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EIGENVALUE PROBLEMS OF A DEGENERATE QUASILINEAR ELLIPTIC EQUATION

C.V. PAO

ABSTRACT. This paper is concerned with positive eigenvalues and positive eigenfunctions of a class of degenerate and nondegenerate quasilinear elliptic equations. The degenerate property of the quasilinear operator can lead to a very different positive eigenvalue distribution when compared with classical linear eigenvalue problems.

1. Introduction. In the eigenvalue problem

$$-\nabla \cdot (D(\phi)\nabla\phi) = \lambda\phi \text{ in } \Omega, \quad \phi(x) = 0 \text{ on } \partial\Omega, \tag{1.0}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega$; if $D(\phi) = D_0$ is a positive constant, then the problem has a countable number of eigenvalues and a positive eigenfunction only associated with the smallest eigenvalue. However, the eigenvalue distribution can be rather different if $D(\phi)$ depends on ϕ, especially in the degenerate case where $D(0) = 0$. In this note we investigate the eigenvalue problem for a slightly more general equation of the form

$$\begin{gathered} -\nabla \cdot (a(x)D(\phi)\nabla\phi) + \mathbf{c}(x) \cdot (D(\phi)\nabla\phi) = \lambda\phi \text{ in } \Omega \\ \phi(x) = 0 \text{ on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $a(x)$ is a strictly positive function in $\overline{\Omega} \equiv \Omega \cup \partial\Omega$, $\mathbf{c}(x) = (c_1(x), \dots, c_n(x))$ is a smooth function in Ω, and $D(\phi)$ is a positive function in $(0, \infty)$ with either $D(0) = 0$ or $D(0) > 0$. We assume that Ω is of class $C^{2+\alpha}$, $a(x)$ and $c_i(x)$, $i = 1, \dots, n$, are in $C^\alpha(\overline{\Omega})$, and $D(\phi)$ satisfies hypothesis (H) in Section 2, where $\alpha \in (0, 1)$. Our aim is to show that, under the above condition, every $\lambda > 0$ is an eigenvalue of (1.1), and corresponding to it there is a positive eigenfunction $\phi(x)$.

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Moreover, the positive eigenfunction is unique if $\mathbf{c} = 0$ and $D(\phi)$ is monotonic. The same conclusions hold true for every $\lambda > \mu_0 D(0)$ if the condition $D(0) = 0$ in hypothesis (H) is replaced by $D(0) > 0$, where μ_0 is the smallest eigenvalue of the linear eigenvalue problem

$$(1.2) \qquad -\nabla \cdot (a(x)\nabla\psi) + \mathbf{c}(x) \cdot \nabla\psi = \mu\psi \text{ in } \Omega, \quad \psi(x) = 0 \text{ on } \partial\Omega.$$

Nonlinear eigenvalue problems have been investigated by many researchers, and some of the earlier works can be found in [**2–7, 10**]. In these papers, the equation under consideration involves either a constant $D(\phi) = D_0$ and a nonlinear function $f(\phi)$ instead of ϕ, cf. [**2–6, 10**] or with $D(\phi)$ depending on ϕ but with $D(0) > 0$, cf. [**2, 3**]. The work in [**7**] is concerned with the existence of a positive solution for a degenerate elliptic system of a slightly different form using the method of upper and lower solutions. In this paper we use the same method of upper and lower solutions as that in [**9**] to study the eigenvalue problem for a class of functions $D(\phi)$ with either $D(0) = 0$ or $D(0) > 0$. This class of functions for the degenerate case $D(0) = 0$ includes the elementary functions

$$(1.3) \qquad \phi^\alpha, \ \sinh(\alpha\phi), \ \cosh(\alpha\phi) - 1, \ \ln(1 + \alpha\phi), \ e^{\alpha\phi} - 1, \quad \alpha > 0,$$

and the products or linear combinations (with positive coefficients) of these functions such as

$$p(\phi) = a_1\phi^{\alpha_1} + \cdots + a_m\phi^{\alpha_m}, \quad \sinh(\alpha\phi)p(\phi), \text{ etc.},$$

where a_i and α_i, $i = 1, \ldots, m$, are positive constants. Some of the constants a_i can be negative so long as $p(\phi) > 0$ for $\phi > 0$ (see Remark 2.1).

2. The main theorems.

To ensure that problem (1.1) has a positive solution for every $\lambda > 0$, we impose the following conditions on $D(\phi)$.

(H) $D(\phi)$ is a continuous function of $\phi \in \mathbf{R}^+$ such that $D(\phi) > 0$ for $\phi > 0$, $D(0) = 0$, and $\lim D(\phi) = \infty$ as $\phi \to \infty$.

The following theorems give our main results.

Theorem 1. *Let $D(\phi)$ satisfy hypothesis* (H). *Then, for every $\lambda > 0$, problem* (1.1) *has a positive solution $\phi(x)$. Moreover, the positive* *solution $\phi(x)$ is unique if $\mathbf{c}(x) \equiv 0$ and $D(\phi)$ is either increasing or decreasing in $\phi > 0$.*

Theorem 2. *Let $D(\phi)$ satisfy hypothesis* (H) *except with the condition $D(0) = 0$ replaced by $D(0) > 0$. Then all the conclusions in Theorem* 1 *hold true for every $\lambda > \mu_0 D(0)$, where μ_0 is the smallest eigenvalue of* (1.2).

Remark 2.1. (a) It is easy to verify that if $D_1(\phi)$ and $D_2(\phi)$ satisfy the conditions in hypothesis (H), then $D(\phi) \equiv D_1(\phi)D_2(\phi)$ and $D(\phi) \equiv a_1 D_1(\phi) + a_2 D_2(\phi)$, where a_1 and a_2 are positive constants, also satisfy hypothesis (H). Moreover, $D(\phi)$ is increasing or decreasing in ϕ if both $D_1(\phi)$ and $D_2(\phi)$ are increasing or decreasing in ϕ. This implies that the elementary functions in (1.3) and their products or linear combinations all satisfy hypothesis (H). Hence, Theorem 1 is applicable to this class of functions $D(\phi)$. Similar elementary functions can be found for the case $D(0) > 0$.

(b) If $D(\phi) < 0$ for $\phi > 0$, then the conclusions in Theorem 1 and Theorem 2 hold true for $\lambda < 0$ and $\lambda < -\mu_0 D(0)$, respectively.

3. Proof of main theorems. We prove Theorem 1 and Theorem 2 by the method of upper and lower solutions. We say that $\tilde{\phi} \in C^2(\Omega) \cap C(\overline{\Omega})$ is an upper solution of (1.1) if

$$\begin{aligned} -\nabla \cdot (aD(\tilde{\phi})\nabla\tilde{\phi}) + \mathbf{c} \cdot (D(\tilde{\phi})\nabla\tilde{\phi}) &\geq \lambda\tilde{\phi} \text{ in } \Omega \\ \tilde{\phi} &\geq 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.1}$$

Similarly, $\hat{\phi}$ is called a lower solution if it satisfies the inequalities in (3.1) in reversed order. The pair $\tilde{\phi}$, $\hat{\phi}$ are said to be ordered if $\tilde{\phi} \geq \hat{\phi}$ in $\overline{\Omega}$. For a given pair of ordered upper and lower solutions $\tilde{\phi}$, $\hat{\phi}$, we set

$$\mathcal{S} \equiv \{\phi \in C(\overline{\Omega});\ \hat{\phi} \leq \phi \leq \tilde{\phi}\}. \tag{3.2}$$

Define

$$w = I(\phi) = \int_0^\phi D(s)\,ds, \quad \phi \geq 0. \tag{3.3}$$

Since $dw/d\phi = D(\phi) > 0$ for $\phi > 0$, the inverse function, denoted by $\phi = q(w)$, exists and is an increasing function of $w > 0$. In view of $\nabla w = D(\phi)\nabla\phi$, problem (1.1) is equivalent to

$$-\nabla \cdot (a\nabla w) + \mathbf{c} \cdot \nabla w = \lambda q(w) \quad \text{in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \tag{3.4}$$

It is easy to see that if $\tilde{\phi}$ and $\hat{\phi}$ are a pair of ordered upper and lower solutions of (1.1), then the pair $\widetilde{w} = I(\tilde{\phi})$, $\widehat{w} = I(\hat{\phi})$ are ordered upper and lower solutions of (3.4). Since $q(w)$ is a monotone nondecreasing function of w for $w \geq 0$ (but not necessarily Lipschitz continuous) the well-known existence theorem for elliptic boundary value problems ensures that problem (3.4) has a maximal solution $\overline{w}$ and a minimal solution $\underline{w}$ such that $\widehat{w} \leq \underline{w} \leq \overline{w} \leq \widetilde{w}$ (cf. [**1, 8**]). This implies that $\overline{\phi} = q(\overline{w})$ and $\underline{\phi} = q(\underline{w})$ are the maximal and minimal solutions of (1.1) in $\mathcal{S}$. Moreover, if $\overline{\phi} = \underline{\phi}$ ($\equiv \phi^*$), then ϕ^* is the unique solution of (1.1) in $\mathcal{S}$. Hence, our goal is to find a pair of ordered upper and lower solutions of (1.1).

Proof of Theorem 1. We first seek a lower solution in the form $\hat{\phi} = q(\delta\psi)$ for a sufficiently small constant $\delta > 0$, where ψ is the (normalized) positive eigenfunction corresponding to the smallest eigenvalue μ_0 of (1.2). Indeed, from $I(\hat{\phi}) = \delta\psi$ and $\nabla(I(\hat{\phi})) = D(\hat{\phi})\nabla\hat{\phi}$ we see that $\hat{\phi}$ satisfies all the reversed inequalities in (3.1) if

$$-\nabla \cdot (a\nabla(\delta\psi)) + \mathbf{c} \cdot \nabla(\delta\psi) \leq \lambda q(\delta\psi) \text{ in } \Omega.$$

In view of (1.2) the above inequality is equivalent to

$$\mu_0(\delta\psi) \leq \lambda q(\delta\psi). \tag{3.5}$$

Since $D(0) = q(0) = 0$ and by the L'Hopital rule,

$$\lim_{\eta\to 0^+} \frac{q(\eta)}{\eta} = \lim_{\eta\to 0^+} q'(\eta) = \lim_{\phi\to 0^+} \frac{1}{D(\phi)} = \infty,$$

we see that given any $\lambda > 0$ there exists a $\delta_0 > 0$ such that $q(\delta\psi)/(\delta\psi) \geq \mu_0/\lambda$ for all $\delta \leq \delta_0$. With this choice of δ, $\hat{\phi} = q(\delta\psi)$ is a lower solution.

To find a positive upper solution $\tilde{\phi}$ we let μ_0' and ψ' be the smallest eigenvalues and its corresponding positive eigenfunction of (1.2) in a larger domain Ω' containing $\overline{\Omega}$, and we seek $\tilde{\phi}$ in the form $q(\rho\psi')$ for a sufficiently large $\rho > 0$. The consideration of Ω' containing $\overline{\Omega}$ ensures that ψ' is strictly positive in $\overline{\Omega}$. In view of $I(\tilde{\phi}) = \rho\psi'$ and $D(\tilde{\phi})\nabla\tilde{\phi} = \nabla(I(\tilde{\phi})) = \nabla(\rho\psi')$, $\tilde{\phi}$ satisfies all the inequalities in (3.1) if

$$-\nabla \cdot (a\nabla(\rho\psi')) + \mathbf{c} \cdot \nabla(\rho\psi') \geq \lambda q(\rho\psi') \text{ in } \Omega.$$

This leads to the requirement

$$\mu_0'(\rho\psi') \geq \lambda q(\rho\psi').$$

Since, by (H),

$$\lim_{\eta\to\infty} \frac{q(\eta)}{\eta} = \lim_{\eta\to\infty} q'(\eta) = \lim_{\phi\to\infty} \frac{1}{D(\phi)} = 0,$$

we see that for any $\lambda > 0$, there exists a $\rho_0 > 0$ such that

$$q(\rho\psi')/(\rho\psi') \leq \mu_0'/\lambda \quad \text{for all } \rho \geq \rho_0.$$

This shows that, for any $\rho \geq \rho_0$, $\tilde{\phi} = q(\rho\psi')$ is a positive upper solution. The ordering relation $\tilde{\phi} \geq \hat{\phi}$ follows by taking either ρ large or δ small. The above construction implies that problem (1.1) has a maximal solution $\overline{\phi}$ and a minimal solution $\underline{\phi}$ such that

$$0 < q(\delta\psi) \leq \underline{\phi} \leq \overline{\phi} \leq q(\rho\psi'), \quad x \in \Omega. \tag{3.6}$$

To show the uniqueness of the solution, we observe from the hypothesis $\vec{c} = 0$ that the functions $\overline{w} = I(\overline{\phi})$ and $\underline{w} = I(\underline{\phi})$ satisfy the equations

$$-\nabla \cdot (a\nabla\overline{w}) = \lambda\overline{\phi} \text{ in } \Omega, \quad \overline{w} = 0 \text{ on } \partial\Omega$$
$$-\nabla \cdot (a\nabla\underline{w}) = \lambda\underline{\phi} \text{ in } \Omega, \quad \underline{w} = 0 \text{ on } \partial\Omega.$$

Multiplication of the first equation by $\underline{w}$, the second equation by $\overline{w}$, subtraction and followed by integration over Ω, yield

$$\int_\Omega [\underline{w}\nabla \cdot (a\nabla\overline{w}) - \overline{w}\nabla \cdot (a\nabla\underline{w})]\, dx = \lambda \int_\Omega (\overline{w}\underline{\phi} - \underline{w}\overline{\phi})\, dx.$$

By Green's theorem, we obtain

$$0 = \lambda \int_\Omega [\overline{w}\underline{\phi} - \underline{w}\overline{\phi}]\, dx = \lambda \int_\Omega \overline{\phi}\underline{\phi}\Big(\frac{I(\overline{\phi})}{\overline{\phi}} - \frac{I(\underline{\phi})}{\underline{\phi}}\Big)\, dx. \tag{3.7}$$

Since

$$\begin{aligned}\frac{d}{d\phi}\Big(\frac{I(\phi)}{\phi}\Big) &= \phi^{-2}[\phi I'(\phi) - I(\phi)] = \phi^{-2}\Big[\phi D(\phi) - \int_0^\phi D(s)\, ds\Big] \\ &= \phi^{-2}\Big[\int_0^\phi (D(\phi) - D(s))\, ds\Big],\end{aligned}$$

the increasing or decreasing property of $D(\phi)$ implies that $I(\phi)/\phi$ is a strictly increasing or strictly decreasing function of ϕ. It follows from the positive property of $\overline{\phi}\underline{\phi}$ that relation (3.7) can hold only if $\overline{\phi} = \underline{\phi}$. Since $\delta > 0$ can be chosen arbitrarily small and ρ arbitrarily large, we conclude that $\overline{\phi}$ (or $\underline{\phi}$) is the unique positive solution. This completes the proof of the theorem. □

Proof of Theorem 2. It is seen from the proof of Theorem 1 that $\tilde{\phi} = q(\rho\psi')$ remains to be an upper solution. Moreover, $\widehat{\phi} = q(\delta\psi)$ is a lower solution if relation (3.5) holds for some $\delta > 0$. Since, by the mean-value theorem and $q(0) = 0$,

$$q(\delta\phi) = q'(\xi)(\delta\phi) = (\delta\phi)/D(\eta),$$

where ξ is an intermediate value between 0 and $\delta\phi$ and $\eta = q(\xi)$, we see that (3.5) holds if $\mu_0 \le \lambda/D(\eta)$. In view of $D(\eta) \to D(0)$ as $\eta \to 0$ this requirement is clearly fulfilled by a sufficiently small $\delta > 0$ when $\lambda > \mu_0 D(0)$. With this choice of δ, $\widehat{\phi} = q(\delta\phi)$ is a lower solution which ensures the existence of a positive solution. The uniqueness of the positive solution follows from the same proof as that in Theorem 1. □

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Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205
Email address: cvpao@math.ncsu.edu