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SOME PROPERTIES OF BASE-MATROIDS OF ARBITRARY CARDINALITY

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ABSTRACT. On a set of arbitrary cardinality, this paper introduces a new construction which is called the base-matroid of arbitrary cardinality. After presenting the base axiom system for a matroid of arbitrary cardinality, it discusses the new construction with the base axiom system and obtains that the new construction is a matroid of arbitrary cardinality. Afterwards, with the assistance of both lattice and matroid theories, it discusses the lattice construction of closed saturated sets relative to a simple matroid of arbitrary cardinality.

1. Introduction and preliminaries. In this paper we will use the techniques of finite base-matroid for reference to produce a base-matroid of arbitrary cardinality. Initially, as a test for what can be achieved by using this approach, we will study the base axioms for a base-matroid of arbitrary cardinality. Welsh in [**9**] and Novetti and White in [**7**] have identified the base axioms for a finite matroid. This paper, however, is the first to show the base axioms for a matroid of arbitrary cardinality.

Mao in [**5**] presents a method to consider the relationship between a geometric lattice and the family of closed sets of a simple matroid of arbitrary-cardinality. Additionally, [**4**] discusses the construction of a Boolean lattice of closed saturated sets relative to a simple matroid of arbitrary cardinality. Combining the ideas of [**4, 5**] this paper also shows some results about the base-matroid of arbitrary cardinality for a simple matroid of arbitrary cardinality.

We will begin by reviewing and presenting the knowledge needed to continue. In what follows, E is assumed to be some arbitrary–possibly infinite–set. For $X \subseteq E$, $|X|$ denotes the cardinality of X.

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Definition 1 [**1**]. Assume $m \in \mathbf{N}_0$ and $\mathcal{F} \subseteq \mathcal{P}(E)$. Then the pair $M := (E, \mathcal{F})$ is called a *matroid of rank* m *with* $\mathcal{F}$ *as its closed sets*, if the following axioms hold:

(F1) $E \in \mathcal{F}$;

(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$;

(F3) Assume $F_0 \in \mathcal{F}$ and $x_1, x_2 \in E \setminus F_0$. Then one has either $\{F \in \mathcal{F} : F_0 \cup x_1 \subseteq F\} = \{F \in \mathcal{F} : F_0 \cup x_2 \subseteq F\}$ or $F_1 \cap F_2 = F_0$ for certain $F_1, F_2 \in \mathcal{F}$ containing $F_0 \cup x_1$ or $F_0 \cup x_2$, respectively;

(F4) $m = \max\{n \in \mathbf{N}_0$: there exist $F_0, F_1, \ldots, F_n \in \mathcal{F}$ with $F_0 \subset F_1 \subset \cdots \subset F_n = E\}$.

The *closure operator* $\sigma : \mathcal{P}(E) \to \mathcal{F}$ of M is defined by $\sigma(A) := \cap_{F \in \mathcal{F}, A \subseteq F} F$. The *rank function* $\rho : \mathcal{P}(E) \to \{0, 1, \ldots, m\}$ of M is defined by $\rho(A) := \max\{k \in \mathbf{N}_0$: there exist $F_0, F_1, \ldots, F_k \in \mathcal{F}$ with $F_0 \subset F_1 \subset \cdots \subset F_k = \sigma(A)\}$.

M is called *simple*, if any subset $A \subseteq E$ with $|A| \leq 1$ lies in $\mathcal{F}$.

One calls $Y \in \mathcal{I} = \{A \subseteq E : x \in A, x \notin \sigma(A \setminus \{x\})\}$ an *independent set* of M, where M is defined as in (1) and σ is the closure operator of M [**6**].

In this paper, $M = (E, \mathcal{F})$ defined in Definition 1, is called a *matroid of arbitrary cardinality* and simply denoted by M. A *basis*(or *base*) of M is a maximal independent set. $\mathcal{I}$ always denotes the collection of independent sets of M. $\mathcal{F}, \sigma$ and ρ denote the set of closed sets of M, closure operator of M and rank function of M, respectively.

Lemma 1 [**1**]. *M has the following properties.*

(1) *For any family $(F_i)_{i \in I}$ of closed sets in M, one has also $F := \cap_{i \in I} F_i \in \mathcal{F}$.*

(2) *For any $A \subseteq E$, the set $\sigma(A)$ is the smallest set in $\mathcal{F}$ containing A. In particular, $\sigma(A) = A$ if and only if $A \in \mathcal{F}$.*

Moreover, σ satisfies the following conditions, which characterize a closure operator:

$$A \subseteq \sigma(A) = \sigma(\sigma(A)) \text{ for all } A \subseteq E;$$

$$\text{for } A \subseteq Y \subseteq E, \text{ one has } \sigma(A) \subseteq \sigma(Y);$$

(F3) *implies either* $\sigma(A \cup x) = \sigma(A \cup y)$ *or* $\sigma(A \cup x) \cap \sigma(A \cup y) = \sigma(A)$ *as claimed.*

(3) $\rho(E) = m$ *equals the rank of* M.

Lemma 2 [**6**]. *A collection* $\mathcal{I}$ *of subsets of* E *is the set of independent sets of a matroid of arbitrary cardinality on* E *if and only if* $\mathcal{I}$ *satisfies the following conditions*:

(i1) $\mathcal{I} \neq \varnothing$;

(i2) *If* $A \in \mathcal{I}$ *and* $Y \subseteq A$, *then* $Y \in \mathcal{I}$;

(i3) *If* $A, Y \in \mathcal{I}$ *and* $|A|, |Y| < \infty$ *with* $|A| = |Y| + 1$, *then there exists an* $a \in A \setminus Y$ *which fits* $Y \cup a \in \mathcal{I}$;

(i4) *If* $A \subseteq E$ *and every finite subset of* A *is a member of* $\mathcal{I}$, *then* $A \in \mathcal{I}$;

(i5) $\max\{k \in \mathbf{N}_0$: *there exist* $I_0, I_1, \ldots, I_k \in \mathcal{I}$ *such that* $I_0 \subset I_1 \subset \cdots \subset I_k\} < \infty$.

Let $\mathcal{I}$ *be the collection of independent sets of* M. *Then for any* $I \in \mathcal{I}, |I| < \infty$ [**6**].

Let $I \subseteq X \subseteq E$ *and* $I \in \mathcal{I}$ *be a maximal one of* $\mathcal{I}$ *in* X. *Then* $\sigma(I) = \sigma(X)$.

Let $I_1, I_2 \in \mathcal{I}$ *with* $I_1 \subset I_2$. *Then* $\sigma(I_1) \subset \sigma(I_2)$.

A family $\mathcal{B}$ *of subsets of* E *is the family of bases of an independence space if and only if the following axioms hold* [**8**]:

(b1) $\mathcal{B} \neq \varnothing$;

(b2) $\mathcal{B}$ *is an antichain in* E, *say* $B_1 \nsubseteq B_2$ *for any* $B_1, B_2 \in \mathcal{B}$ *and* $B_1 \neq B_2$;

(b3) *For every* $X, Y \subseteq E, X \subseteq Y$, *if there exist* $B_1, B_2 \in \mathcal{B}$ *such that* $X \subseteq B_1$ *and* $B_2 \subseteq Y$, *then there exists* $B_3 \in \mathcal{B}$ *such that* $X \subseteq B_3 \subseteq Y$;

(b4) *If* X *is not contained in a basis, then some finite subset of* X *is not contained in a basis.*

If B_1 *and* B_2 *are bases of an independence space, then* $|B_1| = |B_2|$ [**8**].

Remark 1. (1) We refer to Oxley [**8**] for fundamentals of independence spaces and Gräzter [**3**] for that of lattice theory.

(2) According to the definition of independence space (cf. [**8**, page 74]) and Lemma 2, one has that a matroid of arbitrary cardinality on E is an independence space on E.

(3) By Lemma 2, the collection $\mathcal{I}$ of independent sets of a matroid of arbitrary cardinality M determines M uniquely. Therefore, in this paper, sometimes M is denoted by $(E, \mathcal{I})$.

(4) By [**6**, Corollary 2], we have: let $X \subseteq E$ and $I_X \subseteq X$ be a maximal independent set in X; then $\rho(X) = \rho(I_X)$:

$$\rho(Y) < \infty \text{ for all } Y \subseteq E; \qquad I \in \mathcal{I} \Longleftrightarrow \rho(I) = |I|.$$

(5) By Definition 1, Lemma 1 and Lemma 2, we see that for M, $\sigma(X)$ is the set by adding to $X \subseteq E$ all elements $e \in E$ such that $\rho(X \cup e) = \rho(X)$. Moreover, a set $X \subseteq E$ is closed if $\rho(X \cup e) = \rho(X) + 1$ for all $e \in E \setminus X$.

Similarly to [**4**, Definition 1] and [**2**, Definition 1], we give the relative definition for a matroid of arbitrary cardinality as follows.

Definition 2. A set $\theta \subseteq E$ is called *saturated with respect to a base* B of M, or B-*saturated*, if $|\theta \cap B| = \rho(\theta)$.

Remark 2. (1) Any B-saturated closed set θ satisfies the relation $\sigma(\theta \cap B) = \theta$. This is because Lemma 2 and $|\theta \cap B| = \rho(\theta)$ implies that $\theta \cap B$ is a basis of $\theta = \sigma(\theta)$, and finally, it follows $\sigma(\theta \cap B) = \sigma(\theta) = \theta$.

(2) We simply call θ *saturated* when it is clear from the context which base is considered. $\mathcal{F}_B$ denotes the family of all the closed sets of M, saturated with respect to a base B. We also say that $\mathcal{F}_B$ is relative to M.

(3) Let $M_B = (E, \mathcal{I}_B = \{X \subseteq E : |X \cap \theta| \leq \rho(\theta), \text{ for all } \theta \in \mathcal{F}_B\})$.

The aim of this paper is to prove that M_B is a matroid of arbitrary cardinality with $\mathcal{I}_B$ as its set of independent sets; additionally, when M is simple, to discuss the relation between M_B and a Boolean lattice.

2. Base axioms. In this section, we demonstrate a proof that M_B is a matroid of arbitrary cardinality. This approach is different from the way that is used directly to check that $\mathcal{I}_B$ satisfies (i1)–(i5). Firstly, we present the base axioms for a matroid of arbitrary cardinality.

Theorem 1. *$\mathcal{B} \subseteq \mathcal{P}(E)$ is the set of bases of a matroid of arbitrary cardinality on E if and only if $\mathcal{B}$ satisfies* (b1)–(b4) *and the condition* (b5): $|Y| < \infty$ *for every* $Y \in \mathcal{B}$.

Proof. Suppose $\mathcal{B}$ is the family of bases of a matroid M of arbitrary cardinality on E with $\mathcal{I}$ as its independent sets. Then by Remark 2, M is an independence space on E. So by Lemma 2, $\mathcal{B}$ satisfies (b1)–(b4).

Since every basis of M is an independent set of M. Combining this reason with Lemma 2, (b5) is obviously correct.

Conversely, because Lemma 2 and $\mathcal{B}$ satisfy (b1)–(b4), one has that $\mathcal{B}$ is the family of bases of an independence space M^I on E. So by Lemma 2, $|B_1| = |B_2|$ for all $B_1, B_2 \in \mathcal{B}$.

Let $\mathcal{I} = \{X \subseteq E : X \subseteq Y \text{ for some } Y \in \mathcal{B}\}$. By (b5) and the above, one has $|I| < \infty$ for $I \in \mathcal{I}$. Also by (b5) and the maximality of $B \in \mathcal{B}$, one finds that (i5) holds.

We will check whether $\mathcal{I}$ satisfies (i1)–(i4) as follows.

From the definition of $\mathcal{I}$, $\mathcal{B} \subseteq \mathcal{I}$ is derived. Considered this result with (b1), one has $\mathcal{I} \neq \varnothing$. So (i1) is satisfied by $\mathcal{I}$.

If $A \in \mathcal{I}$, then $D \subseteq A$. Because $A \in \mathcal{I}$ shows that there is a $Y_A \in \mathcal{B}$ satisfying $A \subseteq Y_A$, one gets $D \subseteq A \subseteq Y_A$, and so $D \in A$. Thus, (i2) is correct.

If $Y, X \in \mathcal{I}$ with $|Y| = |X| + 1$, then there is a $B_Y, B_X \in \mathcal{B}$ satisfying $X \subseteq B_X$ and $Y \subseteq B_Y$. In view of (b5) and Lemma 2, $|B_X| = |B_Y| < \infty$ holds. Distinguishing two cases will finish the proof of the correctness of (i3).

Case 1. When $B_Y = B_X$. It is obvious that $X \cup y \subseteq X \cup Y \subseteq B_X \in \mathcal{B}$ for any $y \in Y \setminus X$. So $X \cup y \in \mathcal{I}$.

Case 2. When $B_Y \neq B_X$. Similarly to the proof in [**7**, page 30,Proposition 2.1.1], one gets that (b3) is equivalent to

(b3.1): For all $B_1, B_2 \in \mathcal{B}$ and for all $b_1 \in B_1$, there exists a $b_2 \in B_2$ satisfying $(B_1 \setminus b_1) \cup b_2 \in \mathcal{B}$.

Let $X = \{x_1, \dots, x_n\}$, $B_X = \{x_1, \dots, x_n, b_1, \dots, b_q\}$, $Y = \{y_1, \dots y_m\}$, $B_Y = Y \cup \{a_1, \dots, a_s\}$. By Lemma 2 and (b5), one has $n + q = m + s$, and so $s < q$.

In view of (b3.1), for $B_X \setminus b_q$, there exists a $z \in B_Y$ satisfying $(B_X \setminus b_q) \cup z \in \mathcal{B}$.

If $z \in Y$, then $X \cup y \in \mathcal{I}$, and hence (i3) holds. Otherwise, put $((B_x \setminus b_q) \cup z) \setminus b_{q-1} = B'_X$. According to (b3.1), there is a $z_1 \in B_Y$ satisfying $B'_X \cup z_1 \in \mathcal{B}$.

If $z_1 \in Y$, then (i3) is correct. Otherwise, repeat the process above for $(B'_X \cup z_1) \setminus b_{q-2}$ and so on. Because $s < t < \infty$, after at most $s+1$ steps, one can use an element in Y rather than b_i, and hence (i3) is satisfied by $\mathcal{I}$.

Next we prove that (i4) holds to $\mathcal{I}$.

Suppose $A \subseteq E$ and every finite subset of A is a member of $\mathcal{I}$. If $A \notin \mathcal{I}$, this means that for any $B \in \mathcal{B}$, $A \nsubseteq B$ is correct. Using (b4), some finite subset of A is not contained in a basis. This is a contradiction to the supposition. Hence, (i4) holds for $\mathcal{I}$.

Therefore, $\mathcal{I}$ is the collection of independent sets of some matroid $M_{\mathcal{I}}$ of arbitrary cardinality. Evidently, $\mathcal{B}$ is the set of bases of $M_{\mathcal{I}}$. According to Lemma 2, one gets $M_{\mathcal{I}} = M^I$, and so $\mathcal{B}$ is the needed family of bases.

For discussion with the properties of M_B, we now present some preparations.

Lemma 3. *Let M_B be derived from M, $X \in \mathcal{I}_B$, $b \in E \setminus X$ and θ be a closed B-saturated set satisfying $|(X \cup b) \cap \theta| > \rho(\theta)$. Then $b \in \theta$, $|X \cap \theta| = \rho(\theta)$ and $|(X \cup b) \cap \theta| = \rho(\theta) + 1$.*

Proof. Since E is a closed set of M and $|E \cap B| = |B| = \rho(B) = \rho(E)$, one has $E \in \mathcal{F}_B$. Let $X \in \mathcal{I}_B$. Then $|X \cap \theta| \leq \rho(\theta)$ (for all $\theta \in \mathcal{F}_B$), especially, $|X| = |X \cap E| \leq \rho(E) = |B| < \infty$. Hence, every element in $\mathcal{I}_B$ is finite.

$X \in \mathcal{I}_B$ and $\theta \in \mathcal{F}_B$ together shows us $|X \cap \theta| \leq \rho(\theta) < \infty$. In addition, $(X \cup b) \cap \theta = (X \cap \theta) \cup (b \cap \theta)$ tells us $|(X \cup b) \cap \theta| \leq |X \cap \theta| + |b \cap \theta| \leq \rho(\theta) + 1$. Furthermore, by the given $|(X \cup b) \cap \theta| > \rho(\theta)$ and the above, we get $|(X \cup b) \cap \theta| = |X \cap \theta| + |b \cap \theta| = \rho(\theta) + 1$. If $b \notin \theta$, then $|b \cap \theta| = 0$. This induces $|X \cap \theta| = \rho(\theta) + 1$, a contradiction to $|X \cap \theta| \leq \rho(\theta)$. Thus, $b \in \theta$, and so $|X \cap \theta| = \rho(\theta)$.

Corollary 1. *Let $X \in \mathcal{I}_B$, $\{b_1, b_2, \ldots, b_s\} \subseteq E \setminus X$ and θ be a closed B-saturated set satisfying $\rho(\theta) < |(X \cup b_i) \cap \theta|$, $1 \leq i \leq s < \infty$. Then $|(X \cup \{b_1, b_2, \ldots, b_s\}) \cap \theta| = \rho(\theta) + s$.*

Proof. $X \in \mathcal{I}_B$ and $\theta \in \mathcal{F}_B$ together shows us $|X \cap \theta| \leq \rho(\theta) < \infty$. By Lemma 3, $b_i \in \theta$ and $|X \cap \theta| = \rho(\theta)$, $i = 1, 2, \ldots, s$. Besides, $(X \cup \{b_1, b_2, \ldots, b_s\}) \cap \theta = (X \cap \theta) \cup (\{b_1, b_2, \ldots, b_s\} \cap \theta)$. All these results together yield $|(X \cup \{b_1, b_2, \ldots, b_s\}) \cap \theta| = |X \cap \theta| + |\{b_1, b_2, \ldots, b_s\} \cap \theta| = \rho(\theta) + s$. □

Let $\mathcal{B}_B$ denote the set of elements in $\mathcal{I}_B$ having maximal cardinality.

Lemma 4. *$\mathcal{B}_B$ satisfies the following properties.*

(1) *$\mathcal{B}_B \neq \varnothing$.*

(2) *$\mathcal{B}_B$ is an antichain in E.*

(3) *If $X, Y \in \mathcal{B}_B$ and $x \in X \setminus Y$, then there exists a $y \in Y \setminus X$ satisfying $(X \cup y) \setminus x \in \mathcal{B}_B$.*

(4) *$\mathcal{B}_B$ satisfies* (b3).

(5) (b4) *is correct for $\mathcal{B}_B$.*

Proof. (1) By the proof of Lemma 3, $|X| \leq \rho(B) = |B|$ for all $X \in \mathcal{I}_B$. On the other hand, $|B \cap \theta| = \rho(\theta)$ for every $\theta \in \mathcal{F}_B$ implies $B \in \mathcal{I}_B$, and hence, $B \in \mathcal{B}_B$, i.e., $\mathcal{B}_B \neq \varnothing$.

(2) In light of the definition of $\mathcal{B}_B$, the needed result is evidently correct.

(3) Let $X, Y \in \mathcal{B}_B, X \neq Y$ and $x \in X \setminus Y$. Then by (1), $|X|, |Y| \leq |B| < \infty$. Thus, both $X \setminus Y$ and $Y \setminus X$ are not empty. Let $Y \setminus X = \{b_1, b_2, \ldots, b_s\}$. Then $|Y \setminus X| \leq |Y| < \infty$, besides, $s < \infty$. Henceforth, similarly to the proof of [**4**, Theorem 2], the needed consequence follows.

(4) A consequence of (3) above (or similarly to the discussion in [**7**, page 30, Proposition 2.1.1]) is that (b3) is correct for $\mathcal{B}_B$.

(5) Let $X \subseteq E$ and $X \nsubseteq A$ for any $A \in \mathcal{B}_B$. Suppose for any finite subset $Y \subseteq X$, there exists a $D \in \mathcal{B}_B$ such that $Y \subseteq D$. Then $Y \in \mathcal{I}_B$ is evident.

$X \nsubseteq A$ hints $|X \cap \theta_A| \nleq \rho(\theta_A)$ for some $\theta_A \in \mathcal{F}_B$, i.e., $|X \cap \theta_A| > \rho(\theta_A)$, and so $|A \cap \theta_A| < |X \cap \theta_A|$. Especially, $B \in \mathcal{B}_B$ follows $|B \cap \theta_B| < |X \cap \theta_B|$. Set $S = X \cap \theta_B$. We can identify two cases to discuss the properties of S.

Case 1. $n = |B| < |S|$. Let $Z \subseteq S$ and $|Z| = |B| + 1 = n + 1$. Then $Z \subseteq X \cap \theta_B \subseteq X$, and additionally, for any $D \in \mathcal{B}_B, |D| \leq |B| < n + 1 = |Z|$. This implies that Z, and further S, is not contained in any element in $\mathcal{B}_B$, a contradiction to the supposition.

Case 2. $|S| \leq |B| = n$. $|S| \leq n < \infty$ means that S is a finite subset of X, and so there exists $D_S \in \mathcal{B}_B$ such that $S \subseteq D_S$. It follows that $|S \cap \theta| \leq |D_S \cap \theta| \leq \rho(\theta) = |B \cap \theta|$ for every $\theta \in \mathcal{F}_B$, i.e., $|S| = |S \cap \theta_B| \leq |B \cap \theta_B| < |X \cap \theta_B| = |S|$ by $S = (X \cap \theta_B) \cap \theta_B = S \cap \theta_B$, a contradiction.

The above two cases show us that some finite subset of X is not contained in a basis.

Theorem 2. *M_B is a matroid of arbitrary cardinality with $\mathcal{I}_B$ as its collection of independent sets. We call M_B the* base-matroid of arbitrary cardinality *induced by base B.*

Proof. Routine verification by Theorem 1, Lemma 2 and Lemma 4. □

We notice that our discussion for Theorem 2 is based on Theorem 1.

By the definition of a finite matroid (cf. [**9**]) or Theorem 1 and [**7**], it is easily seen that when E is finite, a matroid of arbitrary cardinality on E is a finite matroid on E. Additionally, from the base axioms of a finite matroid (cf. [**7**, page 30]), we see that for $|E| < \infty, \mathcal{B} \subseteq \mathcal{P}(E)$ is the set of bases of a finite matroid on E if and only if it satisfies (b1)–(b3). Hence, all the discussions here are generalizations of that of the finite cases. In addition, it is not surprising that per the generalizations of finite cases, some proofs here are quite similar to that of finite cases. Although [**2, 4**] refer to finite cases, the discussions in this paper are correct for both finite and infinite.

3. Lattice of closed saturated sets. The lattice construction of $(\mathcal{F}_B, \subseteq)$ is presented in this section, where $\mathcal{F}_B$ is relative to M' a simple matroid of arbitrary cardinality on E. In addition, the relation between M' and $\mathcal{F}_B$ will be more evident by the results presented here. But we should notice that both Lemma 5 and Lemma 6 are correct not only for a simple matroid of arbitrary cardinality, but also for a nonsimple matroid of arbitrary cardinality. Beyond Lemma 6, below, all the discussions are only for simple matroids of arbitrary cardinality.

Lemma 5. (1) $\rho(\varnothing) = 0$.

(2) $X \subseteq Y \Rightarrow \rho(X) \leq \rho(Y)$.

(3) $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$.

Proof. By Definition 1, Lemma 1 and Lemma 2, it is obvious that the rank function ρ of M satisfies (1) and (2).

Suppose that $\rho(X \cup Y) = t$ and $\rho(X \cap Y) = s$. Let $I_{X\cap Y}$ be an independent subset of $X \cap Y$ with $|I_{X\cap Y}| = s$. By Lemma 2, $s, t < \infty$ are correct, and so considered with (i3) and Remark 2, there exists A with $I_{X\cap Y} \subseteq A$ such that $A \subseteq X \cup Y, |A| = t$, and A is independent in M. Thus, posit $A = I_{X\cap Y} \cup V \cup W$ where $I_{X\cap Y}, V, W$ are pairwise disjoint and where $V \subseteq X \setminus Y$ and $W \subseteq Y \setminus X$. Then, in view of (i2), $I_{X\cap Y} \cup V$ is an independent subset of X and $I_{X\cap Y} \cup W$ is an independent subset of Y. Hence, $\rho(X) + \rho(Y) \geq |I_{X\cap Y} \cup V| + |I_{X\cap Y} \cup W| = 2|I_{X\cap Y}| + |V| + |W| = |A| + |I_{X\cap Y}| = \rho(X \cup Y) + \rho(X \cap Y)$. □

Lemma 6. *Let $\theta_1, \theta_2, \theta_\alpha, (\alpha \in \mathcal{A})$ be B-saturated sets. Then*

(1) $\theta_1 \cup \theta_2$ is a B-saturated set.

(2) $\theta_1 \cap \theta_2$ is a B-saturated set.

(3) Both $\cap_{\alpha\in\mathcal{A}}\theta_\alpha$ and $\cup_{\alpha\in\mathcal{A}}\theta_\alpha$ are B-saturated. $\cap_{\alpha\in\mathcal{A}}\theta_\alpha \in \mathcal{F}_B$ when $\theta_\alpha \in \mathcal{F}_B$ for all $\alpha \in \mathcal{A}$.

Proof. (1) Since $|\theta_i \cap B| = \rho(\theta_i)$, (i2) and Remark 1 when taken together show us that $\theta_i \cap B$ is a maximal independent set in θ_i, $i = 1, 2$. In view of $(\theta_1 \cup \theta_2) \cap B = (\theta_1 \cap B) \cup (\theta_2 \cap B)$ and (i2), it follows that $(\theta_1 \cup \theta_2) \cap B$ is an independent set of M.

We assert that $(\theta_1 \cap B) \cup (\theta_2 \cap B)$ is a basis in $\theta_1 \cup \theta_2$.

Otherwise, by (i3) and Theorem 1, there exists an $x \in (\theta_1 \cup \theta_2) \setminus ((\theta_1 \cap B) \cup (\theta_2 \cap B))$ such that $((\theta_1 \cup \theta_2) \cap B) \cup x \in \mathcal{I}$. We may also suppose $x \in \theta_1 \setminus \theta_2$. Considered with (i2), $(\theta_1 \cap B) \cup x \in \mathcal{I}$; thus, a contradiction to the maximality of $\theta_1 \cap B \in \mathcal{I}$ in θ_1.

Hence, $|(\theta_1 \cup \theta_2) \cap B| = \rho(\theta_1 \cup \theta_2)$. Namely, $\theta_1 \cup \theta_2$ is B-saturated.

(2) In light of (i2), $(\theta_1 \cap \theta_2) \cap B \in \mathcal{I}$. So $|(\theta_1 \cap \theta_2) \cap B| = \rho((\theta_1 \cap \theta_2) \cap B) = \rho(\theta_1 \cap \theta_2)$. By Lemma 5, $\rho(\theta_1 \cap \theta_2) + \rho(\theta_1 \cup \theta_2) \le \rho(\theta_1) + \rho(\theta_2) = |\theta_1 \cap B| + |\theta_2 \cap B|$. In addition, by (1), $\rho(\theta_1 \cap \theta_2) + \rho(\theta_1 \cup \theta_2) = \rho(\theta_1 \cap \theta_2) + |(\theta_1 \cup \theta_2) \cap B| = \rho(\theta_1 \cap \theta_2) + |(\theta_1 \cap B) \cup (\theta_2 \cap B)| = \rho(\theta_1 \cap \theta_2) + |\theta_1 \cap B| + |(\theta_2 \setminus \theta_1) \cap B| = \rho(\theta_1 \cap \theta_2) + \rho(\theta_1) + |(\theta_2 \setminus \theta_1) \cap B|$, and hence, $\rho(\theta_1 \cap \theta_2) + |(\theta_2 \setminus \theta_1) \cap B| \le |\theta_2 \cap B| = |(\theta_2 \setminus \theta_1) \cap B| + |(\theta_1 \cap \theta_2) \cap B|$; further, $\rho(\theta_1 \cap \theta_2) \le |(\theta_1 \cap \theta_2) \cap B|$. Moreover, $\rho(\theta_1 \cap \theta_2) = |(\theta_1 \cap \theta_2) \cap B|$.

Hence, $\theta_1 \cap \theta_2$ is B-saturated.

(3) $(\cap_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B \subseteq B$ and (i2) together hints $(\cap_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B \in \mathcal{I}$, i.e., $\rho(\cap_{\alpha\in\mathcal{A}}\theta_\alpha) \ge |(\cap_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B|$. Besides, $|(\cap_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B| \le |B| < \infty$ shows that $(\cap_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B = \{b_1, b_2, \dots, b_t\}$ for some $t \in \mathbf{N}_0$. That is to say, there exists a finite subset $\mathcal{J} \subseteq \mathcal{A}$ satisfying $(\cap_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B = (\cap_{j=1}^{|\mathcal{J}|}\theta_j) \cap B$. By induction and (2), $\cap_{j=1}^{|\mathcal{J}|}\theta_j$ is B-saturated. Since $\mathcal{J} \subseteq \mathcal{A}$ tells us that $\cap_{\alpha\in\mathcal{A}}\theta_\alpha \subseteq \cap_{j=1}^{|\mathcal{J}|}\theta_j$. Considering $(\cap_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B, (\cap_{j=1}^{|\mathcal{J}|}\theta_j) \cap B \in \mathcal{I}$ and $(\cap_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B = (\cap_{j=1}^{|\mathcal{J}|}\theta_j) \cap B$ with $|(\cap_{j=1}^{|\mathcal{J}|}\theta_j) \cap B| = \rho(\cap_{j=1}^{|\mathcal{J}|}\theta_j)$, one has that $(\cap_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B$ is a basis of $\cap_{\alpha\in\mathcal{A}}\theta_\alpha$, i.e., $|(\cap_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B| = \rho(\cap_{\alpha\in\mathcal{A}}\theta_\alpha)$. Furthermore, $\cap_{\alpha\in\mathcal{A}}\theta_\alpha$ is B-saturated.

By Lemma 1, $\cap_{\alpha\in\mathcal{A}}\theta_\alpha \in \mathcal{F}$ when $\theta_\alpha \in \mathcal{F}$ for all $\alpha \in \mathcal{A}$, and hence, $\cap_{\alpha\in\mathcal{A}}\theta_\alpha \in \mathcal{F}_B$.

According to (i2) and $(\cup_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B \subseteq B$, it follows that $(\cup_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B \in \mathcal{I}$.

We claim that $(\cup_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B$ is a basis of $\cup_{\alpha\in\mathcal{A}}\theta_\alpha$.

Otherwise, there exists an $x \in (\cup_{\alpha\in\mathcal{A}}\theta_\alpha) \setminus ((\cup_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B)$ satisfying $((\cup_{\alpha\in\mathcal{A}}\theta_\alpha)\cap B)\cup x \in \mathcal{I}$. We may also suppose $x \in \theta_{\alpha_1}$ for some $\alpha_1 \in \mathcal{A}$. This implies $(\theta_{\alpha_1} \cap B) \cup x \in \mathcal{I}$ by (i3) and $(\theta_{\alpha_1} \cap B) \cup x \subseteq \theta_{\alpha_1}$; thus, a contradiction to $|\theta_{\alpha_1} \cap B| = \rho(\theta_{\alpha_1})$.

Hence, $|(\cup_{\alpha\in\mathcal{A}}\theta_\alpha) \cap B| = \rho(\cup_{\alpha\in\mathcal{A}}\theta_\alpha)$.

In what follows, let M' be a simple matroid of arbitrary cardinality on E. Next we will construct the lattice of $\mathcal{F}_B$ which is relative to M'. Let $X \subseteq E$; we denote by $\mathrm{sc}\,(X)$ and call *B-saturated closed set generated by X* the minimum B-saturated closed set which contains X. By Lemma 6 and $X \subseteq E \in \mathcal{F}_B$, $\mathrm{sc}\,(X)$ is well defined. Clearly, $\mathrm{sc}\,(\varnothing) = \varnothing$, $\mathrm{sc}\,(E) = E$, $\varnothing = \min(\mathcal{F}_B, \subseteq)$, and $E = \max(\mathcal{F}_B, \subseteq)$. The atoms of $(\mathcal{F}_B, \subseteq)$ are elements of B. Moreover, let $\mathcal{L}(M')_B = (\mathcal{F}_B, \subseteq)$. Recall that a set P of $\mathcal{F}_B$ is defined as $\sigma(P\cap B)$; thus, P is characterized by the set $P \cap B$.

Let $|B| = n$ and $\mathbf{B}_n$ be the Boolean lattice of subsets of n-set, ordered by inclusion; denote by $\vee, \wedge$ the join and the meet in $\mathbf{B}_n$, respectively.

Lemma 7. (1) *If $P, Q \in \mathcal{F}_B$, then* $\mathrm{sc}\,(P\cup Q)\cap B = (P\cap B)\cup(Q\cap B)$.

(2) *Under the inclusion ordering $\subseteq$, $\mathcal{F}_B$ is a lattice.*

Proof. By Lemma 6, one has that $P \cap Q \in \mathcal{F}_B$ and $P \cup Q$ is B-saturated. But according to [**1**], $P \cup Q$ is not, in general, closed. Thus, $P \wedge Q = P \cap Q$, $P \vee Q = \mathrm{sc}\,(P \cup Q)$ for all $P, Q \in \mathcal{F}_B$. □

Theorem 3. $(\mathcal{F}_B, \subseteq)$ *is a Boolean lattice.*

Proof. Let $\psi : \mathcal{L}(M')_B \to \mathbf{B}_n$ be given by $\psi(P) = P \cap B$ where $P \in \mathcal{L}(M')_B$. Clearly, ψ is a bijection because P is characterized by $P \cap B$. In addition, $\psi(P \wedge Q) = \psi(P \cap Q) = (P \cap Q) \cap B = (P \cap B) \cap (Q\cap B) = \psi(P)\cap\psi(Q) = \psi(P)\wedge\psi(Q)$ and $\psi(P\vee Q) = \psi(sc(P\cup Q)) =$

$sc(P \cup Q) \cap B = (P \cap B) \cup (Q \cap B) = \psi(P) \cup \psi(Q) = \psi(P) \vee \psi(Q)$. Therefore, ψ is a bijection which preserves $\vee$ and $\wedge$, and so it is an isomorphism. □

Remark 3. (1) Since $\theta = \sigma(\theta \cap B)$ is valid for every $\theta \in \mathcal{F}_B$, by Definition 1 and Lemma 2 θ is uniquely defined by $\theta \cap B$. Besides, $|\{X : X \subseteq B\}| < \infty$. Thus, $\mathcal{F}_B$ is a finite set though an element in $\mathcal{F}_B$ is perhaps not finite. So $(\mathcal{F}_B, \vee, \wedge)$, i.e., $\mathcal{L}(M')_B$, is defined as a finite lattice.

Certainly, we also get the same information from Theorem 3.

(2) Actually, by the definition of a finite matroid in [**9**, page 7] and Lemma 2, one has that

(ξ) every matroid of arbitrary cardinality on E is a finite matroid when E is finite.

Additionally,

(α) it is well known that a geometric lattice is not always a Boolean one.

(β) [**9**, page 54, Theorem 2] verifies that the corresponding relation between a finite geometric lattices and a finite simple matroids;

(γ) (1) tells us $\mathcal{L}(M')_B$ is finite.

Hence, from (α)–(γ), we obtain that, up to isomorphism, there does not always exist a finite simple matroid which is isomorphic to $\mathcal{L}(M')_B$.

In addition, Mao in [**5**] points out the corresponding relation between a geometric lattice with finite height and a simple matroid of arbitrary cardinality. Considering this expression with (α) and the above result, we have that, up to isomorphism, there does not always exist a simple matroid of arbitrary cardinality which is isomorphic to $\mathcal{L}(M')_B$.

Therefore, by virtue of [**5**], generally $\mathcal{F}_B$ would not be the family of closed sets of M'_B.

(3) Because $\mathcal{L}(M')_B$ is finite and M' is perhaps infinite, one has that the properties of $\mathcal{L}(M')_B$ are easily obtained, but finding the properties of M' may not be so easy. Based on this, we firmly believe that $\mathcal{L}(M')_B$ could play an appropriate role in dealing with the properties of M'. For example, if the number of atoms of $\mathcal{L}(M')_B$ does not equal that

of $\mathcal{L}(M_1')_{B_1}$, then M' is not isomorphic to M_1', where M_1' is a simple matroid of arbitrary cardinality on E and B_1 is one of bases of M_1'.

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