Let $f \in A_0$. Then the formula*

$$\langle U_f, \phi\rangle = \int_0^a \int_0^a \int_0^a f(x)\phi(x)\, dx_1\, dx_2\, dx_3, \quad \phi \in Eq \tag{4.2}$$

defines a distribution U_f on Eq. A_0 can be embedded in E_q'.

Proof. U_f is clearly linear.

Define

$$\chi_n(x) = \sum_{k=1}^{n} c_{k,q} \aleph(f)(k) \varphi_{k,q}(x).$$

Then $\chi_n \in E_q$ for all values of n. Moreover, $\chi_n \to f$ uniformly on Hq.

$$\begin{aligned}\langle U_f, \phi\rangle &= \int_0^a \int_0^a \int_0^a \Big(\lim_{n\to\infty} \chi_n(x)\Big)\, \phi(x)\, dx_1\, dx_2\, dx_3 \\ &= \lim_{n\to\infty} \int_0^a \int_0^a \int_0^a \chi_n(x)\phi(x)\, dx_1\, dx_2\, dx_3.\end{aligned}$$

The map $f \to U_f$ is linear.

Let, if possible, $U_f = U_g$ for some $f, g \in A_0$; then

$$\langle U_f - U_g, \phi\rangle = \int_0^a \int_0^a \int_0^a (f-g)(x)\phi(x)\, dx_1\, dx_2\, dx_3 = 0,$$

for every $\phi \in E_q$. This implies $f = g$ on E_q.

If $f_n \to f$ uniformly on Eq, then

$$\begin{aligned}\langle U_{f_n - f}, \phi\rangle &= \int_0^a \int_0^a \int_0^a (f_n - f)(x)\phi(x)\, dx_1\, dx_2\, dx_3 \\ &\le \alpha_{0,q}(\phi) \int_0^a \int_0^a \int_0^a |(f_n - f)(x)|\, dx_1\, dx_2\, dx_3 \longrightarrow 0\end{aligned}$$

as $n \to \infty$, for all $\phi \in Eq$. This proves the map $f \to U_f$ is continuous. Finally, if $U_{f_n} \to U_f$ in the image of A_0, then

$$\int_0^a \int_0^a \int_0^a (f_n - f)(x)\phi(x)\, dx_1\, dx_2\, dx_3 = \langle U_{f_n} - U_f, \phi\rangle \longrightarrow 0$$

as $n \to \infty$ for all $\phi \in Eq$.

This implies $f_n \to f$ as $n \to \infty$. Thus, A_0 can be embedded in E_q'. □

There are distributions that do not have the form (4.2) with $f \in A_0$.

Example 4.5. Dirac function δ_x centered at $x \in Hq$

$$\langle \delta_x, \phi\rangle = \phi(x), \quad \phi \in Eq. \tag{4.3}$$

It is easy to prove δ_x is linear. Take

$$\chi_n(y) = \sum_{k=1}^{n} c_{k,q}\varphi_{k,q}(x)\varphi_{k,q}(y),$$

$x, y \in Hq$ and x is fixed. Then χ_n is in Eq for each $n \in N$.

$$\begin{aligned}\lim_{n\to\infty} \int_0^a\int_0^a\int_0^a \chi_n(y)\phi(y)\, dy_1\, dy_2\, dy_3 &= \lim_{n\to\infty} \sum_{k=1}^{n} c_{k,q} \aleph(\phi)(k)\varphi_{k,q}(x) \\ &= \phi(x) = \langle \delta_x, \phi\rangle. \qquad \square\end{aligned}$$

Proposition 4.6. *For each $f \in E'_q$, there exist a nonnegative integer r and a positive constant C_0 such that*

$$|\langle f, \phi\rangle| \le C_0 \max_{0\le m\le r} \alpha_{m,q}(\phi). \tag{4.4}$$

Here C_0 and r depend on f but not on ϕ.

The proof is similar to the proof of Theorem 3.1.

5. Generalized finite symmetric trilinear integral transform. The generalized finite symmetric trilinear integral transform $\aleph'$ of $f \in E'_q$ is defined by

(5.1)

$$\langle \aleph'(f), (a_n)_{n\in N}\rangle = \left\langle f(x), \sum_{n=1}^{\infty} c_{n,q} a_n \varphi_{n,q}(x)\right\rangle, \quad (a_n)_{n\in N} \in Bq.$$

If $\phi(x) = \sum_{n=1}^{\infty} c_{n,q} a_n \varphi_{n,q}(x)$, then by Theorem 3.3, $(a_n)_{n\in N} = (\aleph(\phi)(n))_{n\in N} = \aleph(\phi)$ and (5.1) can be written as

$$\langle \aleph'(f), \aleph(\phi)\rangle = \langle f, \phi\rangle. \tag{5.2}$$

Theorem 5.1. *$\aleph'$ is a homeomorphism from E'_q onto the space B'_q.*

Proof. $\aleph'$ is a mapping from E'_q into the space B'_q. Indeed, for any $(a_n)_{n\in N}, (b_n)_{n\in N} \in B_q$ and $\alpha, \beta \in C$,

$$\begin{aligned}\langle \aleph' f, \alpha(a_n)_{n\in N} + \beta(b_n)_{n\in N}\rangle &= \left\langle f, \sum_{n=1}^{\infty} c_{n,q}(\alpha a_n + \beta b_n)\varphi_{n,q}\right\rangle \\ &= \alpha\langle \aleph' f, (a_n)_{n\in N}\rangle + \beta\langle \aleph' f, (b_n)_{n\in N}\rangle,\end{aligned}$$

which shows $\aleph' f$ is a linear functional on B_q. Furthermore, let $(a_n^{(v)})_{n\in N}$ converge in B_q to zero. Then, as $v \to \infty$,

$$\aleph^{-1}((a_n^{(v)})_{n\in n}) = \sum_{n=1}^{\infty} c_{n,q} a_n^{(v)} \varphi_{n,q} \longrightarrow 0 \text{ in } E_q$$

and

$$\langle \aleph' f, (a_n^{(v)})_{n\in N}\rangle = \left\langle f, \sum_{n=1}^{\infty} c_{n,q} a_n^{(v)} \varphi_{n,q} \right\rangle \longrightarrow 0.$$

Thus, $\aleph' f$ is a continuous linear functional on B_q.

We now prove $\aleph'$ is linear, one-to-one and onto.

Let $(a_n)_{n\in N} \in B_q$; $f, g \in E_q'$ and $\alpha, \beta \in C$. Then,

$$\begin{aligned}\langle \aleph'(\alpha f + \beta g), (a_n)_{n\in N}\rangle &= \left\langle \alpha f + \beta g, \sum_{n=1}^{\infty} c_{n,q} a_n \varphi_{n,q} \right\rangle \\ &= \langle \alpha\aleph' f + \beta\aleph' g, (a_n)_{n\in N}\rangle,\end{aligned}$$

which shows $\aleph'$ is linear.

If $\langle \aleph' f, (a_n)_{n\in N}\rangle = \langle \aleph' g, (a_n)_{n\in N}\rangle$, for every $(a_n)_{n\in N} \in Bq$, then

$$\left\langle f, \sum_{n=1}^{\infty} c_{n,q} a_n \varphi_{n,q} \right\rangle = \left\langle g, \sum_{n=1}^{\infty} c_{n,q} a_n \varphi_{n,q} \right\rangle.$$

Since $\aleph$ is a homeomorphism, this is equivalent to

$$\langle f, \phi\rangle = \langle f, \phi\rangle, \text{ for every } \phi \in E_q,$$

which proves $\aleph'$ is one-to-one.

Let $\hbar$ be an arbitrary member of B_q'. By Theorem 3.2, there exists a sequence $(\alpha_n)_{n\in N}$ of slow growth such that

$$\langle \hbar, (a_n)_{n\in N}\rangle = \sum_{n=1}^{\infty} c_{n,q} \alpha_n a_n, \text{ for every } (a_n)_{n\in N} \in Bq.$$

Define $f : E_q \to C$ by the formula

$$\langle f, \phi\rangle = \sum_{n=1}^{\infty} c_{n,q} \alpha_n \aleph(\phi)(n).$$

$$\langle f, \phi\rangle = \lim_{n\to\infty} \sum_{k=1}^{n} c_{k,q} \alpha_k \aleph(\phi)(k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} c_{k,q}\alpha_k \int_0^a \int_0^a \int_0^a \phi(x)\varphi_{k,q}(x)\,dx_1\,dx_2\,dx_3$$
$$= \lim_{n\to\infty} \int_0^a \int_0^a \int_0^a \phi(x)\Big(\sum_{k=1}^{n} c_{k,q}\alpha_k\varphi_{k,q}(x)\Big)\,dx_1\,dx_2\,dx_3$$
$$= \lim_{n\to\infty} \int_0^a \int_0^a \int_0^a \phi(x)\psi_n(x)\,dx_1\,dx_2\,dx_3,$$

where $\psi_n(x) = \sum_{k=1}^{n} c_{k,q}\alpha_k\varphi_{k,q}(x)$.

This implies $f \in E'_q$. Further,

$$\begin{aligned}\langle \aleph' f, (a_n)_{n\in N}\rangle &= \Big\langle f, \sum_{n=1}^{\infty} c_{n,q}a_n\varphi_{n,q}(x)\Big\rangle\\ &= \sum_{n=1}^{\infty} c_{n,q}\alpha_n \aleph\Big(\sum_{n=1}^{\infty} c_{n,q}a_n\varphi_{n,q}(x)\Big)(n)\\ &= \sum_{n=1}^{\infty} c_{n,q}\alpha_n \int_0^a \int_0^a \int_0^a \Big(\sum_{n=1}^{\infty} c_{n,q}a_n\varphi_{n,q}(x)\Big)\\ &\quad\times \varphi_{n,q}(x)\,dx_1\,dx_2\,dx_3\\ &= \sum_{n=1}^{\infty} c_{n,q}\alpha_n a_n \text{ (using orthogonality relations)}\\ &= \langle \hbar, (a_n)_{n\in N}\rangle,\end{aligned}$$

which proves $\aleph'$ is onto. Hence, $(\aleph')^{-1}$ exists.

We now prove $\aleph'$ and $(\aleph')^{-1}$ are continuous. Let $f_v \to 0$ in E'_q. Then for every $(a_n)_{n\in N} \in Bq$,

$$\langle \aleph' f_v, (a_n)_{n\in N}\rangle = \Big\langle f_v, \sum_{n=1}^{\infty} c_{n,q}a_n\varphi_{n,q}\Big\rangle \longrightarrow 0.$$

That is, $\aleph' f_v \to 0$ in B'_q. Consequently, $\aleph'$ is continuous.

Let $g_v \to 0$ in B'_q. Then, for every $\phi \in Eq$,

$$\langle (\aleph')^{-1}g_v, \phi\rangle = \Big\langle (\aleph')^{-1}g_v, \sum_{n=1}^{\infty} c_{n,q}\aleph(\phi)(n)\varphi_{n,q}\Big\rangle,$$

(by Corollary 2.6)

$$= \langle g_v, (\aleph(\phi)(n))_{n\in N}\rangle \longrightarrow 0.$$

This implies $(\aleph')^{-1}$ is continuous.

Proposition 5.2. *The finite symmetric trilinear integral transform $\aleph$ is a special case of the generalized transform $\aleph'$. That is, $\aleph' f = \aleph f$ for every $f \in Eq$.*

Proof. Since Eq is a subspace of E'_q, for every $f \in Eq$ we have

$$\begin{aligned}\langle \aleph' f, (a_n)_{n\in N}\rangle &= \left\langle f, \sum_{n=1}^{\infty} c_{n,q} a_n \varphi_{n,q}\right\rangle \\ &= \sum_{n=1}^{\infty} c_{n,q}\aleph(f)(n) a_n \\ &= \langle (\aleph(f)(n))_{n\in N}, (a_n)_{n\in N}\rangle,\end{aligned}$$

$(a_n)_{n\in N} \in Bq$. This implies $\aleph'(f) = (\aleph(f)(n)_{n\in N}$ in the sense of equality in B'_q. □

Motivated by the above result, we define generalized integral transform $\aleph' f$ of $f \in E'_q$ as

$$\aleph'(f) = (\langle f(x), \varphi_{n,q}(x)\rangle)_{n\in N},$$

and we set

$$\aleph'(f)(n) = \langle f(x), \varphi_{n,q}(x)\rangle, \quad \varphi_{n,q} \in Eq \text{ and } n \in N. \tag{5.3}$$

We now state and prove an inversion theorem for the elements of E'_q that can be seen as an inversion formula for the $\aleph'$-transformation.

Theorem 5.3. *Let $f \in E'_q$. Then*

$$f = \lim_{n\to\infty} \sum_{k=1}^{n} c_{k,q}\langle f, \varphi_{k,q}\rangle \varphi_{k,q} \tag{5.4}$$

where the limit is taken in the sense of E'_q.

Proof. Let

$$F_n(x) = \sum_{k=1}^{n} c_{k,q} \langle f, \varphi_{k,q} \rangle \varphi_{k,q}.$$

Since $F_n \in Eq$ for every n, by Proposition 4.4,

$$\langle F_n, \varphi_{m,q} \rangle = \begin{cases} \langle f, \varphi_{m,q} \rangle & \text{if } m \leq n \\ 0 & \text{if } m > n. \end{cases} \tag{5.5}$$

By using Theorem 2.7, Parseval's identity, we have

$$\begin{aligned} \langle F_n, \phi \rangle &= \sum_{k=1}^{\infty} c_{k,q} \langle F_n, \varphi_{k,q} \rangle \aleph(\phi)(k) \\ &= \sum_{k=1}^{n} c_{k,q} \langle f, \varphi_{k,q} \rangle \aleph(\phi)(k), \text{ (by (5.5))} \\ &= \langle f, G_n(\phi) \rangle \text{ for every } \phi \in Eq, \end{aligned}$$

where

$$G_n(\phi) = \sum_{k=1}^{n} c_{k,q} \aleph(\phi)(k) \varphi_{k,q}.$$

By Corollary 2.6, $G_n(\phi) \to \phi$ for all $\phi \in Eq$. Therefore,

$$\lim_{n\to\infty} \langle F_n, \phi \rangle = \lim_{n\to\infty} \langle f, G_n(\phi) \rangle = \langle f, \phi \rangle \text{ for all } \phi \in Eq. \quad \square$$

The following example illustrates the inversion theorem.

Example 5.4. The Dirac function δ_x centered at $x \in H_q$ is given by

$$\langle \delta_x, \phi \rangle = \phi(x), \quad \phi \in Eq.$$

The finite symmetric trilinear integral transform of δ_x is given as

$$\aleph'(\delta_x)(n) = \langle \delta_x(t), \varphi_{n,q}(t) \rangle = \varphi_{n,q}(x).$$

By virtue of Proposition 4.4, for all $\phi(t) \in Eq$,

$$\left\langle \sum_{m=0}^{N} c_{m,q}\varphi_{m,q}(x)\varphi_{m,q}(t), \phi(t) \right\rangle = \sum_{m=1}^{N} c_{m,q}\aleph(\phi)(m)\varphi_{m,q}(x) \longrightarrow \phi(x) \quad \text{as } N \to \infty.$$

But $\phi(x) = \langle \delta_x(t), \phi(t) \rangle$. Therefore,

$$\delta_x(t) = \lim_{N\to\infty} \sum_{m=1}^{N} c_{m,q}\langle \delta_x, \varphi_{m,q}\rangle \varphi_{m,q}(t). \qquad \square$$

6. Operational calculus. Integrating by parts and using boundary conditions, one can easily prove that if $f \in A_0$, then

$$\langle Lf, \phi \rangle = \langle f, L\phi \rangle \quad \text{for every } \phi \in Eq.$$

This allows us to define for any $f \in E'_q$

$$\langle Lf, \phi \rangle = \langle f, L\phi \rangle, \quad \phi \in Eq.$$

It is clear that $Lf \in E'_q$.

It can also be seen inductively that for any integer m

$$\langle L^m f, \phi \rangle = \langle f, L^m \phi \rangle \quad \text{for every } \phi \in Eq \text{ and } L^m f \in E'q.$$

Therefore,

$$\langle L^m f, \varphi_{n,q} \rangle = \langle f, L^m \varphi_{n,q} \rangle = (-1)^m \lambda_{n,q}^{2m} \langle f, \varphi_{n,q} \rangle.$$

That is,

$$\aleph'(L^m f)(n) = (-1)^m \lambda_{n,q}^{2m} \aleph'(f)(n), \tag{6.1}$$

which gives an operation transform formula.

Now consider the partial differential equation of the form

$$\Omega(L)f = g \tag{6.2}$$

where given g and unknown f are required to be in E'_q and Ω is a polynomial such that

$$\Omega(-\lambda^2_{n,q}) \neq 0, \quad n = 1, 2, 3, \dots .$$

By applying the operation transform formula (6.1) to equation (6.2), we obtain

$$\Omega(-\lambda^2_{n,q})F(n) = G(n), \text{ where } F(n) = (\aleph' f)(n) \text{ and } G(n) = (\aleph' g)(n)$$

$$F(n) = \frac{G(n)}{\Omega(-\lambda^2_{n,q})}.$$

By applying the inversion theorem 5.3, we get

(6.3)

$$f = \sum_{n=1}^{\infty} c_{n,q} \frac{G(n)}{\Omega(-\lambda^2_{n,q})} \varphi_{n,q}, \text{ where } \Omega(-\lambda^2_{n,q}) \neq 0 \text{ for } n = 1, 2, \dots .$$

7. Application. In this section we apply the present theory to find the temperature inside a hexagonal prism of semi-infinite length. The formulation of the problem is given below.

Find the conventional function $v(x, z)$ on the domain

$$D \equiv \{(x, z) = (x_1, x_2, x_3, z)/0 < x_i < a, \ i = 1, 2, 3, \\ x_1 + x_2 + x_3 = p, \ 0 < z < \infty\}$$

that satisfies the Laplace equation

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_3^2} - \frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 v}{\partial x_1 \partial x_3} - \frac{\partial^2 v}{\partial x_2 \partial x_3} + \frac{\partial^2 v}{\partial z^2} = 0 \tag{7.1}$$

in D and the following boundary conditions:

(i) As $z \to 0+$, $-K(\partial v(x, z)/\partial z) \to f(x) \in E'_q$ in the sense of convergence in E'_q, where K is a positive constant.

(ii) As $x_i \to 0+$, $x_i \to a-$, $i = 1, 2, 3$, $(\partial v(x,z)/\partial\eta)$ converges to zero uniformly on $Z \le z < \infty$ for each $Z > 0$.

(iii) As $z \to \infty$, $v(x,z)$ converges uniformly to zero on $0 < x_i < a$, $i = 1, 2, 3$.

Every section of the hexagonal prism by a plane perpendicular to the $z-$ axis is a hexagon with its centroid on the z-axis.

Set $V(n,z) = \aleph'(v(x,z)) = \langle v, \varphi_{n,q}\rangle$. By applying the finite symmetric trilinear integral transform $\aleph'$ to (7.1), we arrive at

$$-\lambda_{n,q}^2 V(n,z) + \frac{\partial^2}{\partial z^2} V(n,z) = 0,$$

whose general solution is

$$V(n,z) = A(n)e^{\lambda_{n,q}z} + B(n)e^{-\lambda_{n,q}z}, \tag{7.2}$$

where $A(n)$ and $B(n)$ do not depend on z.

In view of boundary condition (iii) it is reasonable to choose $A(n) = 0$ and $B(n) = (F(n)/K\lambda_{n,q})$ because of boundary condition (i). Therefore,

$$V(n,z) = \frac{F(n)}{K\lambda_{n,q}} e^{-\lambda_{n,q}z}. \tag{7.3}$$

Applying inversion theorem 5.3 to the above equation, we get

$$v(x,z) = \sum_{n=1}^{\infty} c_{n,q} \frac{F(n)}{K\lambda_{n,q}} e^{-\lambda_{n,q}z} \varphi_{n,q}(x). \tag{7.4}$$

We now verify that (7.4) is truly a solution of (7.1) that satisfies the given boundary conditions.

From Proposition 4.6, it is clear that

$$|F(n)| \le C\lambda_{n,q}^{2l},$$

where C is a positive constant and l is a positive integer. For $Z \le z < \infty$ where $Z > 0$, the nth term of the series (7.4) satisfies the condition

$$\left| c_{n,q} \frac{F(n)}{K\lambda_{n,q}} e^{-\lambda_{n,q}z} \varphi_{n,q}(x) \right| \le \frac{3c_{n,q}}{K} C\lambda_{n,q}^{2l-1} e^{-\lambda_{n,q}Z}.$$

Using

$$c_{n,q} = 2(q/kp)^3/3,$$
$$\lambda_{n,q} = (q2n\pi/p) \text{ and}$$
$$e^{-\lambda_{n,q}Z} < \frac{(2l+2)!}{\lambda_{n,q}^{2l+2} Z^{2l+2}},$$

we get

$$\left|\frac{c_{n,q}F(n)e^{-\lambda_{n,q}z}\varphi_{n,q}(x)}{K\lambda_{n,q}}\right| < C^* \frac{1}{n^3}, \text{ where } C^* = \frac{(2l+2)!2C}{(2\pi k)^3 Z^{2l+2} K}.$$

By the Weierstrass M-test, the series on the righthand side of (7.4) converges uniformly over D. The factor $e^{-\lambda_{n,q}z}$ ensures the uniform convergence of any series obtained by term-by-term differentiation of (7.4) with respect to x_i, $i = 1, 2, 3$ or z. We may apply the operator $L + (\partial^2/\partial z^2)$ under the summation sign in (7.4). Since $e^{-\lambda_{n,q}z}\varphi_{n,q}(x)$ satisfies the Laplace equation, so does v. Thus, the differential equation (7.1) is satisfied in the conventional sense.

To verify the boundary condition (i), we have

$$\text{(7.5)} \qquad \left[-K\frac{\partial v}{\partial z}\right] = \left[\sum_{n=1}^{\infty} -K\frac{\partial}{\partial z}\left(\frac{c_{n,q}F(n)e^{-\lambda_{n,q}z}\varphi_{n,q}(x)}{K\lambda_{n,q}}\right)\right].$$

Now, for any fixed $z > 0$, the series (7.5) defines a function in A_0, and by Proposition 4.2 we have for every $\phi \in Eq$,

(7.6)

$$\left\langle -K\frac{\partial v}{\partial z}, \phi\right\rangle = \int_0^a\int_0^a\int_0^a \left[\sum_{n=1}^{\infty} c_{n,q}F(n)e^{-\lambda_{n,q}z}\varphi_{n,q}(x)\right]\phi(x)\, dx_1dx_2dx_3$$
$$= \sum_{n=1}^{\infty} c_{n,q}F(n)e^{-\lambda_{n,q}z}\aleph(\phi)(n).$$

The series in (7.6) converges uniformly for all $z > 0$. By taking the limit as $z \to 0+$, one has

$$\lim_{z\to 0+}\left\langle -K\frac{\partial v}{\partial z}, \phi\right\rangle = \sum_{n=1}^{\infty} c_{n,q}F(n)\aleph(\phi)(n)$$
$$= \lim_{N\to\infty}\left\langle \sum_{n=1}^{N} c_{n,q}F(n)\varphi_{n,q}(x), \phi(x)\right\rangle$$
$$= \langle f, \phi\rangle \text{ by virtue of Theorem 5.3.}$$

Next we verify (ii). For any $z > 0$,

$$\frac{\partial v(x,z)}{\partial \eta} = \sum_{n=1}^{\infty} c_{n,q} \frac{F(n)}{K\lambda_{n,q}} e^{-\lambda_{n,q} z} \frac{\partial \varphi_{n,q}(x)}{\partial \eta}. \tag{7.7}$$

The series in (7.7) converges uniformly on H_q. So we may take the limit as $x_i \to 0$, $x_i \to a-$ under the summation sign and arrive at the conclusion.

Finally, we have

$$|v(x,z)| \leq 3 \sum_{n=1}^{\infty} \frac{c_{n,q}}{K\lambda_{n,q}} |F(n)| e^{-\lambda_{n,q} z}. \tag{7.8}$$

The series in (7.8) converges uniformly on $0 < z < \infty$. By taking the limit as $z \to \infty$ under the summation sign, one verifies boundary condition (iii).

Particular cases.

Case I. Regular hexagonal prism of semi-infinite length. The domain is

$$D \equiv \Big\{ (x,z) = (x_1, x_2, x_3, z)/0 < x_i < \frac{2p}{3},\ i = 1,2,3,\\ x_1 + x_2 + x_3 = p,\ 0 < z < \infty \Big\}.$$

The solution is

$$v(x,z) = \sum_{n=1}^{\infty} c_{n,3} \frac{F(n)}{K\lambda_{n,3}} e^{-\lambda_{n,3} z} \varphi_{n,3}(x), \tag{7.9}$$

where

$$c_{n,3} = \frac{2}{3} \left(\frac{3}{2p} \right)^3,$$

$$\lambda_{n,3} = \frac{6n\pi}{p}, \quad n = 1,2,3,\dots,$$

$$\varphi_{n,3}(z_1) = \cos \lambda_{n,3} x_1 + \cos \lambda_{n,3} x_2 + \cos \lambda_{n,3} x_3.$$

Case II. Equilateral triangular prism of semi-infinite length. The domain is

$$D \equiv \{(x,z) = (x_1, x_2, x_3, z)/0 < x_i < p,\ i = 1,2,3,\\ x_1 + x_2 + x_3 = p,\ 0 < z < \infty\}.$$

The solution is

$$v(x,z) = \sum_{n=1}^{\infty} c_{n,1} \frac{F(n)}{K\lambda_{n,1}} e^{-\lambda_{n,1} z} \varphi_{n,1}(x), \tag{7.10}$$

where

$$c_{n,1} = \frac{2}{3p^3}, \quad \lambda_{n,1} = \frac{2n\pi}{p}, \qquad n = 1,2,3,\dots,$$
$$\varphi_{n,1}(x_1) = \cos\lambda_{n,1}x_1 + \cos\lambda_{n,1}x_2 + \cos\lambda_{n,1}x_3.$$

Acknowledgments. I am thankful to the referee for the valuable suggestions.

REFERENCES

1. J.J. Betancor, M. Linares and J.M.R. Mendez, *A distributional convolution for a generalized finite Fourier transformation*, Pro. Amer. Math. Soc. **128** (1999), 547–556.

2. R.J. Iorio and V.M. Iorio, Jr., *Fourier analysis and partial differential equations*, Cambridge University Press, Cambridge, 2001.

3. P.B. Patil, *Methods of mathematical physics and their applications*, Ph.D. thesis, Marathwada University (M.S.), India, 1975.

4. W. Rudin, *Functional analysis*, Tata McGraw-Hill Publishing Co. Ltd., New Delhi, 1976.

5. B. Sen, *Trilinear coordinates and boundary value problems*, Bull. Calcutta Math. Soc. **60** (1968), 25–30.

6. G.L. Waghmare and S.V. More, *On finite symmetric trilinear integral transform of distributions*, J. Indian Academy Math. **26** (2004), 137–166.

7. ———, *Finite symmetric trilinear integral transform of distributions*, Part II, Inter. J. Math. Math. Sci. **2006**, Article ID 41620, 1–16.

8. A.H. Zemanian, *Generalized integral transformations*, Wiley Interscience, New York, 1968.

Department of Mathematics, Institute of Science, Nagpur – 440 001, (M.S.) India
Email address: glwaghmare@rediff.com