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SOME REMARKS ON SPECIAL SUBORDINATORS

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ABSTRACT. A subordinator is called *special* if the restriction of its potential measure to $(0,\infty)$ has a decreasing density with respect to the Lebesgue measure. In this note we investigate what type of measures μ on $(0,\infty)$ can arise as Lévy measures of special subordinators and what type of functions $u : (0,\infty) \to [0,\infty)$ can arise as potential densities of special subordinators. As an application of the main result, we give examples of potential densities of subordinators which are constant to the right of a positive number.

1. Introduction. A function $\phi : (0,\infty) \to (0,\infty)$ is called a Bernstein function if it admits a representation

$$\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda x})\,\mu(dx), \tag{1.1}$$

where $a \geq 0$ is the *killing term*, $b \geq 0$ the *drift* and μ a measure on $(0,\infty)$ satisfying $\int_0^\infty (x \wedge 1)\,\mu(dx) < \infty$, called the *Lévy measure*. By defining $\mu(\{\infty\}) = a$, the measure μ is extended to a measure on $(0,\infty]$. The function $\overline{\mu}(x) := \mu((x,\infty])$ on $(0,\infty)$ is called the *tail* of the Lévy measure. Using integration by parts, formula (1.1) becomes

$$\phi(\lambda) = b\lambda + \lambda \int_0^\infty e^{-\lambda x}\overline{\mu}(x)\,dx. \tag{1.2}$$

The function ϕ is called a special Bernstein function if the function $\psi : (0,\infty) \to (0,\infty)$ defined by $\psi(\lambda) := \lambda/\phi(\lambda)$ is again a Bernstein function. Let

$$\psi(\lambda) = \tilde{a} + \tilde{b}\lambda + \int_0^\infty (1 - e^{-\lambda x})\,\nu(dx) \tag{1.3}$$

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be the corresponding representation. It is shown in [**16**] that

$$\tilde{b} = \begin{cases} 0 & b > 0 \\ 1/(a + \mu((0,\infty))) & b = 0, \end{cases} \qquad \tilde{a} = \begin{cases} 0 & a > 0 \\ 1/(b + \int_0^\infty t\,\mu(dt)) & a = 0; \end{cases} \tag{1.4}$$

here we are using the convention that $1/\infty = 0$.

Bernstein functions are closely related to subordinators. Let $S = (S_t : t \geq 0)$ be a (killed) subordinator, that is, an increasing Lévy process starting from zero, possibly killed at an exponential time. Then

$$\mathbf{E}(\exp -\lambda S_t) = \exp(-t\phi(\lambda)),$$

where ϕ is a Bernstein function. The potential measure U of the subordinator S is defined by

$$U(A) = \mathbf{E}\int_0^\infty 1_{\{S_t \in A\}}\,dt, \quad A \subset [0,\infty).$$

It is well known that $\mathcal{L}U(\lambda) = 1/\phi(\lambda)$, where $\mathcal{L}$ denotes the Laplace transform. Similarly, for $a > 0$, one defines the a-potential measure U^a of the subordinator S by

$$U^a(A) = \mathbf{E}\int_0^\infty e^{-at}1_{\{S_t \in A\}}\,dt, \quad A \subset [0,\infty).$$

Let ϕ and ψ be a pair of special Bernstein functions such that $\phi(\lambda)\psi(\lambda) = \lambda$ for all $\lambda > 0$, and let $S = (S_t : t \geq 0)$ and $T = (T_t : t \geq 0)$ be the corresponding (killed) subordinators with potential measures U and V. By Theorem 2.1 in [**16**] (see also [**2**, Corollaries 1, 2] for an earlier account), ϕ is special if and only if $U_{|(0,\infty)}$ has a decreasing density $u : (0,\infty) \to [0,\infty)$. In this case

$$U(dt) = \tilde{b}\,\delta_0(dt) + u(t)\,dt,$$

where δ_0 denotes the Dirac measure at 0. Similarly, $V(dt) = b\,\delta_0(dt) + v(t)\,dt$ for a decreasing function $v : (0,\infty) \to [0,\infty)$. Moreover, we have $v(t) = \overline{\mu}(t)$, the tail of the Lévy measure of ϕ.

In this note we are going to investigate what type of measures μ on $(0,\infty)$ can arise as Lévy measures of special subordinators and what type of functions $u:(0,\infty)\to[0,\infty)$ can arise as potential densities of special subordinators. More precisely, we would like to know whether the potential density of a special subordinator is necessarily continuous, and whether it can be constant near the origin or constant to the right of a positive number (or equivalently, whether the Lévy measure of a special subordinator can be supported away from the origin or can have bounded support). Our main tool is an extended version of Hawkes' Theorem 2.1 from [**6**] which essentially says that nonincreasing log-convex functions are tails of Lévy measures of subordinators. Besides providing a proof of an extended version of Hawkes' theorem, we also describe two alternative approaches known from the literature. As an application of the main result, we give examples showing that the potential density of a special subordinator can be constant to the right of a positive number, a fact that is quite surprising to many people in potential theory and certainly surprising to the authors. At the end of this note we also give an application to delayed subordinators studied in [**17**].

Throughout the paper we use the above introduced notation.

2. Hawkes' result revisited. In this section we will give an extended version of Theorem 2.1 in [**6**]. We begin with a well-known result and sketch a proof following [**14**].

Lemma 2.1. *Let $(v_n : n\ge 0)$ be a sequence satisfying $v_0=1$ and $0<v_n\le 1$, $n\ge 1$. Assume that $(v_n : n\ge 0)$ is a Kaluza sequence, that is, $v_n^2\le v_{n-1}v_{n+1}$ for all $n\ge 1$. Then there exists a nonincreasing sequence $(r_n : n\ge 0)$ such that $r_n\ge 0$ for all n, and*

$$1=\sum_{j=0}^{n} r_j\, v_{n-j} \quad \textit{for all } n\ge 0. \tag{2.1}$$

Proof. Note that the inequality $v_n^2\le v_{n-1}v_{n+1}$ is equivalent to $v_n/v_{n-1}\le v_{n+1}/v_n$. Therefore, the sequence $(v_n/v_{n-1} : n\ge 1)$ is increasing.

Define $f_1 := v_1$, and inductively

$$f_n := v_n - \sum_{j=1}^{n-1} f_j\, v_{n-j}. \tag{2.2}$$

It is shown in [**14**] that $f_n \geq 0$ for every $n \geq 0$, and that $\sum_{n=1}^{\infty} f_n \leq 1$.

Define the sequence $(r_n : n \geq 0)$ by $r_0 := 1$ and $r_n := 1 - \sum_{j=1}^{n} f_k$, $n \geq 1$. Clearly, $(r_n : n \geq 0)$ is a nonincreasing sequence of nonnegative numbers. Note that $r_{n-1} - r_n = f_n$ for all $n \geq 1$. Hence, for all $n \geq 1$,

$$\begin{aligned} v_n &= \sum_{j=1}^{n} f_j\, v_{n-j} = \sum_{j=1}^{n} (r_{j-1} - r_j)\, v_{n-j} \\ &= \sum_{j=1}^{n} r_{j-1}\, v_{n-j} - \sum_{j=1}^{n} r_j\, v_{n-j} \\ &= \sum_{j=0}^{n-1} r_j\, v_{n-1-j} - \sum_{j=1}^{n} r_j\, v_{n-j}. \end{aligned}$$

This implies that $\sum_{j=0}^{n} r_j\, v_{n-j} = \sum_{j=0}^{n-1} r_j\, v_{n-1-j}$ for all $n \geq 1$. But for $n = 1$ we have that $\sum_{j=0}^{n-1} r_j\, v_{n-1-j} = r_0 v_0 = 1$. This proves that $\sum_{j=0}^{n} r_j\, v_{n-j} = 1$ for all $n \geq 0$. □

Remark 2.2. The above lemma appears also in [**6**] with a proof having a minor gap (namely, it works only for the case $\sum_{j=1}^{\infty} f_j = 1$). This is why a proof of Lemma 2.1 is included. We note that a sequence $(v_n : n \geq 0)$ satisfying $v_n^2 \leq v_{n-1} v_{n+1}$ for all $n \geq 1$ is also called a log-convex sequence. The conclusion of the lemma, equation (2.1), is a discrete version of Chung's equation (see [**3**]).

The following result is one of those folkloric results whose proofs are difficult to locate. When the subordinator has no killing, it is basically contained in [**3**, page 63 and pages 89–91]. In the compound Poisson case, it is contained in [**15**, Remark 27.3] and [**5**, page 278].

Lemma 2.3. *Suppose that $x \mapsto \overline{\mu}(x)$ is absolutely continuous on $(0, \infty)$. If $\mu((0,\infty)) = \infty$ or $b > 0$, then the potential measure U is*

absolutely continuous. If $\mu((0,\infty)) < \infty$ *and* $b = 0$*, then* $U_{|(0,\infty)}$ *is absolutely continuous.*

Proof. Assume first that the killing term $a = 0$. If $b > 0$, then it is well known that U is absolutely continuous. Assume that $\mu((0,\infty)) = \infty$. Since $\overline{\mu}(x)$ is absolutely continuous, by [**15**, Theorem 27.7] the transition probabilities of S are absolutely continuous and therefore U is absolutely continuous.

If the killing term $a > 0$, then the potential measure of the killed subordinator is equal to the a-potential measure of the (nonkilled) subordinator, hence again absolutely continuous (see, e.g., [**15**, Remark 41.12]).

Assume now that $\mu((0,\infty)) < \infty$ and $b = 0$. Since $x \mapsto \overline{\mu}(x) = \mu((x,\infty))$ is absolutely continuous, so $\mu(dx) = \mu(x)\,dx$, where μ by abuse of notation denotes the density of the measure μ. Let $c := \mu((0,\infty))$. The transition operator at time t of the nonkilled subordinator is given by

$$P_t = \sum_{k=0}^{\infty} e^{-tc}\,\frac{t^k}{k!}\mu^{*k}.$$

Therefore, the potential operator of the killed subordinator is equal to

$$\begin{aligned} U &= \int_0^\infty e^{-at} P_t\,dt \\ &= \int_0^\infty e^{-at}\left(\sum_{k=0}^{\infty} e^{-tc}\frac{t^k}{k!}\mu^{*k}\right) dt \\ &= \sum_{k=0}^{\infty}\frac{\mu^{*k}}{k!}\int_0^\infty e^{-(a+c)t}t^k\,dt \\ &= \frac{1}{a+c}\,\delta_0 + \frac{1}{a+c}\sum_{k=1}^{\infty}\left(\frac{\mu}{a+c}\right)^{*k}. \end{aligned}$$

This shows that $U_{|(0,\infty)}$ is absolutely continuous with the density

$$u(x) = \frac{1}{a+c}\sum_{k=1}^{\infty}\left(\frac{\mu}{a+c}\right)^{*k}(x). \qquad (2.3)$$

□

The following result is an extended version of Theorem 2.1 of [**6**]. It extends Theorem 2.1 of [**6**] in the following three directions: (1) the drift b may be positive; (2) the killing term a may be positive; (3) the Lévy measure may be finite or equivalently the subordinator may be a compound Poisson process. The proof is a slight modification of the original proof in [**6**].

Theorem 2.4. *Suppose that $x \mapsto \overline{\mu}(x)$ is log-convex on $(0,\infty)$. If $\mu((0,\infty)) = \infty$ or $b > 0$, then the potential measure U has a nonincreasing density u. If $\mu((0,\infty)) < \infty$ and $b = 0$, then the restriction $U_{|(0,\infty)}$ has a nonincreasing density u.*

Proof. The log-convexity of $\overline{\mu}(x)$ implies that $\overline{\mu}(x)$ is absolutely continuous on $(0,\infty)$, hence by Lemma 2.3, we know that in both cases densities do exist. We choose the version of u such that

$$\limsup_{h\to 0} \frac{U((x,x+h))}{h} = u(x), \text{ for all } x > 0. \tag{2.4}$$

Note that the log-convexity of $\overline{\mu}$ implies that it is strictly positive everywhere. This excludes the case where $a + \mu((x,\infty)) = 0$ for some $x > 0$. Fix $c > 0$ and define a sequence $(v_n(c) : n \geq 0)$ by

$$v_0(c) := \frac{b/c + \overline{\mu}(c)}{b/c + \overline{\mu}(c)} = 1, \qquad v_n(c) := \frac{\overline{\mu}(nc+c)}{b/c + \overline{\mu}(c)}, \quad n \geq 1.$$

Then clearly $0 < v_n(c) \leq 1$ for all $n \geq 0$. Moreover, $v_n(c)^2 \leq v_{n-1}(c)v_{n+1}(c)$ for all $n \geq 1$. Indeed, for $n \geq 2$, this is equivalent to $\overline{\mu}(nc+c)^2 \leq \overline{\mu}((n-1)c+c)\overline{\mu}((n+1)c+c)$, which is a consequence of the log-convexity of $\overline{\mu}$. For $n = 1$, we have $\overline{\mu}(2c) \leq \overline{\mu}(c)\overline{\mu}(3c) \leq (b/c + \overline{\mu}(c))\overline{\mu}(3c)$.

By Lemma 2.1, there exists a nonincreasing sequence $(r_n(c) : n \geq 0)$ such that

$$\sum_{j=0}^{n} r_j(c)v_{n-j}(c) = 1 \text{ for all } n \geq 0. \tag{2.5}$$

Define

$$u_n(c) := \frac{r_n(c)}{b/c + \overline{\mu}(c)}, \quad n \geq 0.$$

By rewriting (2.5) we get that, for all $n \geq 0$,

$$\frac{b}{c}\, u_n(c) + \sum_{j=0}^{n} u_j(c)\overline{\mu}((n-j)c + c) = 1. \tag{2.6}$$

By multiplying (2.6) by $c\lambda e^{-(n+1)c\lambda}$ and summing over all $n \geq 0$, we obtain

$$b\lambda \sum_{n=0}^{\infty} e^{-(n+1)c\lambda}\, u_n(c) + \sum_{n=0}^{\infty}\sum_{j=0}^{n} c\lambda e^{-(n+1)c\lambda} u_j(c)\overline{\mu}((n-j)c + c) = \sum_{n=0}^{\infty} c\lambda e^{-(n+1)c\lambda}.$$

This can be simplified to

$$b\lambda e^{-c\lambda} \sum_{n=0}^{\infty} e^{-nc\lambda}\, u_n(c) + \left(\sum_{n=0}^{\infty} e^{-nc\lambda} u_n(c) \right) \left(c\lambda \sum_{n=1}^{\infty} e^{-nc\lambda} \overline{\mu}(nc) \right) = \frac{c\lambda e^{-c\lambda}}{1 - e^{-c\lambda}}. \tag{2.7}$$

Define a measure U_c on $(0, \infty)$ by $U_c := \sum_{n=0}^{\infty} u_n(c)\, \delta_{nc}$. Then (2.7) reads

$$\left(\int_0^{\infty} e^{-\lambda t}\, dU_c(t) \right) \left(b\lambda e^{-c\lambda} + \sum_{n=1}^{\infty} c\lambda e^{-nc\lambda} \overline{\mu}(nc) \right) = \frac{c\lambda e^{-c\lambda}}{1 - e^{-c\lambda}}.$$

Let $c \downarrow 0$. The righthand side converges to 1, while

$$\lim_{c \downarrow 0} \left(b\lambda e^{-c\lambda} + \sum_{n=1}^{\infty} c\lambda e^{-nc\lambda} \overline{\mu}(nc) \right) = b\lambda + \int_0^{\infty} \lambda e^{-\lambda t} \overline{\mu}(t)\, dt = \phi(\lambda).$$

Therefore,

$$\lim_{c \downarrow 0} \int_0^{\infty} e^{-\lambda t}\, dU_c(t) = \frac{1}{\phi(\lambda)} = \int_0^{\infty} e^{-\lambda t}\, dU(t).$$

Hence, U_c converges vaguely to U. Since U is absolutely continuous, this implies that, for all $x > 0$ and all $h > 0$,

$$\lim_{c \downarrow 0} U_c((x, x+h)) = \int_x^{x+h} u(t)\, dt.$$

Now suppose that $0 < x < y$, and choose $h > 0$ such that $x < x+h < y$. Moreover, let c be such that none of the endpoints x, $x+h$, y, $y+h$ is a multiple of c. By the monotonicity of $(u_n(c) : n \geq 0)$, it follows that

$$U_c((y, y+h)) \leq U_c((x, x+h)).$$

Let c go to zero along values such that the endpoints x, $x+h$, y, $y+h$ are not multiples of c. It follows that

$$U((y, y+h)) \leq U((x, x+h)).$$

Now from (2.4) it follows that $u(y) \leq u(x)$. □

Corollary 2.5. *Suppose that $v : (0,\infty) \to (0,\infty)$ is decreasing, log-convex and satisfies $\int_0^1 v(t)\, dt < \infty$. Then there exists a special subordinator $T = (T_t : t \geq 0)$ with potential measure V such that v is a density of V.*

Proof. Put $a := v(+\infty)$ and define a measure μ on $(0,\infty)$ by $\mu((x,\infty)) := v(x) - a$. Then μ is a Lévy measure and $\overline{\mu}(x) = v(x)$ is log-convex.

Define $\phi(\lambda) := a + \int_0^\infty (1 - e^{-\lambda t})\, \mu(dt)$. By Theorem 2.4, the restriction $U|_{(0,\infty)}$ to $(0,\infty)$ of the potential measure of U has a decreasing density u. Therefore, ϕ is a special Bernstein function. Let $\psi(\lambda) := \lambda/\phi(\lambda)$ with corresponding special subordinator $T = (T_t : t \geq 0)$. Since the drift b of ϕ is zero, the potential measure V of T has a density equal to $x \to a + \mu((x,\infty))$. But this is precisely v. □

Remark 2.6. If we defined $\phi(\lambda) := a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \mu(dt)$, with $b > 0$, then the same argument would show that v is the density of $V_{|(0,\infty)}$.

Corollary 2.5 is basically Corollary 14.9 in [**1**], a result originally due to [**7, 8, 9**]. We now give another approach to proving Corollary 2.5. This approach comes from [**4**] and is based on random covering. Let λ denote the Lebesgue measure on $[0,\infty)$, and let π be a σ-finite measure on $(0,\infty]$. Let N be a Poisson random measure on $[0,\infty)\times(0,\infty]$ with the characteristic measure $\lambda\times\pi$, and by abuse of notation we denote the set of random points in $[0,\infty)\times(0,\infty]$ also by N. For $(s,t)\in N$, the interval $(s,s+t)$ is called a covering interval. Let

$$R = [0,\infty) \setminus \bigcup_{(s,t)\in N} (s,s+t)$$

be the subset of $[0,\infty)$ of uncovered points. It is proved in [**4**, Theorem 1], that R is a regenerative set. Since every regenerative set is the closure of the image of a subordinator (see [**12**]), one can speak about the potential measure V of the regenerative set R. The second part of Theorem 1 from [**4**] states that if

$$\int_0^1 \exp\left\{\int_x^1 \pi((s,\infty])\,ds\right\} dx < \infty,$$

then the potential measure V has a density v given by the formula

$$v(x) = \exp\left\{\int_x^1 \pi((s,\infty])\,ds\right\}, \quad x>0. \tag{2.8}$$

The function v is clearly decreasing and log-convex. In particular, the regenerative set R is special (in the sense that the corresponding subordinator is special).

Assume now that v satisfies the assumptions of Corollary 2.5. Without loss of generality, we may assume that $v(1)=1$. The function $w(x) := \log v(x)$ is convex so we can define a measure π on $(0,\infty]$ by $\pi((x,\infty]) = -w'(x+)$, the righthand side derivative of w. Let R be the regenerative set of uncovered points corresponding to the Poisson random measure with the characteristic measure $\lambda\times\pi$. Since

$$\int_0^1 \exp\left\{\int_x^1 \pi((s,\infty])\,ds\right\} dx = \int_0^1 v(x)\,dx < \infty,$$

the potential measure of R has a density given by

$$\exp\left\{\int_x^1 \pi((s,\infty])\,ds\right\} = v(x).$$

3. Examples. We recall that a Bernstein function is a complete Bernstein function if its Lévy measure has a completely monotone density. A great deal of literature exists which is devoted to complete Bernstein functions (see, e.g.,[**10, 13**]), and the Laplace exponents of most of the subordinators one encounters are in fact complete Bernstein functions. It is known that complete Bernstein functions are special, and the corresponding potential densities are completely monotone functions. One of the goals of this section is to provide examples of subordinators that are special, but not complete Bernstein.

Example 3.1. Define

$$v(x) := \begin{cases} x^{-\alpha} & 0 < x < 1, \\ x^{-\beta} & 1 \le x < \infty. \end{cases}$$

Assume that $0 < \beta < \alpha < 1$. Then v is decreasing, log-convex and satisfies $\int_0^1 v(t)\,dt < \infty$. By Corollary 2.5, v is the potential density of a special subordinator. Since v is not completely monotone (it is clearly not C^∞), the corresponding Laplace exponent is not a complete Bernstein function.

Proposition 3.2. *Suppose that $S = (S_t : t \ge 0)$ is a subordinator with Laplace exponent $\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\,\mu(dt)$. If μ has bounded support, then S cannot be special.*

Proof. Assume that S is special and $\mu((x_0,\infty)) = 0$. Let $\psi(\lambda) := \lambda/\phi(\lambda)$ with corresponding subordinator $T = (T_t : t \ge 0)$. Let V be the potential measure of T, v the density of $V_{|(0,\infty)}$. Then $v(x) = \mu((x,\infty)) = 0$ for all $x \ge x_0$. But this implies $V((x_0,\infty)) = 0$, which is impossible. □

The following two examples show that a special subordinator may have a Lévy measure with bounded support provided the killing term is

strictly positive. Thus, we have examples of special Bernstein functions $\phi(x) = a + \int_0^\infty (1 - e^{-\lambda t})\,\mu(dt)$ for which $x \mapsto \int_0^\infty (1 - e^{-\lambda t})\,\mu(dt)$ is not a special Bernstein function. This is in contrast to the case of complete Bernstein functions since a Bernstein function is a complete Bernstein function if and only if its Lévy measure has a completely monotone density, and this has nothing to do with the drift term nor the killing term.

Example 3.3. For $0 < \alpha < 1$, define

$$v(x) := \begin{cases} x^{-\alpha} & 0 < x < 1, \\ 1 & 1 \le x < \infty. \end{cases}$$

Again, v is decreasing, log-convex and satisfies $\int_0^1 v(t)\,dt < \infty$. Hence, there exists a special subordinator $T = (T_t : t \ge 0)$ with potential measure V such that v is the density of V. Let ψ be the Laplace exponent of T, and define $\phi(\lambda) := \lambda/\psi(\lambda)$. Then $\phi(\lambda) = 1 + \int_0^\infty (1 - e^{-\lambda t})\,\mu(dt)$ with the Lévy measure $\mu(dx) = \mu(x)\,dx$, where

$$\mu(x) := \begin{cases} \alpha x^{-\alpha-1} & 0 < x < 1, \\ 0 & 1 \le x < \infty. \end{cases}$$

The following example is similar to the previous one but with a finite Lévy measure μ.

Example 3.4.

$$v(x) := \begin{cases} e^{1-x} & 0 < x < 1, \\ 1 & 1 \le x < \infty. \end{cases}$$

Again, v is decreasing, log-convex and satisfies $\int_0^1 v(t)\,dt < \infty$. Hence, there exists a special subordinator $T = (T_t : t \ge 0)$ with potential measure V such that v is the density of V. Let ψ be the Laplace exponent of T, and define $\phi(\lambda) := \lambda/\psi(\lambda)$. Then $\phi(\lambda) = 1 + \int_0^\infty (1 - e^{-\lambda t})\,\mu(dt)$ with the Lévy measure $\mu(dx) = \mu(x)\,dx$, where

$$\mu(x) := \begin{cases} e^{1-x} & 0 < x < 1, \\ 0 & 1 \le x < \infty. \end{cases}$$

Remark 3.5. In the last example, the density of the Lévy measure $\mu(x)$ is discontinuous at $x = 1$. Put

$$f(x) := \begin{cases} e^{-x} & 0 < x < 1, \\ 0 & 1 \le x < \infty. \end{cases}$$

It is easy to see that

$$f^{*2}(x) = \begin{cases} xe^{-x} & 0 < x < 1, \\ (2-x)e^{-x} & 1 \le x < 2, \\ 0 & 2 \le x < \infty, \end{cases}$$

and that all convolutions f^{*n}, $n \ge 2$, are continuous everywhere. It is easy to check that, for $x \in (0,1)$,

$$f^{*k}(x) = \frac{x^{k-1}}{(k-1)!}e^{-x}, \quad k \ge 1.$$

Using the above formula, one can check that for $x \in [1,(5/4))$,

$$f^{*k}(x) \le \frac{5}{8}\left(\frac{1}{2}\right)^{k-3} e^{-x}, \quad k \ge 3.$$

Using the above two displays, we can easily see that the series

$$\frac{1}{e}\sum_{k=1}^{\infty} f^{*k}(x)$$

is uniformly convergent for $x \in (0,(5/4))$. By formula (2.3), we get that $u(x)$ is discontinuous at $x = 1$. But $u(x) = \tilde{a} + \nu((x,\infty))$, implying that $x \mapsto \nu((x,\infty))$ has a discontinuity at $x = 1$. Hence, ν has an atom at 1. This shows that the Lévy measure of a special subordinator may have atoms. As a consequence, the tail is not log-convex. In particular, this shows that the family of special Bernstein functions is larger than the family of Bernstein functions with Lévy measure having a log-convex tail.

Note that potential densities similar to those given in Examples 3.3 and 3.4 can also be constructed from formula (2.8) by choosing the measure π so that $\pi((1,\infty]) = 0$.

We end this section by showing that a special subordinator with no drift cannot have its Lévy measure supported away from zero.

Proposition 3.6. *Suppose that* $\phi(\lambda) = a + \int_0^\infty (1 - e^{-\lambda t})\,\mu(dt)$ *and that the Lévy measure* μ *is nontrivial and that* $\mu((0, t_0]) = 0$ *for* $t_0 > 0$. *Then* ϕ *is not special.*

Proof. Suppose, on the contrary, that ϕ is special. Let $\psi(\lambda) = \lambda/\phi(\lambda) = \widetilde{a} + \tilde{b}\lambda + \int_0^\infty (1 - e^{-\lambda x})\,\nu(dx)$. The potential density of the corresponding subordinator T is given by the formula $v(t) = a + \mu((t, \infty))$. In particular, $v(t) = a + \mu((t_0, \infty)) =: \gamma$ for $0 < t < t_0$, and $V(t) := V([0, t]) = \gamma t$. It follows from (1.4) that $\tilde{b} = 1/\gamma > 0$. Choose $x \leq t_0$. Then $V(x) = \gamma x = x/\tilde{b}$. Suppose that $\nu \neq 0$. Let τ be an exponential random variable with parameter $\widetilde{a}$ independent of T when $\widetilde{a} > 0$ and let $\tau = +\infty$ when $\widetilde{a} = 0$. Then

$$V(x) = \mathbf{E} \int_0^\tau 1_{(\tilde{b}t + \sum_{0 < s \leq t} \Delta T_s \leq x)}\, dt.$$

Since $\mathbf{P}(\sum_{0<s\leq t} \Delta T_s > 0) > 0$, it follows that

$$\begin{aligned} V(x) &= \mathbf{E} \int_0^\tau 1_{\{\tilde{b}t + \sum_{0 < s \leq t} \Delta T_s \leq x\}}\, dt \\ &< \mathbf{E} \int_0^\tau 1_{\{\tilde{b}t \leq x\}}\, dt = \mathbf{E}\left[\tau \wedge \frac{x}{\tilde{b}}\right] \leq \frac{x}{\tilde{b}}. \end{aligned}$$

This contradicts the fact $V(x) = x/\tilde{b}$. If $\nu = 0$ and $\widetilde{a} > 0$, then

$$V(x) = \mathbf{E} \int_0^\tau 1_{\{\tilde{b}t \leq x\}}\, dt = \mathbf{E}\left[\tau \wedge \frac{x}{\tilde{b}}\right] < \frac{x}{\tilde{b}}.$$

The above display again gives a contradiction. Hence, $\nu = 0$ and $\widetilde{a} = 0$, implying that $\psi(\lambda) = \tilde{b}\lambda$, and hence $\phi(\lambda) = 1/\tilde{b}$. This is a contradiction with $\mu \neq 0$. □

4. Delayed subordinators. Assume that T is a subordinator with no killing such that the restriction $V|_{(0,\infty)}$ of its potential measure V has a decreasing density v satisfying $v(x) = a$ for all $x \geq 1$. Examples

3.3 and 3.4 show that this is possible. Note that for any interval $(x,y) \subset (1,\infty)$, $V((x,y)) = a(y-x)$ holds. This means that $V_{|(1,\infty)}$ is proportional to the Lebesgue measure. The Laplace transform of V is

$$\mathcal{L}V(\lambda) = \int_0^1 e^{-\lambda x}\, V(dx) + \int_1^\infty e^{-\lambda x} a\, dx = \int_0^1 e^{-\lambda x}\, V(dx) + \frac{a}{\lambda} e^{-\lambda}.$$

Since $e^{-\lambda} V([0,1]) \le \int_0^1 e^{-\lambda x}\, V(dx) \le V([0,1])$, we have that

$$\frac{1}{\lambda V([0,1]) + a e^{-\lambda}} \le \frac{1}{\lambda \mathcal{L}V(\lambda)} \le \frac{1}{\lambda e^{-\lambda} V([0,1]) + a e^{-\lambda}}.$$

Since $\psi(\lambda)/\lambda = 1/(\lambda \mathcal{L}V(\lambda))$, by letting $\lambda \to 0$ we obtain that

$$\psi'(0+) = \lim_{\lambda \to 0} \frac{\psi(\lambda)}{\lambda} = \frac{1}{a}.$$

Therefore, the subordinator T has finite expectation and $\mathbf{E}T_1 = 1/a$.

Define the first passage time $\tau_x := \inf\{t > 0 : T_t > x\}$. The Laplace transform of T_{τ_x} is given by the formula

$$\mathbf{E}[e^{-\lambda T_{\tau_x}}] = \psi(\lambda) \int_{(x,\infty)} e^{-\lambda z}\, V(dz) \tag{4.1}$$

(see for example [**11**, Exercise 5.5]). Therefore, the Laplace transform of the overshoot distribution for T with the above potential density is for all $x \ge 1$ given by

$$\mathbf{E}[e^{-\lambda(T_{\tau_x} - x)}] = e^{\lambda x} \psi(\lambda) \int_{(x,\infty)} e^{-\lambda z} v(z)\, dz = a\, \frac{\psi(\lambda)}{\lambda}. \tag{4.2}$$

Proposition 4.1. *Suppose that $T = (T_t : t \ge 0)$ is a subordinator with Laplace exponent ψ and potential measure V such that*

$$\mathbf{E}[e^{-\lambda(T_{\tau_x} - x)}] = a\, \frac{\psi(\lambda)}{\lambda} \tag{4.3}$$

for $x = x_0 > 0$. Then $V_{|(x_0,\infty)}(dz) = a\, dz$, and moreover, (4.3) is valid for all $x \ge x_0$.

Proof. From (4.1) and the assumption it follows that

$$\begin{aligned}\frac{a}{\lambda} &= e^{\lambda x_0}\int_{(x_0,\infty)} e^{-\lambda z}\,V(dz)\\ &= \int_{(0,\infty)} e^{-\lambda(z-x_0)}1_{(0,\infty)}(z-x_0)\,V(dz)\\ &= \int_{(0,\infty)} e^{-\lambda z}V_{x_0}(dz),\end{aligned}$$

where $V_{x_0}(dz)$ is the image of $V(dz)$ under the map $z \mapsto z - x_0$. By uniqueness of the Laplace transform, $V_{x_0}(dz) = a\,dz$. This implies that $V_{|(x_0,\infty)}(dz) = a\,dz$. The last statement now follows from the discussion preceding the proposition (with 1 replaced by x_0). □

Let $X = (X_t : t \geq 0)$ be a subordinator with Laplace exponent Φ, no killing, drift $d \geq 0$, Lévy measure Π and finite mean equal to $1/a$. The limiting overshoot distribution of X as $x \to \infty$ is given by

$$F(y) = \lim_{x\to\infty} \mathbf{P}(X_{\tau_x} - x \leq y) = a\bigg(d + \int_0^v \Pi(t,\infty)\,dt\bigg),$$

see, for example, [**17**]. The Laplace transform of F is equal to

(4.4)
$$\begin{aligned}\int_0^\infty e^{-\lambda x}\,F(dx) &= a\bigg(\int_0^\infty e^{-\lambda x}d\,\delta_0(dx) + \int_0^\infty e^{-\lambda x}\Pi(x,\infty)\,dx\bigg)\\ &= a\bigg(d + \int_0^\infty e^{-\lambda x}\Pi(x,\infty)\,dx)\bigg)\\ &= a\frac{\Phi(\lambda)}{\lambda}.\end{aligned}$$

In [**17**] van Harn and Steutel discussed delayed subordinators. Let Y be a random variable independent of the subordinator X. The process $\overline{X} = (\overline{X}_t : t \geq 0)$ defined by

$$\overline{X}_t := Y + X_t$$

is called a delayed subordinator. Let $\overline{H}(x) = \mathbf{E}\int_0^\infty 1_{(\overline{X}_t \leq x)}$, and let $\overline{W}(x)$ be the overshoot distribution of $\overline{X}$ over the level x. It is shown

in [**17**] that the following properties are equivalent: (1) the distribution of Y is equal to F, (2) $\overline{H}(x) = ax$ for all $x \geq 0$, and (3) $\overline{W}(x)$ has distribution F for all $x > 0$. In particular, in this case, the delayed subordinator $\overline{X}$ is stationary in the sense that the expected occupation time is proportional to Lebesgue measure.

Going back to the subordinator T, formula (4.4) compared with (4.2) shows that the overshoot distribution of $T_{\tau_x} - x$ for all $x \geq 1$ is constant and equal to the limiting overshoot distribution F of the subordinator T. This suggests that T is "close" to the delayed subordinator. To make this precise, consider the following decomposition:

$$T_{\tau_1+t} - 1 = (T_{\tau_1+t} - T_{\tau_1}) + (T_{\tau_1} - 1).$$

The process $(T_{\tau_1+t} - T_{\tau_1} : t \geq 0)$ is a copy of the subordinator T, independent of the random variable $T_{\tau_1} - 1$. The distribution of $T_{\tau_1} - 1$ is equal to F. Hence the process $(T_{\tau_1+t} - 1 : t \geq 0)$ with state space $[0, \infty)$ can be considered as a delayed subordinator T with the delay distributed as $T_{\tau_1} - 1$.

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