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THE SMALLEST AMOUNT OF CHAOS

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ABSTRACT. In this paper we argue that the property that all nonapproximately periodic points form asymptotic pairs can be used to characterize those among chaotic dynamical systems having the least complicated behavior. Here, for a dynamical system (X, T, Φ), we say that a point $x \in X$ is *approximately periodic* if, for every $\varepsilon > 0$, there is a periodic point p such that $\limsup_{t\to\infty} \mathrm{d}(\Phi(t,x),\Phi(t,p)) < \varepsilon$, and we say that points $x, y \in X$ form an *asymptotic pair* if $\lim_{t\to\infty} \mathrm{d}(\Phi(t,x),\Phi(t,y)) = 0$. In support of this thesis we introduce a preorder in the class of properties of dynamical systems and show that, under certain conditions, this property is "smaller" than several other notions naturally related to the idea of chaotic behavior.

1. Introduction. Sometimes (see, e.g., [**5**, page 8]) a *dynamical system* (shortly, DS) is defined as a triplet (X, T, Φ) consisting of an additive semi-group T of $\mathbf{R}$ (the *set of times*), a set X (the *phase space*) and a map $\Phi : T \times X \to X$ (the *flow*) satisfying $\Phi(0, x) = x$ for every $x \in X$ and $\Phi(t, \Phi(s, x)) = \Phi(t+s, x)$ for every $t, s \in T$ and every $x \in X$. Depending on the point of view we are interested in, it is customary to establish some minimal requirements for both X and Φ. For instance, in the probabilistic setting X is a measure space and Φ is measurable. In this paper we adhere to the purely topological approach; hence, we assume that X is a topological space and Φ is continuous.

As it is, there is nothing wrong with the map Φ but one immediately sees that some additional restrictions must still be imposed both on T and X. For instance, since we aim to investigate the behavior of the *orbits* $\Phi_x(t) := \Phi(t, x)$ (we also use the word "orbit" to refer to the set $\Phi_x(T)$) of the system in the far future, T must accumulate at ∞, that is, sets of times like $\mathbf{Z}^- \cup \{0\}$ or $\mathbf{R}^- \cup \{0\}$ should not be considered. Besides, as a minimum technical requirement (see for instance Lemma 3.2), one

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would expect some additional topological structure on T, say T must be closed. Furthermore, recall that a *periodic orbit* is defined by the property $\Phi_x(r) = x$ for some $r > 0$ (when all points of the orbit are called *periodic points* and the number r is called a *period* of the orbit and its points; in the particular case when all numbers r are periods of the orbit, that is, the orbit consists of just the point x, then we also call it a *stationary point*). It is natural to expect that if y is a point of this periodic orbit, say $y = \Phi_x(t)$ with $nr < t < (n+1)r$ for some integer n, then y should return to the starting point x after the time $(n+1)r - t$. In other words, T must satisfy $s - t \in T$ whenever $t, s \in T$ and $t < s$. This, together with the rest of the properties of T, easily implies that T is one of the sets $\mathbf{R}^+ \cup \{0\}$, $\mathbf{R}$ (when we say that the DS is *continuous*) or $k(\mathbf{Z}^+ \cup \{0\})$, $k\mathbf{Z}$ for some $k > 0$ (when we say that the DS is *discrete*). We use the notation CDS and DDS to shortly refer to, respectively, a continuous or a discrete DS.

In the latter case all the information we need is comprised in the map $f(x) = \Phi_k(x) := \Phi(k, x)$ (which becomes a homeomorphism in the case $T = k\mathbf{Z}$) for then $\Phi(kn, x) = f^n(x)$ for every $x \in X$ and every n belonging to, respectively, $\mathbf{Z}^+ \cup \{0\}$ or $\mathbf{Z}$. Thus, when dealing with discrete dynamics, one can assume without loss of generality that $T = \mathbf{Z}^+ \cup \{0\}$ or $T = \mathbf{Z}$ and referring to them as, respectively, the dynamics of continuous maps and homeomorphisms from the space X into itself. Notice in passing that every CDS can be "discretized" via the maps $\Phi_t(x)$. Less trivially, if $f : X \to X$ is a continuous map (respectively, a homeomorphism), then there is a DS $(Y, \mathbf{R}^+ \cup \{0\}, \Phi)$ (respectively, $(Y, \mathbf{R}, \Phi)$) such that the DDS generated by f can be embedded into that generated by the one-time map $g(y) = \Phi_1(y)$, that is, there is a continuous one-to-one map $i : X \to Y$ such that $gi = if$ [**1**, page 4]. Here both X and Y are compact metric spaces.

Assuming that the phase space of a DS is compact and metrizable is pretty natural and very convenient from the technical point of view, and so we will do it in the sequel (when $\mathrm{d}(\cdot,\cdot)$ will denote a fixed distance in X). Indeed, this is the standard assumption in discrete dynamics. A minor problem arises when dealing with flows generated by ordinary differential equations (say, defined in open subsets of the Euclidean space $\mathbf{R}^n$) since the phase space X is no longer compact and, moreover, the flow may be defined in just an open subset of $\mathbf{R} \times X$. Yet it is well known (see, e.g., [**9**, Lemma 2.3]) that if X is a locally compact subspace of a compact metric space Y (a paradigmatic case occurs when Y is the one-point compactification of X), then, after reparametrizing time, we get a full-defined flow on $\mathbf{R} \times Y$, all points from $Y \setminus X$ becoming stationary points (or *singular points*, as they are usually called in this context) of the new flow. Hence, even in the continuous setting, our standing assumption does not involve an essential loss of generality.

In the theory of DSs the dialectics of order and chaos is a fundamental one and there is a vast literature concerning what "order" and "chaos" exactly mean. However, to the best of our knowledge, the question of finding the property characterizing those among nonregular DSs that are the "least chaotic" of all has never been addressed. Ideally such a property should be full-range, equally meaningful regardless the type of dynamical system one is dealing with. This is the aim of the present paper.

2. Statement and discussion of the results. We have a threefold task ahead. Firstly we need to define what a "property" is. This is pretty simple.

Definition 2.1. We say that a family $\mathcal{P}$ of DSs is a (*dynamical systems*) *property* if whenever a DS $\mathcal{X} = (X, T, \Phi)$ belongs to $\mathcal{P}$ and $\mathcal{Y} = (Y, T, \Psi)$ is topologically conjugate to $\mathcal{X}$, then $\mathcal{Y}$ also belongs to $\mathcal{P}$.

If $\mathcal{X} \in \mathcal{P}$, then we also say that $\mathcal{X}$ *has the property* $\mathcal{P}$ or $\mathcal{X}$ is a $\mathcal{P}$*-system*.

Here we say that $\mathcal{X}$ and $\mathcal{Y}$ are *topologically conjugate* if there is a homeomorphism $h : X \to Y$ such that $h\Phi_t = \Psi_t h$ for every $t \in T$, that is, the dynamics of $\mathcal{X}$ and $\mathcal{Y}$ are exactly the same up to a topological deformation.

Secondly we must find a way to compare two properties in order to say which of two, if any, involves a larger dynamical complexity. Our approach, while somewhat complicated, will prove its usefulness later.

Definition 2.2. Let $\mathcal{X} = (X, T, \Phi)$ and $\mathcal{Y} = (Y, W, \Psi)$ be DSs. We say that $\mathcal{Y}$ *enriches* $\mathcal{X}$ if there are DSs $\mathcal{P} = (P, U, \Gamma)$ and $\mathcal{Q} = (Q, V, \Upsilon)$, continuous one-to-one maps $i : X \to Q$ and $j : P \to Y$, and continuous

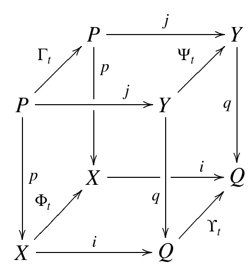

FIGURE 1. (Y, W, Ψ) enriches (X, T, Φ).

onto maps $p : P \to X$ and $q : Y \to Q$, such that $T \subset U, V \subset W$ and all squares commute in Figure 1 whenever t makes sense.

In the particular case when $\mathcal{P} = \mathcal{X}$, $\mathcal{Q} = \mathcal{Y}$, $p = \mathrm{Id}_X$, $q = \mathrm{Id}_Y$, and $i = j$ (respectively, $\mathcal{P} = \mathcal{Y}$, $\mathcal{Q} = \mathcal{X}$, $i = \mathrm{Id}_X$, $j = \mathrm{Id}_Y$, and $p = q$) we say that $\mathcal{Y}$ *extends* (respectively, *blows up*) $\mathcal{X}$.

Remark 2.3. Notice that "$\mathcal{Y}$ *extends* (respectively, *blows up*) $\mathcal{X}$" just means that $i\Phi_t = \Psi_t i$ (respectively, $p\Psi_t = \Phi_t p$) for every $t \in T$. Also, observe that if in Definition 2.2 all involved DSs are discrete, then we can replace Φ_t, Ψ_t, Γ_t and Υ_t by the maps f, g, u, and v generating, respectively, the systems $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{P}$ and $\mathcal{Q}$.

Roughly speaking, enriching a DS consists of two steps. First we embed it into a larger one. Then we blow up each of the points of the new phase space into a compact set and define a flow on the union of these sets, taking care that points belonging to the same set are transported by the flow into the same set at every given time t. The set of times, by the way, may also increase in the process.

Clearly, if $\mathcal{Y}$ enriches $\mathcal{X}$, then its dynamics is, at the very least, as complicated as that of $\mathcal{X}$. Hence, the following way to compare the "dynamical complexity" of two properties $\mathcal{P}$ and $\mathcal{Q}$ suggests itself:

Definition 2.4. We say that *the property $\mathcal{P}$ involves less dynamical complexity than the property $\mathcal{Q}$* (written $\mathcal{P} \Rrightarrow \mathcal{Q}$) if every $\mathcal{P}$-system can be enriched to a $\mathcal{Q}$-system.

Recall that every DDS can be extended to a CDS. Thus, $\mathcal{P} \Rrightarrow \mathcal{Q}$ for the properties $\mathcal{P}$ and $\mathcal{Q}$ of being, respectively, a discrete and a continuous DS. Notice that we are not claiming that *every* CDS has a more complicated dynamics than that *every* DDS, which is of course nonsense, but only that the dynamics of CDSs are, *potentially*, more complex than those of DDSs, which is pretty reasonable. Also, observe that if $\mathcal{P} \subset \mathcal{Q}$ for some properties $\mathcal{P}$ and $\mathcal{Q}$, that is, every DS having the property $\mathcal{P}$ also has the property $\mathcal{Q}$ (shortly, $\mathcal{P} \Rightarrow \mathcal{Q}$ in the usual sense), then $\mathcal{P} \Rrightarrow \mathcal{Q}$. This, again, is in accord with intuition.

While it is clearly unrealistic to expect that the relation "$\Rrightarrow$" (or, for that matter, any plausible relation) totally orders the class of all properties, it should at least induce a preorder in it. As our first result shows, this is fortunately the case.

Proposition A. *If $\mathcal{P} \Rrightarrow \mathcal{Q}$ and $\mathcal{Q} \Rrightarrow \mathcal{R}$, then $\mathcal{P} \Rrightarrow \mathcal{R}$.*

The last stage of our construction involves finding a plausible definition of "minimal chaos," which of course requires a definition of "nonchaoticity" in advance. Our source of inspiration comes from discrete interval dynamics, but the key notions can be formulated in full generality:

Definition 2.5. Let (X, T, Φ) be a DS, and let $x \in X$. We say that x is *approximately periodic* if for every $\varepsilon > 0$ there is a periodic point p such that

$$\limsup_{t\to\infty} \mathrm{d}(\Phi(t,x), \Phi(t,p)) < \varepsilon.$$

Let $x, y \in X$. We say that $\{x, y\}$ is an *asymptotic* (respectively, *distal*) *pair* if

$$\lim_{t\to\infty} \mathrm{d}(\Phi(t,x), \Phi(t,y)) = 0$$

(respectively,

$$\liminf_{t\to\infty} \mathrm{d}(\Phi(t,x), \Phi(t,y)) > 0).$$

We say that $\{x, y\}$ is a *Li-Yorke pair* if it is neither asymptotic nor distal.

Approximate periodicity was introduced by Smítal in [**11**]. There he proves (see also [**6**]) that if $f : [0,1] \to [0,1]$ is continuous, then either all points are approximately periodic, or there is an uncountable set S such that each pair of points in S is Li-Yorke (sets S with this property are called *scrambled*). Scrambled sets were famously brought to the fore by Li and Yorke in their paper, *Period three implies chaos*, [**7**] where, among other things, they proved that if a continuous interval map has a periodic point of (minimal) period three, then it possesses an uncountable scrambled set.

It is worth noticing that the properties described above (full approximate periodicity and existence of an uncountable scrambled set) cannot be simultaneously satisfied by any continuous map regardless of the compact metric space it is defined in. The reason is that no pair of approximately periodic points can be Li-Yorke. This was first noticed in [**8**, pages 117–118] and [**3**, pages 144–145]. As a second result of this paper (Proposition B) emphasizes, the nonexistence of Li-Yorke pairs of approximately periodic points is a general feature of DSs. Hence, although the dichotomy order-chaos just explained only works in the interval setting, we may reasonably identify regular DSs with those just consisting of approximately periodic points. To stress the asymptotical simplicity of an approximately periodic point, we also prove in Proposition B that either it is *asymptotically periodic*, that is, there is a periodic point p such that $\lim_{t\to\infty} \mathrm{d}(\Phi(t,x),\Phi(t,p)) = 0$, or the restricted flow to the set $\omega_\Phi(x)$ of limit points of its orbit as t goes to ∞ can be discretized, up to a topological conjugacy, to an adding machine map. We next introduce this notion and give a simple description of the dynamics of this type of map.

For a sequence of integers $\alpha = (p_m)_{m=1}^\infty$ such that $p_m \geq 2$ for every m, the *α-adic adding machine* Δ_α is the set of sequences (x_m) such that $x_i \in \{0,1,\dots,p_m-1\}$ for every m. We use the product topology in Δ_α and define the *adding machine map* $f_\alpha : \Delta_\alpha \to \Delta_\alpha$ by writing

$$f_\alpha((x_m)) = \begin{cases} (0,\dots,0,x_m+1,x_{m+1},\dots) & \\ \qquad\qquad \text{if } x_m < p_m - 1 \text{ and } x_j = p_j - 1 \\ \qquad\qquad \text{for every } j < m; \\ (0,0,\dots) \qquad \text{if } x_m = p_m - 1 \text{ for every } m. \end{cases}$$

Let X be an infinite compact metric space, and let $h : X \to X$ be a continuous map. It is well known that h is topologically conjugate

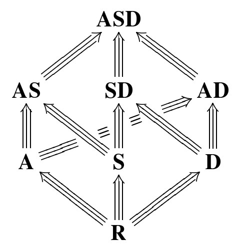

FIGURE 2. Some relations between the properties introduced below.

to an adding machine map if and only if, for every $\varepsilon > 0$, X can be decomposed into finitely many pairwise disjoint compact sets with diameters less than ε which are cyclically permuted by h [**4**]. Moreover, it is easy to show that X is in fact a Cantor set, h is a homeomorphism and there are a decreasing sequence $X = X_0 \supset X_1 \supset \cdots \supset X_m \supset \cdots$ of Cantor sets and an increasing sequence (r_m) of positive integers such that r_m divides r_{m+1} for every m, $h^{r_m}(X_m) = X_m$, X is the disjoint union of the sets $C_m, \ldots, h^{r_m-1}(C_m)$ and the diameters of these sets tend uniformly to zero as m goes to ∞.

Proposition B. *Let (X, T, Φ) be a* DS*. Then the following statements hold*:

(i) *If $x \in X$ is approximately periodic, then either x is asymptotically periodic or there is an $r \in T$ such that the restriction of $f = \Phi_r$ to $\omega_f(x)$ is topologically conjugate to an adding machine map.*

(ii) *If $x, y \in X$ are approximately periodic and*

$$\liminf_{t\to\infty} \mathrm{d}(\Phi(t, x), \Phi(t, y)) = 0,$$

then

$$\lim_{t\to\infty} \mathrm{d}(\Phi(t, x), \Phi(t, y)) = 0.$$

In particular, $\{x, y\}$ cannot be a Li-Yorke pair.

Let $\mathcal{X} = (X, T, \Phi)$ be a DS, and denote by $R(\mathcal{X})$ its set of approximately periodic points. We say that $\mathcal{X}$ *has the property* **R** if $R(\mathcal{X}) = X$. We say that $\mathcal{X}$ *has the property* **A** (respectively, **S**, **D**) if $R(\mathcal{X}) \neq X$ and every pair of points in $X \setminus R(\mathcal{X})$ is asymptotic (respectively, Li-Yorke, distal). Properties **ASD**, **AS**, **SD** and **AD** are defined in a similar vein: for instance, $\mathcal{X}$ has the property **SD** if $R(\mathcal{X}) \neq X$ and every pair of points in $X \setminus R(\mathcal{X})$ is Li-Yorke or distal.

It can be checked that the implications from Figure 2 hold. Indeed all of them are trivial except $\mathbf{R} \not\Rightarrow \mathbf{A}$, $\mathbf{R} \not\Rightarrow \mathbf{S}$ and $\mathbf{R} \not\Rightarrow \mathbf{D}$. To prove any of these last implications assume that (X, T, Φ) has the property **R**. Take a DS $(Z, \mathbf{R}, \Theta)$ having the corresponding property **A**, **S** or **D** and preserving it when seen as a DS with a set of times T (those in the proof of Theorem D, for instance, have this feature). Now it suffices to extend (X, T, Φ) to a system (Y, T, Ψ) where Y is the disjoint union of X and Z and Ψ is defined in the obvious way. Notice that we have used the symbol **R** to denote simultaneously the property **R** and the set **R** of real numbers. This should not lead to confusion.

Thus, in accordance with expectation, **R** involves the least amount of complexity; the rest of the properties imply chaos up to some extent, with the incompatible properties **A**, **S** and **D** being the "least chaotic" at all. If one of them implied the other two, then it could be rightly used as the indicator of "minimal chaos" we are looking for.

To begin with, $\mathbf{A}, \mathbf{S}, \mathbf{D} \not\Rightarrow \mathbf{R}$, thus consolidating the lowest place of **R** at the complexity scale. We also have $\mathbf{S}, \mathbf{D} \not\Rightarrow \mathbf{A}$ and $\mathbf{D} \not\Rightarrow \mathbf{S}$. Indeed all these "nonimplications" (except those with the property **D** at the left side) are the strongest possible, in the sense that no DS having the left-side property can be enriched to a DS having the right-side property.

These statements easily follow from Proposition C below. We emphasize that the additional condition in (iii) (satisfied, for instance, for all points of the discrete irrational rotation in the circle or the continuous irrational rotation in the torus) cannot be, unfortunately, disposed of. A simple counterexample is provided by any adding machine map h. Clearly, for such a map h all pairs of points are distal and no point is approximately periodic (because there are not periodic points at all). Hence h has the property **D**. Notice, by the way, that the sequence $(h^{rn}(x))_{n=0}^{\infty}$ (but not the sequence $(h^{rn+1}(x))_{n=0}^{\infty}$) accumulates at x for every point x and every positive integer r. It is well known that h can be realized, up to a homeomorphism, as a subsystem X of the interval map $f(x) = \alpha x(1-x)$ for an appropriate parameter α. Moreover, with the notation before Proposition B, f admits a family (P_m) of periodic orbits of respective minimal periods r_m such that $X' = X \cup \cup_m P_m$ is compact and all points of X' become approximately periodic for the restriction of f to X'. See [**10**, Theorem 4.1, page 118 and Proposition 4.5, page 242] for details. We have shown that the system generated by h can be extended to an **R**-system.

Proposition C. *Let* (X, T, Φ) *be a* DS*, and let* $x \in X$ *be a non-approximately periodic point of* X*. Assume that* (Y, W, Ψ) *enriches* (X, T, Φ) *and, with the notation of Definition* 2.2*, let* $y \in Y$ *be such that* $q(y) = i(x)$*. Also, assume that one of the following statements holds*:

(i) *all pairs of points of the orbit of* x *are asymptotic*;

(ii) *all pairs of points of the orbit of* x *are Li-Yorke*;

(iii) *all pair of points of the orbit of* x *are distal and for every positive* $r \in T$ *the sequence* $(\Phi(rn+1, x))_{n=0}^{\infty}$ *accumulates at* x*.*

Then y *is not approximately periodic for* (Y, W, Ψ)*. If, moreover,* (ii)*, respectively* (iii)*, holds, then no pair of points of the orbit of* y *can be asymptotic, respectively asymptotic or Li-Yorke.*

At this point one realizes why we must retort to enrichments, rather than just extensions or blow-ups, to implement our preorder relation. Extensions are simply not operative enough: it is obvious that no system having exactly one of the properties **A**, **S** or **D** can be extended to a system having exactly one of the other two. In general, blow-ups fare better than extensions (compare to Theorem D) but they just fall short of working: a system consisting of a two-point periodic orbit is a trivial **R**-system that cannot be blown up to an **A**-system.

After Proposition C only $\mathbf{A} \Rrightarrow \mathbf{S} \Rrightarrow \mathbf{D}$ remains as a feasible alternative, leading to **A** as the characterization of "minimal chaoticity" which we aimed for. Notice that, contrary to initial expectations, distality promises more complexity than the mere existence of Li-Yorke pairs.

Proving these implications remains beyond our present capabilities but we conjecture that both of them are true. Next, we explain why this is a reasonable conjecture.

Among all nontrivial DSs, those generated by interval maps and of the type $(X, \mathbf{R}, \Phi)$, with X a compact surface, are by far the most

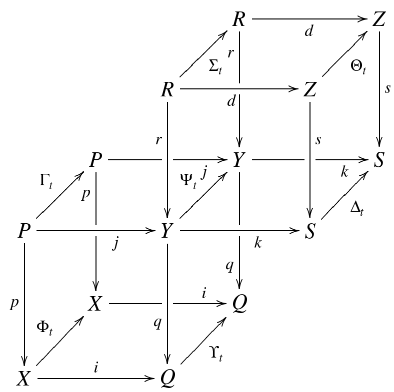

FIGURE 3. (Y, W, Ψ) enriches (X, T, Φ) (via (P, U, Γ) and (Q, V, Υ)) and (Z, K, Θ) enriches (Y, W, Ψ) (via (R, I, Σ) and (S, J, Δ)).

accessible and best known. In the first setting our discussion is void of content because in Smítal's papers [**6, 11**] we mentioned earlier is implicitly proved that either a continuous map $f : [0,1] \to [0,1]$ has the property **A**, or not only possesses uncountably Li-Yorke pairs, but also uncountable asymptotic and distal pairs outside the set of approximately periodic points, that is, f has the property **ASD**.

Thus, we are naturally led to surface flows. Although we are unable to prove, even in this favorable setting, that every **A**-system (respectively **S**-system) can be enriched to an **S**-system (respectively a **D**-system), we can at least provide some examples supporting this view. Such is the content of our last theorem, the main result of the paper:

Theorem D. *Let X be a compact surface. Then there is an* **A***-system (respectively, an* **A***-system, an* **S***-system) $(X, \mathbf{R}, \Phi)$ that can be blown up to a* **D***-system (respectively, an* **S***-system, a* **D***-system).*

Admittedly this may seem a rather unambitious effort. Yet, besides some obvious and nonnegligible technical difficulties, we very much expect that the ideas we employ will essentially suffice to prove the validity of our conjecture, at least in the surface setting.

3. Proofs.

Proof of Proposition A. The statement is a trivial consequence of the following fact: if $\mathcal{Y}$ enriches $\mathcal{X}$ and $\mathcal{Z}$ enriches $\mathcal{Y}$, then $\mathcal{Z}$ enriches $\mathcal{X}$.

To prove the latter fact we use the same notational conventions as in Definition 2.2 to generate the diagram in Figure 3, where all squares commute.

Now we complete the diagram as follows. Let $M = r^{-1}(j(P))$, and let $c : M \to R$ be the inclusion map. Also, we define a surjective map $m : M \to P$ by $m(x) = z$, where if $x \in M$, then z is the only element of P such that $j(z) = r(x)$. Clearly m is continuous. Finally, if U is the set of times for the DS (P, U, Γ), then we define a flow $\Lambda : U \times M \to M$ by $\Lambda(t, x) = \Sigma(t, x)$ for every $(t, x) \in U \times M$. Notice that Λ is well defined. In fact, if $t \in U$ and $x \in M$, then

$$(r\Lambda_t)(x) = (r\Sigma_t)(x) = (\Psi_t r)(x) = (\Psi_t j)(z) = (j\Gamma_t)(z) \in j(P);$$

hence, $\Lambda(t, x) \in M$.

Next we define in S an equivalence relation "$\sim$" by writing $x \sim y$ if either $x = y$ or there is a $z \in Q$ such that $x, y \in k(q^{-1}(\{z\}))$. Since the equivalence class $[x]$ of every $x \in S$ is compact, the quotient space $N = S/\sim$ inherits in the natural way the compact metric space structure of S. Let $n : S \to N$ be the projection map, and define a one-to-one map $d : Q \to N$ by putting $d(z) = k(q^{-1}(\{z\}))$. It is easy to check that d is continuous. Now, if V is the set of times for the DS (Q, V, Υ), then we construct a flow $\Omega : V \times N \to N$ by writing $\Omega(t, [x]) = [\Delta(t, x)]$. This definition makes sense because if $x \sim y$, then $\Delta(t, x) \sim \Delta(t, y)$. Indeed, say $x, y \in k(q^{-1}(\{z\}))$ for some $z \in Q$, and let $a, b \in q^{-1}(\{z\})$ such that $k(a) = x$, $k(b) = y$. Then $(k\Psi_t)(a) = (\Delta_t k)(a) = \Delta_t(x)$ and analogously $(k\Psi_t)(b) = \Delta_t(y)$. Since $q(a) = q(b) = z$, we have that $(\Upsilon_t q)(a) = (\Upsilon_t q)(b) = \Upsilon_t(z) \in Q$. Since $(q\Psi_t)(a) = (\Upsilon_t q)(a)$ and $(q\Psi_t)(b) = (\Upsilon_t q)(b)$, we get $\Psi_t(a), \Psi_t(b) \in q^{-1}(\{\Upsilon_t(z)\})$. Hence, both $\Delta_t(x) = (k\Psi_t)(a)$ and $\Delta_t(y) = (k\Psi_t)(b)$ belong to $k(q^{-1}(\{\Upsilon_t(z)\}))$, that is, $\Delta_t(x) \sim \Delta_t(y)$.

Now it is routine to check that all squares and rectangles in Figure 4 commute. This implies that $\mathcal{Z}$ enriches $\mathcal{X}$ as we desired to prove. □

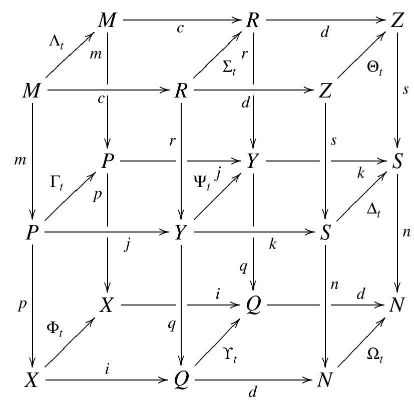

FIGURE 4. All squares and rectangles commute here.

Proof of Proposition B(i). We assume that x is approximately but not asymptotically periodic.

Let $(p_j)_{j=1}^{\infty}$ be a sequence of periodic points satisfying $\limsup_{t\to\infty} \times \mathrm{d}(\Phi(t,x),\Phi(t,p_j)) < 1/j$. Since the orbit of x does not converge to a stationary point, we can assume that the periodic points p_j are not stationary either. Let r_j be a period of the orbit Γ_j of p_j for every j.

Observe that $\limsup_{t\to\infty} \mathrm{d}(\Phi(t,p_j),\Phi(t,p_m)) < 1/j+1/m$ for every j and m. If r_m/r_j is irrational, then we easily get that $\mathrm{d}(y,z) < 1/j+1/m$ for every $x \in \Gamma_j$ and $y \in \Gamma_m$, which in particular implies that the diameter of both Γ_j and Γ_m is less than $2(1/j+1/m)$. Since $\omega_\Phi(x)$ has a positive diameter and the diameter of Γ_j is very close to the diameter of $\omega_\Phi(x)$ if j is large enough, we conclude that $r_m/r_j \in \mathbf{Q}$ whenever j and m are sufficiently large. We can assume $r_m/r_j \in \mathbf{Q}$ for every j and m. Let $f = \Phi_{r_1}$. Then all points p_j are periodic for f and x is approximately periodic for f because

$$\limsup_{n\to\infty} \mathrm{d}(f^n(x), f^n(p_j)) \leq \frac{1}{j}$$

for every j. Let P_j be the periodic orbit of p_j for f and say that its minimal period for f is s_j. Let $A = \omega_f(x)$. Then A is clearly the limit set of the union $P = \cup_j P_j$.

Now the proof goes exactly as that of [**2**, Proposition 5.1], where a similar result was obtained for interval maps. For the convenience of the reader we repeat the argument here.

Let $y \in A$, and fix j. Then there is a $0 \leq k < s_j$ such that y is a limit point of the sequence $(f^{ms_j+k}(x))_m$. Hence, if we write $q_j = f^k(p_j)$, we get that $\mathrm{d}(f^n(y), f^n(q_j)) \leq 1/j$ for every j. Therefore, $\omega_f(y)$ is also the limit set of P, that is, $\omega_f(y) = A$ for every $y \in A$. Maps with the property that the limit set of each of the orbits of the space are the whole space are usually called *minimal*; we have just proved that $f|_A$ is minimal.

Let $\varepsilon > 0$, fix $j > 2/\varepsilon$, rename $s = s_j$, and let $C_i = \omega_{f^s}(f^i(x))$, $0 \leq i < s$. Then $f(C_i) = C_{i+1}$ for every i (we mean $C_s = C_0$). Since each $f^i(x)$ is approximately periodic for f^s, we apply the previous reasoning to conclude that every map $f^s|_{C_i}$ is minimal. Let $0 \leq l, m < s$, $l \neq m$, and assume that $C_l \cap C_m \neq \varnothing$. Then $f^s(C_l \cap C_m) \subset C_l \cap C_m$, so the minimality of $f^s|_{C_l}$ and $f^s|_{C_m}$ forces $C_l = C_m$. Further, observe that if $y \in C_i$, then $d(y, f^i(p_j)) \leq 1/j$, and hence the diameter of C_i is at most $2/j$. If we define $C = C_0$ and k is the minimal positive number satisfying $f^k(C) = C$, then the partition $\{C, f(C), \dots, f^{k-1}(C)\}$ satisfies the required properties in the definition of adding machine for the number ε. □

The proof of Proposition B(ii) requires two preparatory lemmas.

Lemma 3.1. *Let $r, s > 0$ and $0 < \delta < s/2$. Then there are sequences $(a_n)_{n=0}^\infty$, $(b_n)_{n=0}^\infty$ of positive integers such that $|a_n r - b_n s| < \delta$ for every n, (a_n) is strictly increasing, and $(a_{n+1} - a_n)$ is bounded.*

Proof. If $r/s \in \mathbf{Q}$, then there are positive integers a_0, b_0 such that $r/s = b_0/a_0$. After defining $a_n = (n+1)a_0$ and $b_n = (n+1)b_0$ for $n \geq 1$ we get $|a_n r - b_n s| = |(n+1)a_0 s b_0/a_0 - (n+1)b_0 s| = 0$ and the statement trivially holds.

Suppose that $r/s \notin \mathbf{Q}$. It is sufficient to inductively define the sequences (a_n) and (b_n) so that, besides

$$|a_n r - b_n s| < \delta, \tag{1}$$

we have

$$a_{n+1} = a_n + a_1 + \alpha a_0, \text{ with } \alpha \in \{-1, 1\} \tag{2}$$

for every n.

The numbers $0 < a_0 < a_1$ and $0 < b_0 < b_1$ are just chosen so that (1) holds when $n = 0, 1$, for property (2) immediately holds when $n = 0$ after writing $\alpha = -1$.

Assume that the numbers a_n and b_n have defined for every $n \leq k$ so that (1) and (2) holds when, respectively, $n \leq k$ and $n < k$. Since

$$-\delta < a_0 r - b_0 s < \delta$$
$$-\delta < a_1 r - b_1 s < \delta$$
$$-\delta < a_k r - b_k s < \delta,$$

the numbers $A = (a_k + a_1 + a_0)r - (b_k + b_1 + b_0 + 1)s$ and $B = (a_k + a_1 - a_0)r - (b_k + b_1 - b_0 - 1)s$ satisfy $-s - 3\delta < A < -s + 3\delta$ and $s - 3\delta < B < s + 3\delta$. Moreover, $A - B = 2(a_0 r - b_0 s + s) > 2(s - \delta) > s > 0$, hence either $A > 0$ or $B < 0$. Suppose that $A > 0$. Then $a_{k+1} = a_k + a_1 + a_0$ and $b_{k+1} = b_k + b_1 + b_0 + 1$ satisfy (1) for $n = k+1$, since otherwise $A \geq \delta$ and then $\delta \leq A < -s + 3\delta < \delta$, a contradiction. We similarly define $a_{k+1} = a_k + a_1 - a_0$ and $b_{k+1} = b_k + b_1 - b_0 - 1$ in the case $B < 0$. □

Lemma 3.2. *Let (X, T, Φ) be a* DS*, let p be a periodic point, and let $\varepsilon > 0$. Then there is a $\delta > 0$ such that if $t, w \in T$, $|t - w| < \delta$ and $z \in \Phi_p(T)$, then* $\mathrm{d}(\Phi(t, z), \Phi(w, z)) < \varepsilon$.

Proof. Since the periodic orbit $P = \Phi_p(T)$ is compact, the restriction of Φ to $([-r, 2r] \cap T) \times P$ is uniformly continuous (here r denotes a period of P). In particular, there is a $0 < \delta < r$ such that if $t', w' \in [-r, 2r] \cap T$, $|t' - w'| < \delta$, and $z \in P$, then $\mathrm{d}(\Phi(t', z), \Phi(w', z)) < \varepsilon$. Now, if $t, w \in T$, $|t - w| < \delta$, then there is a $t' \in [0, r]$ (and hence $w' \in [-r, 2r]$) such that $|t' - w'| = |t - w|$ and $\Phi(t', z) = \Phi(t, z)$, $\Phi(w', z) = \Phi(w, z)$ for every $z \in P$. Thus, $\mathrm{d}(\Phi(t, z), \Phi(w, z)) < \varepsilon$ for every $z \in P$. □

Proof of Proposition B(ii). Let $\varepsilon > 0$. Then there are periodic points p, q of respective periods r and s such that

$$\limsup_{t \to \infty} \mathrm{d}(\Phi(t, x), \Phi(t, p)) < \frac{\varepsilon}{5},$$
$$\limsup_{t \to \infty} \mathrm{d}(\Phi(t, y), \Phi(t, q)) < \frac{\varepsilon}{5}.$$

According to Lemma 3.2 there is a number $\delta > 0$ such that if $|t-w| < \delta$, then $\mathrm{d}(\Phi(t,z), \Phi(w,z)) < \varepsilon/5$ for every $z \in \Phi_p(T) \cup \Phi_q(T)$. We can assume $\delta < s/2$ to apply Lemma 3.1 to r, s and δ, and find corresponding sequences (a_n) and (b_n) such that, if we write $(t_n) = (a_n r)$, then

$$\Phi(t_n, u) = u$$

and

$$\mathrm{d}(\Phi(t_n, v), v) = \mathrm{d}(\Phi(a_n r, v), \Phi(b_n s, v)) < \frac{\varepsilon}{5}$$

for every n and every $u \in \Phi_p(T)$, $v \in \Phi_q(T)$.

Recall that the sequence $(t_{n+1} - t_n)$ is bounded, say by κ. Let $\delta' > 0$ be such that if $\mathrm{d}(u,v) < \delta'$, then $\mathrm{d}(\Phi(t,u), \Phi(t,v)) < \varepsilon/5$ for every $t \in [0,\kappa] \cap T$. Since $\liminf_{t\to\infty} \mathrm{d}(\Phi(t,x), \Phi(t,y)) = 0$, there is a t' such that $\mathrm{d}(\Phi(t',x), \Phi(t',y)) < \delta'$. Further, we can assume

$$\mathrm{d}(\Phi(t,x), \Phi(t,p)) < \frac{\varepsilon}{5},$$
$$\mathrm{d}(\Phi(t,y), \Phi(t,q)) < \frac{\varepsilon}{5}$$

for every $t \geq t'$.

Let $t \geq t'$. Then $t = t' + t_n + \tilde{t}$ for some integer n and some $\tilde{t} \in [0,\kappa]$. Hence,

$$\begin{aligned}
&\mathrm{d}(\Phi(t,x), \Phi(t,y)) \\
&\quad \leq \mathrm{d}(\Phi(t,x), \Phi(t,p)) + \mathrm{d}(\Phi(t,p), \Phi(t'+\tilde{t}, p)) \\
&\qquad + \mathrm{d}(\Phi(t'+\tilde{t}, p), \Phi(t'+\tilde{t}, x)) + \mathrm{d}(\Phi(t'+\tilde{t}, x), \Phi(t'+\tilde{t}, y)) \\
&\qquad + \mathrm{d}(\Phi(t'+\tilde{t}, y), \Phi(t'+\tilde{t}, q)) + \mathrm{d}(\Phi(t'+\tilde{t}, q), \Phi(t,q)) \\
&\qquad + \mathrm{d}(\Phi(t,q), \Phi(t,y)) \\
&\quad < \frac{\varepsilon}{5} + 0 + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} \\
&\quad = \epsilon.
\end{aligned}$$

We have shown that $\lim_{t\to\infty} \mathrm{d}(\Phi(t,x), \Phi(t,y)) = 0$ as we desired to prove. □

Proof of Proposition C. Let $x \in X$ be a nonapproximately periodic point of X for Φ, and assume that $y \in Y$ is such that $q(y) = i(x)$ and it is an approximately periodic point of Y for Ψ. Since the diagram in Figure 1 commutes, q maps periodic points of Y to periodic points of Q for Υ. Thus, since q is continuous and y is approximately periodic, the point $z = q(y) = i(x)$ is approximately periodic. We emphasize that, if $t \neq s$, then $\Upsilon(t,z) \neq \Upsilon(s,z)$; otherwise, z would be periodic and x would be periodic as well.

Assume that condition (i) holds, i.e., all pairs of points of the orbit of x are asymptotic. Then all pairs of points of the orbit of z are asymptotic. Clearly, $\omega_\Upsilon(z)$ is then a continuum of stationary points. Since z is approximately periodic, Proposition B(i) implies that, in fact, $\omega_\Upsilon(z)$ consists of just one point, that is, $\Upsilon_z(t)$ converges to a stationary point. Hence, $\Phi_x(t)$ also converges to a stationary point, a contradiction.

Now we assume that condition (ii) holds, i.e., all pairs of points of the orbit of x are Li-Yorke. Then the same happens to the orbit of z, which contradicts Proposition B(ii).

Finally, assume that condition (iii) holds. Then all pairs of points of the orbit of z are distal and, for every positive $r \in T$, the sequence $(\Upsilon(rn+1,z))_{n=0}^\infty$ accumulates at z. Since $z \neq \Upsilon(1,z)$, the distality hypothesis applies and we get

$$\liminf_{t\to\infty} \mathrm{d}(\Upsilon(t,z),\Upsilon(t+1,z)) = \delta > 0.$$

Now we use the approximate periodicity of z to find a periodic point p such that

$$\limsup_{t\to\infty} \mathrm{d}(\Upsilon(t,z),\Upsilon(t,p)) < \delta/4.$$

Let r be a period of p. Then $\limsup_{n\to\infty} \mathrm{d}(\Upsilon(rn,z),p)) < \delta/4$ so, if n_0 is large enough and we write $z' = \Upsilon(rn_0,z)$, we have

$$\mathrm{d}(\Upsilon(rn,z'),z')) < \delta/2$$

for every nonnegative integer n. According to the hypothesis $(\Upsilon(rn+1,z'))_{n=0}^\infty$ accumulates at z' and hence there is a number m as large as required such that

$$\mathrm{d}(\Upsilon(rm+1,z'),z') < \delta/4.$$

We conclude that $\mathrm{d}(\Upsilon(r(m+n_0),z),\Upsilon(r(m+n_0)+1,z)) = \mathrm{d}(\Upsilon(rm,z'), \Upsilon(rm+1,z')) < 3\delta/4$, thus arriving at the desired contradiction.

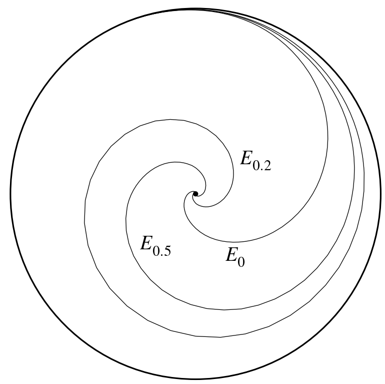

FIGURE 5. The phase portrait of $(\mathbf{D}, \mathbf{R}, \Phi)$.

To conclude the proof of Proposition C, notice that if condition (ii) is satisfied, then the orbit of y cannot contain an asymptotic pair of points, for then their image by q would also be an asymptotic pair of distinct points and then the orbit of x would contain an asymptotic pair. Similarly, if (iii) is satisfied, then every pair of points in the orbit of y must be distal, since otherwise we would be able to locate a nondistal pair of points in the orbit of x. □

Proof of Theorem D. We begin by constructing an $\mathbf{A}$-system $(\mathbf{D}, \mathbf{R}, \Phi)$ that can be blown up to a $\mathbf{D}$-system. Here $\mathbf{D}$ denotes the unit disk $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$ (later we explain how to construct a similar example for an arbitrary compact surface). We also write $\mathbf{D}^* = \{(x, y) \in \mathbf{R}^2 : 0 < x^2 + y^2 < 1\}$.

Let $\varphi(r, \theta) = (r \cos\theta, r \sin\theta)$. Clearly,

$$F(s, \mu) = \varphi\left(\frac{e^s}{1 + e^s}, 2\pi\mu^{1/(1+e^s)} + \log\log(1 + e^s)\right)$$

maps continuously and bijectively $\mathbf{R} \times (0, 1]$ onto $\mathbf{D}^*$; moreover, it can be continuously extended to an onto map from $\mathbf{R} \times [0, 1]$ to $\mathbf{D}^*$.

Now we define $\Phi : \mathbf{R} \times \mathbf{D} \to \mathbf{D}$ by

$$\Phi(t, (x, y)) = \begin{cases} F(F^{-1}(x, y) + (t, 0)) & \text{if } (x, y) \in \mathbf{D}^*, \\ (x, y) & \text{if } (x, y) \in \mathbf{D} \setminus \mathbf{D}^*. \end{cases}$$

In this way we construct a flow on $\mathbf{D}$ for which $(0,0)$ and all points from the unit circle $\mathbf{S}^1 = \{(x,y) \in \mathbf{R}^2 : x^2+y^2=1\}$ are singular points. The rest of the orbits are the spirals $E_\mu = \{F(s,\mu) : s \in \mathbf{R}\}$, $\mu \in (0,1]$. They go to $(0,0)$ as $t \to -\infty$, and accumulate at $\mathbf{S}^1$ as $t \to \infty$. Notice that all pairs of points from $\mathbf{D}^*$ are asymptotic (which, incidentally, ensures the continuity of Φ at $\mathbf{S}^1$) because

$$\lim_{s\to\infty} \frac{e^s}{1+e^s} = 1,$$
$$\lim_{s\to\infty} 2\pi\mu^{1/(1+e^s)} = 2\pi$$

for every $\mu \in (0,1]$, and

$$\lim_{t\to\infty} \log\log(1+e^{s+t}) - \log\log(1+e^{s'+t}) = 0$$

for every $s, s' \in \mathbf{R}$. Hence, $(\mathbf{D}, \mathbf{R}, \Phi)$ is an $\mathbf{A}$-system. See Figure 5.

Let $Y \subset \mathbf{R}^3$ be defined by

$$Y = \{(x,y,z) : z \in [0,1], (x,y) \in E_z\} \cup (\{(0,0)\} \times [0,1])(\mathbf{S}^1 \times [0,1])$$

(here we mean $E_0 := E_1$). Thus Y is the compact set arising after lifting every spiral E_z to the height z and adding the vertical line $\{(0,0)\} \times [0,1]$ and the cylinder $\mathbf{S}^1 \times [0,1]$. See Figure 6.

After defining $\Psi(t,(x,y,z)) = (\Phi(t,(x,y)),z)$ we get a system $(Y, \mathbf{R}, \Psi)$ blowing up $(X, \mathbf{R}, \Phi)$. Apart from the singular points at the vertical line and the cylinder, the only orbits of the new system are the liftings S_z of the corresponding spirals E_z, accumulating at the point $(0,0,z)$ and the circle $\mathbf{S}^1 \times \{z\}$ as $t \to \pm\infty$. Notice that if two points belong to the same orbit S_z, then they form an asymptotic pair; otherwise, they are distal. Hence $(Y, \mathbf{R}, \Psi)$ is an $\mathbf{AD}$-system.

Finally, we blow up $(Y, \mathbf{R}, \Psi)$ to a $\mathbf{D}$-system as follows. Let $(A, \mathbf{R}, \Gamma)$ be an $\mathbf{R}$-system such that, for some $p \in A$, each pair of points $\{\Gamma_p(t), \Gamma_p(s)\}$, $s \neq t$, is distal. (An example of such a system is the map $f : X' \to X'$ described before the statement of Proposition C in Section 2–a discrete $\mathbf{R}$-system having the required property–after being canonically extended to a CDS.) Observe that every spiral S_z contains exactly one point $p_z = (x,y,z)$ such that $x^2+y^2 = 1/4$. Now, after

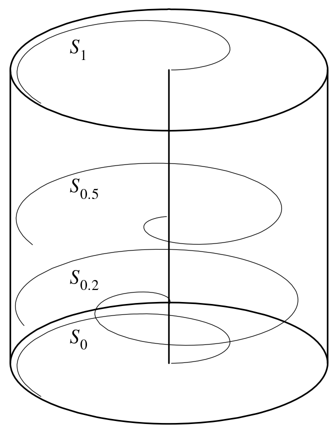

FIGURE 6. The set Y.

defining $B_z \subset Y \times A = \{(\Psi(s, p_z), \Gamma(s, p)) : s \in \mathbf{R}\}$ for every $z \in [0, 1]$, we easily see that

$$Z = (\{(0, 0)\} \times [0, 1] \times A) \cup (\mathbf{S}^1 \times [0, 1] \times A) \cup \bigcup_z B_z$$

is a compact subset of $Y \times A$.

Let $\Theta : \mathbf{R} \times Z \to Z$ be defined by $\Theta(t, u) = (\Psi(t, v), \Gamma(t, w))$ for every $u = (v, w) \in Z$. The system $(Z, \mathbf{R}, \Theta)$ so constructed blows up every singular point of $(Y, \mathbf{R}, \Psi)$ to a subsystem topologically conjugate to $(A, \mathbf{R}, \Gamma)$, hence consisting of approximately periodic orbits. Every orbit S_z is blown up to the orbit B_z which, due to the way it has been defined, just contains distal pairs of points. Since pairs of points belonging to different B_z, $B_{z'}$ keep being distal, all asymptotic pairs have been eliminated: $(Z, \mathbf{R}, \Theta)$ is a $\mathbf{D}$-system blowing up $(Y, \mathbf{R}, \Psi)$, and hence the $\mathbf{A}$-system $(\mathbf{D}, \mathbf{R}, \Phi)$.

If the surface X we are dealing with is other than the disk $\mathbf{D}$, then we proceed in exactly the same fashion, just using a topological disk inside X instead of $\mathbf{D}$ and defining the initial $\mathbf{A}$-system $(X, \mathbf{R}, \Phi)$ so that all points outside the topological disk are also singular points of the flow Φ. We have completed the proof of the part of Theorem D regarding the blow-up of an $\mathbf{A}$-system to a $\mathbf{D}$-system.

Our next task is constructing an $\mathbf{S}$-system $(X, \mathbf{R}, \Upsilon)$ for every given compact surface X that can be blown up to a $\mathbf{D}$-system. There is again no loss of generality in assuming $X = \mathbf{D}$.

The flow on $\mathbf{D}$ we are going to define will have the singular point $(0,0)$ and the same orbits E_μ as the previous one. The circle $\mathbf{S}^1$ will consist now of the singular point $(-1,0)$ and a homoclinic orbit starting from and ending at $(-1,0)$. (By the way, when carrying on our construction to another surface X, we must first add similar pairs of orbits to each circle $\{(x,y) : x^2+y^2=r^2\}$, $1<r<2$, the flow being slower as r increases, and then a circle $\{(x,y) : x^2+y^2=4\}$ of singular points. This larger disk is what we embed in X.)

Although we used a different method earlier, the standard way to generate a continuous-time plane flow Σ is by defining a vector field $f(x,y) = (P(x,y), Q(x,y))$ so that $\Upsilon(t,(x_0,y_0))$ is the solution of $x' = P(x,y)$, $y' = Q(x,y)$ with initial condition $x(0) = x_0$, $y(0) = y_0$. We already know what the orbits look like, so we only need to specify, at every point $p = (x,y)$, the orientation of the vector $f(p)$ (counterclockwise) and its modulus $\|f(p)\|$.

To do this, we proceed as follows. Clearly, there are a segment S_0 in $S = \{(x,0) : 0 < x < 1\}$ and a increasing homeomorphism $h : [0,1) \to S_0$ such that $p_\mu := h(\mu) \in E_\mu$ for every μ. In fact, if $(p_\mu^n)_{n=-\infty}^{\infty}$ is the sequence of consecutive intersection points of E_μ with the segment S ($p_\mu^0 = p_\mu$), then h can be extended to a homeomorphism $h : \mathbf{R} \to S$ so that $h(\mu+n) = p_\mu^n$ for every $\mu \in (0,1]$ and $n \in \mathbf{Z}$. Let a_μ^n denote the intersection point of E_μ with $\{(-x,0) : 0 < x < 1\}$ between p_μ^n and p_μ^{n+1}.

We are ready to define $\|f(p)\|$ for every p. To begin with, $\|f(p)\| = x$ if $p = (x,0) \in S$. Now we define $\|f(p)\|$ for all points of an arbitrary arc T in E_μ between p_μ^n and p_μ^{n+1}. Namely, if $\alpha : [0,2\pi] \to T$ is the angle homeomorphism mapping 0 to p_μ^n and 2π to p_μ^{n+1}, then we put

$$\|f(\alpha(\theta))\| = \frac{\pi-\theta}{\pi}\|f(\alpha(0))\| + \frac{\theta}{\pi}\|f(\alpha(\pi))\|$$

whenever $\theta \in [0,\pi]$, and

$$\|f(\alpha(\theta))\| = \frac{2\pi-\theta}{\pi}\|f(\alpha(\pi))\| + \frac{\theta-\pi}{\pi}\|f(\alpha(2\pi))\|$$

whenever $\theta \in [\pi, 2\pi]$. Of course the definition of $\|f\|$ at the point $a_\mu^n = \alpha(\pi)$ is still pending. We fix it so that the time needed by the flow to travel from p_μ^n to p_μ^{n+1} is $1 + |\mu + n|$. Observe that, since the time required to travel from p_μ^n to p_μ^{n+1} goes to ∞ as $n \to \pm\infty$, the vector field so defined can be extended to a continuous vector field on $\mathbf{D}$ whose only singular points are $(0,0)$ and $(-1,0)$.

We show that $(\mathbf{D}, \mathbf{R}, \Upsilon)$ is an $\mathbf{S}$-system. Fix a small neighborhood U of $(-1,0)$ and recall that $\mathbf{S}^1 \setminus \{(-1,0)\}$ is an orbit of the system. Due to the continuity of Φ there is a number $\delta > 0$ such that if $p \in \mathbf{D}^*$ and t is large enough, then $\Upsilon(t+s, p) \in U$ for every $s \in [\delta, 2\delta]$. This implies $\liminf_{t\to\infty} \mathrm{d}(\Upsilon(t,p), \Upsilon(t,p')) = 0$ for every $p, p' \in \mathbf{D}^*$.

If $p \neq p'$ belong to the same orbit E_μ, then $\limsup_{t\to\infty} \mathrm{d}(\Upsilon(t,p), \Upsilon(t,p')) > 0$ is obvious. Let $p \in E_\mu$ and $p' \in E_{\mu'}$ with $\mu \neq \mu'$ and assume that $\lim_{t\to\infty} \mathrm{d}(\Upsilon(t,p), \Upsilon(t,p')) = 0$. Then there are a small $\varepsilon > 0$ and large positive integers n, n' such that $\sup_{t>0} \mathrm{d}(\Upsilon(t,p_\mu^n), \Upsilon(t,p_{\mu'}^{n'})) < \varepsilon$. This, in particular, means that the difference of times required to travel from p_μ^n to p_μ^{n+k}, and from $p_\mu'^{n'}$ to $p_{\mu'}^{n'+k}$, remains bounded regardless how large is the number k. But these times are, respectively, $k(1+\mu+n) + k(k+1)/2$ and $k(1+\mu'+n') + k(k+1)/2$. We have arrived at a contradiction.

We have proved that $(\mathbf{D}, \mathbf{R}, \Upsilon)$ is an $\mathbf{S}$-system. It can be blown up to a $\mathbf{D}$-system in exactly the same way we did before.

Finally we show how the $\mathbf{A}$-system $(\mathbf{D}, \mathbf{R}, \Phi)$ we constructed at the beginning of the proof of Theorem D can be blown up to an $\mathbf{S}$-system (a similar example for a general compact surface X can be easily derived from this one). To do this, the $\mathbf{S}$-system $(\mathbf{D}, \mathbf{R}, \Upsilon)$ we have just constructed will prove very useful. We keep using the notation E_μ and p_μ for every $\mu \in [0,1]$ (of course we mean $p_1 = p_0^1$) and also write $q_\mu = p_{\mu/2}$.

Let $Y' \subset \mathbf{R}^3$ be defined by

$$\begin{aligned} Y' = \{(x,y,z) : (x,y) = \Phi(s, p_\mu), z \\ = \mu + (1-\mu)e^s/(1+e^s), \quad s \in \mathbf{R},\ \mu \in [0,1]\} \\ \cup (\{(0,0)\} \times [0,1]) \cup (\mathbf{S}^1 \times \{1\}) \end{aligned}$$

and write $\Psi'(t,(x,y,z)) = (\Phi(t,(x,y)), z')$, where if $(x,y) = \Phi(s,p_\mu)$ and $z = \mu + (1-\mu)e^s/(1+e^s)$, then $z' = \mu + (1-\mu)e^{s+t}/(1+e^{s+t})$. In this way we get a blow-up $(Y', \mathbf{R}, \Psi')$ of $(\mathbf{D}, \mathbf{R}, \Phi)$ where, in contrast to the blow-up $(Y, \mathbf{R}, \Psi)$ from the first construction, the spirals E_μ are not lifted to the horizontal spirals S_μ but they go up as time increases to reach the level $z = 1$ as $t \to \infty$ and the level μ as $t \to -\infty$. Thus $(Y', \mathbf{R}, \Psi')$ is still an $\mathbf{A}$-system, the only difference being that $E_0 = E_1$ has been blown up to two different orbits.

For every $\mu \in [0,1]$, let $A_\mu \subset Y' \times \mathbf{D} = \{(\Psi'(s, p_\mu), \Upsilon(s, q_\mu)) : s \in \mathbf{R}\}$. Then

$$Z' = (\{(0,0)\} \times [0,1] \times \{(0,0)\}) \cup (\mathbf{S}^1 \times \{1\} \times \mathbf{S}^1) \cup \bigcup_\mu A_\mu$$

is a compact subset of $Y' \times \mathbf{D}$.

Let $\Omega : \mathbf{R} \times Z' \to Z'$ be defined by $\Omega(t, u) = (\Psi'(t, v), \Upsilon(t, w))$ for every $u = (v, w) \in Z'$. Realize that all points of the segment $\{(0,0)\} \times [0,1] \times \{(0,0)\}$ are singular for the flow Ω, while every orbit of the invariant torus $\mathbf{S}^1 \times \{1\} \times \mathbf{S}^1$ converges both in positive and negative time to some singular point $(z, 1, (0, -1))$. As for the rest of orbits, notice that if $\mu \neq \mu'$ or $s \neq s'$, then $\Upsilon(s, q_\mu)$ and $\Upsilon(s', q_{\mu'})$ form a Li-Yorke pair (observe that this may not be true if we replace q_μ and $q_{\mu'}$ by p_0 and p_1; this is the reason why we use the points $q_\mu = p_{\mu/2}$). Since all pairs $\Psi'(s, p_\mu)$ and $\Psi'(s', p_{\mu'})$ are asymptotic, we conclude that $(Z', \mathbf{R}, \Omega)$ is an $\mathbf{S}$-system blowing up the $\mathbf{A}$-system $(\mathbf{D}, \mathbf{R}, \Phi)$.

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