ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 1, 2010

LEGENDRE SURFACES WITH HARMONIC MEAN CURVATURE VECTOR FIELD IN THE UNIT 5-SPHERE

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ABSTRACT. We obtain the explicit representation of Legendre surfaces in the unit 5-sphere with harmonic mean curvature vector field, under the condition that the mean curvature function is constant along a certain special direction.

1. Introduction. It is well known that an odd-dimensional unit sphere S^{2n+1} is equipped with the standard Sasakian structure (g, ϕ, η, ξ), see Section 2. The study of minimal Legendre submanifolds in S^{2n+1} is a very active field and closely related to one of minimal Lagrangian submanifolds in complex projective space.

A natural generalization of a minimal submanifold is a submanifold with parallel mean curvature vector field. However, Legendre submanifolds in S^{2n+1} with parallel mean curvature vector field are in fact minimal, see [**9**]. So, in case the ambient space is S^{2n+1}, as a generalization of minimal Legendre submanifolds, it is natural to consider Legendre submanifolds whose mean curvature vector field H is harmonic with respect to the normal Laplacian Δ^D, that is,

$$\Delta^D H = 0. \tag{1.1}$$

The purpose of this paper is to study the class of nonminimal Legendre surfaces satisfying (1.1) in the unit 5-sphere. On nonminimal Legendre submanifolds, there exists a special vector field: ϕH. Moreover, in case the dimension is 2, up to signs, there is a unique unit vector field normal to ϕH. We denote it by $(\phi H)^{\perp}$. In this paper, under the condition that the square of the mean curvature function is constant along one of ϕH and $(\phi H)^{\perp}$, we completely determine nonminimal Legendre surfaces satisfying (1.1) in the unit 5-sphere.

2010 AMS *Mathematics subject classification.* Primary 53C42, Secondary 53B25.
Received by the editors on December 20, 2005, and in revised form on September 12, 2007.
DOI:10.1216/RMJ-2010-40-1-313

Theorem 1. *Let $f : M^2 \to S^5 \subset \mathbf{C}^3$ be a nonminimal Legendre surface satisfying $\Delta^D H = 0$ in the unit 5-sphere. Then the mean curvature function of M^2 is not constant.*

If H satisfies $\phi H ||H||^2 = 0$, then f is locally given by

(1.2)
$$f(x,y) = \left(\frac{1}{\sqrt{2}}\exp\left(\frac{1+\sqrt{5}}{2}iy\right)\cos x, \frac{1}{\sqrt{2}}\exp\left(\frac{1-\sqrt{5}}{2}iy\right)\cos x, \sin x\right).$$

If H satisfies $(\phi H)^{\perp}||H||^2 = 0$, then f is locally given by

(1.3) $f(x,y)$
$$= \frac{1}{\sqrt{2}}\left(i+\sin x, (\sec x+\tan x)^i \cos x \cos y, (\sec x+\tan x)^i \cos x \sin y\right).$$

2. Legendre submanifolds in the unit sphere. Let $\mathbf{C}^{n+1}$ be the complex Euclidean $(n+1)$-space together with the canonical complex structure J. Denote by S^{2n+1} the unit sphere with the standard induced metric g in $\mathbf{C}^{n+1}$. The position vector field $\mathbf{x}$ of S^{2n+1} is a unit normal vector field of S^{2n+1} in $\mathbf{C}^{n+1}$ and the vector field $\xi := -J\mathbf{x}$ is tangent to S^{2n+1}. Define a 1-form η and an endomorphism field φ on M by the formula:

$$JX = \phi X + \eta(X)\mathbf{x}, X \in TM.$$

It is easy to see that (g, φ, η, ξ) satisfies

$$\varphi^2 = -I + \eta \otimes \xi, \qquad d\eta(X,Y) = g(X, \varphi Y).$$

Thus (g, φ, η, ξ) is a contact metric structure of S^{2n+1}.

Denote by $\overline{\nabla}$ the Levi-Civita connection of S^{2n+1}. Then

$$\overline{\nabla}_X \xi = -\varphi X, \qquad (\overline{\nabla}_X \varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

These formulas imply that S^{2n+1} is a Sasakian manifold.

An immersed n-submanifold $x : M^n \to S^{2n+1}$ is said to be a *Legendre submanifold* if $x^*\eta = 0$. The formulas of Gauss and Weingarten of x are given respectively by

(2.1)
$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\overline{\nabla}_X V = -A_V X + D_X V,$$

where $X, Y \in TM^n$, $V \in T^\perp M^n$, ∇, h, A and D are the Levi-Civita connection of M^n, the second fundamental form, the shape operator and the normal connection. The mean curvature vector H is given by $H = (1/n)\,\mathrm{trace}\, h$. Its length $\|H\|$ is called the *mean curvature function* of M^n. The normal Laplacian is defined by $\Delta^D = -\sum_{i=1}^n (D_{e_i} D_{e_i} - D_{\nabla_{e_i} e_i})$, where $\{e_i\}$ is a local orthonormal frame of M^n.

For Legendre submanifolds we have [**1**]

$$A_{\phi Y} X = -\phi h(X,Y) = A_{\phi X} Y, \quad A_\xi = 0. \tag{2.2}$$

Moreover, a straightforward computation shows that the equations of Gauss, Codazzi, Ricci of Legendre submanifolds in the unit sphere are equivalent to

$$\langle R(X,Y)Z, W\rangle = \langle [A_{\phi Z}, A_{\phi W}]X, Y\rangle + \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle, \tag{2.3}$$

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z), \tag{2.4}$$

where $\overline{\nabla} h$ is defined by $(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

3. The proof of Theorem 1. We assume that the mean curvature function is nowhere zero. Let $\{e_i\}$, $i = 1, \dots, 5$, be an orthonormal frame along M^2 such that e_1 and e_2 are tangent to M^2, $\phi e_1 = e_3$, $\phi e_2 = e_4$, $\xi = e_5$ and $H = (\alpha/2)\phi e_1$, with $\alpha > 0$. Then, it follows from (2.2) that the second fundamental form takes the form:

$$\begin{aligned} h(e_1, e_1) &= (\alpha - c)\phi e_1 + b\phi e_2, \\ h(e_1, e_2) &= b\phi e_1 + c\phi e_2, \\ h(e_2, e_2) &= c\phi e_1 - b\phi e_2, \end{aligned} \tag{3.1}$$

for some functions b and c.

We put $\omega_i^j(e_k) = \langle \nabla_{e_k} e_i, e_j\rangle$. From (2.4) we get

$$e_1 c + 3b\omega_1^2(e_1) = e_2 b + (\alpha - 3c)\omega_1^2(e_2), \tag{3.2}$$

$$-e_1 b + 3c\omega_1^2(e_1) = e_2 c + 3b\omega_1^2(e_2), \tag{3.3}$$

$$e_2(\alpha - c) - 3b\omega_1^2(e_2) = e_1 b + (\alpha - 3c)\omega_1^2(e_1). \tag{3.4}$$

Suppose that M^2 satisfies $\Delta^D H = 0$. Then we have the following three lemmas, see [**8**, pages 290, 291].

Lemma 2.

$$\Delta_M \alpha + \alpha\{1 + (\omega_1^2(e_1))^2 + (\omega_1^2(e_2))^2\} = 0, \tag{3.5}$$

$$2(e_1\alpha)\omega_1^2(e_1) + 2(e_2\alpha)\omega_1^2(e_2) + \alpha\{e_1(\omega_1^2(e_1)) + e_2(\omega_1^2(e_2))\} = 0, \tag{3.6}$$

$$e_1\alpha + \alpha\omega_1^2(e_2) = 0, \tag{3.7}$$

where Δ_M is the Laplace operator acting on $C^\infty(M)$.

Note that α is not constant by (3.5), since α is nowhere zero.

Lemma 3. *There exist local coordinates x and y such that*

$$\begin{aligned} e_1 &= \alpha\partial_x, \\ e_2 &= \alpha\partial_y, \\ \omega_1^2(e_1) &= \alpha_y, \\ \omega_1^2(e_2) &= -\alpha_x. \end{aligned}$$

Lemma 4. *The following relation holds*:

$$b^2 = \frac{\alpha c}{2} - c^2. \tag{3.8}$$

The allied mean curvature vector $a(H)$ is defined by

$$\sum_{r=4}^{5}(\operatorname{trace} A_H A_{e_r})e_r,$$

see [**2**, page 197]. If $a(H)$ vanishes identically on M^2, it is called a *Chen* surface.

Suppose that M^2 is not Chen surface, i.e., $b \neq 0$. Then we have $c \neq 0$ and $\alpha \neq 2c$ from (3.8). We may assume that $b > 0$, if necessary by changing the sign of e_2. By differentiating (3.8) we get

$$b_i = \frac{(\alpha - 4c)c_i + \alpha_i c}{4b}, \tag{3.9}$$

where $i = x, y$.

Using (3.9) we replace (3.2) and (3.3) by the derivatives with respect to x and y as follows:

$$\begin{pmatrix} \alpha & -(\alpha(\alpha - 4c))/(4b) \\ (\alpha(\alpha - 4c))/(4b) & \alpha \end{pmatrix} \begin{pmatrix} c_x \\ c_y \end{pmatrix} = \begin{pmatrix} -(\alpha - 3c)\alpha_x - ((12b^2 - \alpha c)/(4b))\alpha_y \\ ((12b^2 - \alpha c)/(4b))\alpha_x + 3c\alpha_y \end{pmatrix}. \tag{3.10}$$

First we investigate the case of $\phi H\|H\|^2 = 0$, i.e., $\alpha_x = 0$. Then using (3.8) and (3.10) we get

$$c_x = -\frac{8bc}{\alpha^2}\alpha_y, \tag{3.11}$$

$$c_y = \frac{5\alpha c - 8c^2}{\alpha^2}\alpha_y. \tag{3.12}$$

By a long but straightforward computation, we obtain

$$\begin{aligned} bc_{xy} = {} & (-24\alpha_y^2 - 4\alpha\alpha_{yy})\left(\frac{c}{\alpha}\right)^2 + (112\alpha_y^2 + 8\alpha\alpha_{yy})\left(\frac{c}{\alpha}\right)^3 \\ & - 128\alpha_y^2\left(\frac{c}{\alpha}\right)^4, \end{aligned} \tag{3.13}$$

$$bc_{yx} = -20\alpha_y^2\left(\frac{c}{\alpha}\right)^2 + 104\alpha_y^2\left(\frac{c}{\alpha}\right)^3 - 128\alpha_y^2\left(\frac{c}{\alpha}\right)^4. \tag{3.14}$$

Since $bc_{xy} = bc_{yx}$, we find that $\alpha_y^2 + \alpha\alpha_{yy} = 0$ from (3.13) and (3.14). But it contradicts (3.5).

Next, we investigate the case of $(\phi H)^{\perp}\|H\|^2 = 0$, i.e., $\alpha_y = 0$. Similarly, as in the case of $\alpha_x = 0$, we have

$$c_x = \frac{-3\alpha c + 8c^2}{\alpha^2}\alpha_x, \tag{3.15}$$

$$c_y = \frac{4\alpha b - 8bc}{\alpha^2}\alpha_x, \tag{3.16}$$

$$bc_{xy} = -6\alpha_x^2\left(\frac{c}{\alpha}\right) + 56\alpha_x^2\left(\frac{c}{\alpha}\right)^2 - 152\alpha_x^2\left(\frac{c}{\alpha}\right)^3 + 128\alpha_x^2\left(\frac{c}{\alpha}\right)^4, \tag{3.17}$$

(3.18)

$$\begin{aligned} bc_{yx} = {} & (-4\alpha_x^2 + 2\alpha\alpha_{xx})\left(\frac{c}{\alpha}\right) + (48\alpha_x^2 - 8\alpha\alpha_{xx})\left(\frac{c}{\alpha}\right)^2 \\ & - (144\alpha_x^2 - 8\alpha\alpha_{xx})\left(\frac{c}{\alpha}\right)^3 + 128\alpha_x^2\left(\frac{c}{\alpha}\right)^4. \end{aligned}$$

Using $bc_{xy} = bc_{yx}$ we get $\alpha_x^2 + \alpha\alpha_{xx} = 0$ from (3.17) and (3.18); however, it contradicts (3.5).

Therefore, we conclude that b must be 0, i.e., M^2 must be a Chen surface if $\phi H\|H\|^2 = 0$ or $(\phi H)^{\perp}\|H\|^2 = 0$. Applying the classification of Legendre Chen surfaces satisfying $\Delta^D H = 0$, see [**7**, Theorem 8 and Corollary 9], we can prove the statement.

4. Other examples of Legendre surfaces satisfying $\Delta^D H = 0$. In this section, we show a way to construct Legendre surfaces satisfying $\Delta^D H = 0$, $(\phi H)^{\perp}\|H\|^2 \neq 0$ and $\phi H\|H\|^2 \neq 0$.

One can obtain the following existence and uniqueness theorem by the similar way to those given in [**4, 5**], cf. [**3**].

Theorem 5. *Let $(M^n, \langle\cdot,\cdot\rangle)$ be an n-dimensional simply connected Riemannian manifold. Let σ be a symmetric bilinear TM^n-valued form on M^n satisfying*

(1) *$\langle\sigma(X,Y), Z\rangle$ is totally symmetric,*

(2) *$(\nabla\sigma)(X,Y,Z) = \nabla_X\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ is totally symmetric,*

(3) $R(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y + \sigma(\sigma(Y,Z),X) - \sigma(\sigma(X,Z),Y)$.

Then there exists a Legendre isometric immersion $x : (M^n, \langle\cdot,\cdot\rangle) \to S^{2n+1}$ *such that the second fundamental form* h *satisfies* $h(X,Y) = \phi\sigma(X,Y)$.

Theorem 6. *Let* $x^1, x^2 : M^n \to S^{2n+1}$ *be two Legendre isometric immersions of a connected Riemannian* n*-manifold into the unit sphere* S^{2n+1} *with second fundamental forms* h^1 *and* h^2. *If*

$$\langle h^1(X,Y), \phi x^1_* Z\rangle = \langle h^2(X,Y), \phi x^2_* Z\rangle$$

for all vector fields X, Y *and* Z *tangent to* M^n, *there exists an isometry* A *of* S^{2n+1} *such that* $x^1 = A \circ x^2$.

Let $f(t)$ be a solution of the following ODE:

$$\frac{d^2 f}{dt^2} = \frac{1}{2}e^{-2f}. \tag{4.1}$$

We put $\alpha(x,y) := e^{f(x-y)}$. Let $(M^2, g = (1/\alpha^2)(dx^2+dy^2))$ be a Riemannian 2-manifold. We define a symmetric bilinear form σ on M^2 by

$$\begin{aligned} \sigma(e_1,e_1) &= \frac{3}{4}\alpha e_1 + \frac{\alpha}{4}e_2, \\ \sigma(e_1,e_2) &= \frac{\alpha}{4}e_1 + \frac{\alpha}{4}e_2, \\ \sigma(e_2,e_2) &= \frac{\alpha}{4}e_1 - \frac{\alpha}{4}e_2, \end{aligned} \tag{4.2}$$

where $e_1 = \alpha\partial_x$ and $e_2 = \alpha\partial_y$. By a straightforward computation, we find that $((M^2,g),\sigma)$ satisfies (1), (2) and (3) of Theorem 5. Therefore, there exists a unique Legendre surfaces in S^5 whose second fundamental form h is given by $h = \phi\sigma$. Moreover, such a surface satisfies $\Delta^D H = 0$, $(\phi H)^\perp \|H\|^2 \neq 0$ and $\phi H\|H\|^2 \neq 0$.

REFERENCES

1. D.E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress Math. **203**, Birkhauser Boston, Inc., Boston, 2002.

2. B.Y. Chen, *A report on submanifolds of finite type*, Soochow J. Math. **22** (1996), 117–337.

3. B.Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *An exotic totally real minimal immersion of S^3 in CP^3 and its characterization*, Proc. Royal Soc. Edinburgh **126** (1995), 153–165.

4. B.Y. Chen and L. Vrancken, *Addendum to*: *Existence and uniqueness theorem for slant immersions and its applications*, Result. Math. **39** (2001), 18–22.

5. ———, *Existence and uniqueness theorem for slant immersions and its applications*, Result. Math. **31** (1997), 28–39.

6. J. Inoguchi, *Submanifolds with harmonic mean curvature vector field in contact 3-manifolds*, Colloq. Math. **100** (2004), 163–179.

7. T. Sasahara, *Legendre surfaces whose mean curvature vectors are eigenvectors of the Laplace operator*, Note. Mat. **22** (2003/04), 49–58.

8. ———, *Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors*, Publ. Math. Debrecen. **67** (2005), 285–303.

9. K. Yano and M. Kon, *Structures on manifolds*, Series Pure Math. **3**, World Scientific, Singapore, 1984.

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