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NIKOL'SKII INEQUALITIES FOR LORENTZ SPACES

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ABSTRACT. A general approach is given for establishing Nikol'skii-type inequalities for various Lorentz spaces. The key ingredient for the proof is either a Bernstein-type inequality or a Remez-type inequality. Applications of our results to trigonometric polynomials on the torus T^d, algebraic polynomials on $[-1,1]$, spherical harmonic polynomials on the unit sphere S^{d-1} in $\mathbf{R}^d$, algebraic polynomials on $\mathbf{R}$ with Freud's weights and others will be presented.

1. Introduction. The Lorentz space $L_{p,q} \equiv L_{p,q}(\Omega,\mu)$, $0 < p,q \le \infty$, is the class of measurable functions on Ω with respect to the nonnegative measure μ satisfying $\|f\|_{p,q} < \infty$ where

$$(1.1)\qquad \begin{gathered} \|f\|_{p,q} := \|f\|_{L_{p,q}(\Omega)} := \left\{ \frac{q}{p} \int_0^\infty t^{(q/p)-1} f^*(t)^q \, dt \right\}^{1/q}, \\ 0 < p, \quad q < \infty, \\ \|f\|_{p,\infty} := \|f\|_{L_{p,\infty}(\Omega)} := \sup_{0<t<\mu(\Omega)} t^{1/p} f^*(t), \quad 0 < p \le \infty, \end{gathered}$$

and f^* is the nonincreasing rearrangement of f. Note that for $p = \infty$ the space $L_{\infty,q}$ is defined only for $q = \infty$.

We recall that the distribution function $\mu_f(\lambda)$ is given by

$$(1.2)\qquad \mu_f(\lambda) = \mu(x \in \Omega : |f(x)| > \lambda),$$

and f^*, the rearrangement of f, is given by

$$(1.3)\qquad f^*(t) = \inf(\lambda : \mu_f(\lambda) \le t).$$

For the classes of functions $\{\mathcal{N}_\nu\}_{\nu \in \mathbf{R}_+}$ for which $\mathcal{N}_\nu \subset L_{p,q}(\Omega,\mu)$ for $0 < p,q \le \infty$, a Nikol'skii-type inequality is

$$(1.4)\qquad \|f\|_{p_2,q_2} \le C\Psi(\nu) \|f\|_{p_1,q_1} \quad \text{for } f \in \mathcal{N}_\nu.$$

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Here, we will have $\Psi(\nu) = \nu^{\beta((1/p_1)-(1/p_2))_+}$ or $\Psi(\nu) = (\log(\nu + 1))^{((1/q_2)-(1/q_1))_+}$ where $a_+ = \max(a,0)$. Both $\Psi(\nu)$ and the constant C depend on p_1, p_2, q_1 and q_2. (C also depends on $\{\mathcal{N}_\nu\}$ but not on ν.) We emphasize the fact that in this paper various classes of functions $\{\mathcal{N}_\nu\}$ will be described that will fit different domains and measures for which different constants β will be appropriate. (The classes of functions $\{\mathcal{N}_\nu\}$ are not uniquely determined by the domain Ω and the measure μ.) While in general we would not obtain the best possible constants, we will attempt to differentiate between cases or ranges for which an absolute constant exists and situations in which the constant C is unbounded as $p_1 - p_2 \to 0$ or as $p_1 \to 0$.

For the case when $\mathcal{N}_\nu$ is the class of trigonometric polynomials of degree $\le \nu$ on $\Omega = T$ (the circle $[-\pi,\pi]$) with μ the Lebesgue measure on T, Sherstneva [**6**] achieved (1.4). In most other cases, the Nikol'skii-type inequality proved was restricted to $p_1 = q_1$ and $p_2 = q_2$, i.e., to the case $L_{p_i,q_i} = L_{p_i}$.

In Section 2 we establish the general framework of our treatment. Inequalities will be proved for the class of functions $\mathcal{A}_\sigma(\gamma)$, where $\mathcal{A}_\sigma(\gamma)$ with $\sigma,\gamma > 0$, is the collection of all functions $f \in L_{p,q}$ satisfying

$$\gamma f^*(\sigma^{-1}) \ge f^*(0). \tag{1.5}$$

The condition $f \in \mathcal{A}_\sigma(\gamma)$ (for various $\mathcal{A}_\sigma(\gamma)$) will later be related to the classes $\{\mathcal{N}_\nu\}$ in applications. Applications in case $\mu(\Omega) < \infty$ are given in Sections 3 and 4, where Remez and Bernstein-type inequalities are used respectively to obtain (1.5). In Section 5, applications when $\mu(\Omega) = \infty$ are dealt with. Further remarks conclude the paper in Section 6.

2. Basic inequalities. Relations between $\|f\|_{p_1,q_1}$ and $\|f\|_{p_2,q_2}$ in general and under the restriction $f \in \mathcal{A}_\sigma(\gamma)$, see (1.5), given in Theorems 2.1 and 2.3 respectively form the framework for proving Nikol'skii-type estimates in the present work. The results in Theorem 2.1, though known or straightforward, will be given for completeness. Furthermore, we state here explicitly the constants that stem from the exact definition in (1.1).

Theorem 2.1. *Let* $\|f\|_{p,q}$ *be given by* (1.1) *with respect to some* Ω *and* μ, *and let* $0 < p, p_i, q, q_i \le \infty$. *Then*:

$$\|f\|_{p,\infty} \le \|f\|_{p,q} \quad \textit{for } p, q < \infty; \tag{2.1}$$

$$\|f\|_{p,q_2} \le \left(\frac{q_2}{q_1}\right)^{1/q_2} \|f\|_{p,q_1} \tag{2.2}$$

for $p < \infty$ *and* $q_1 < q_2 < \infty$;

$$\|f\|_{p_2,q_2} \le \left(\frac{p_1}{p_1 - p_2}\right)^{1/q_2} \mu(\Omega)^{1/p_2-(1/p_1)} \|f\|_{p_1,q_1} \tag{2.3}$$

for $p_2 < p_1 < \infty$ *and* $q_2 < q_1$;

$$\|f\|_{p_2,q_2} \le \left(\frac{q_2}{q_1}\right)^{1/q_2} \left(\frac{p_1}{p_2}\right)^{1/q_1} \mu(\Omega)^{\frac{1}{p_2}-\frac{1}{p_1}} \|f\|_{p_1,q_1} \tag{2.4}$$

for $p_2 < p_1 < \infty$ *and* $q_1 \le q_2$; *and*

$$\|f\|_{p_2,q_2} \le \mu(\Omega)^{1/p_2} \|f\|_\infty \quad \textit{for } p_2 < \infty. \tag{2.5}$$

Remark 2.2. In proving (2.3), (2.4) and (2.5) we assume that $\mu(\Omega) < \infty$ as otherwise these inequalities are meaningless. When $\mu(\Omega) < \infty$, $f^*(t) = 0$ for $t > \mu(\Omega)$ and $\int_0^\infty \cdots = \int_0^{\mu(\Omega)} \cdots$ in (1.1). The condition $q_2 < q_1$ in (2.3) is redundant (not used), but when $q_2 \ge q_1$, the superior estimate (2.4) holds.

Note that when we write p (instead of p_i), we use it for the case $p_1 = p_2 = p$.

Proof. For any $0 < t < \mu(\Omega)$, we have

$$t^{1/p} f^*(t) = f^*(t)\left\{\frac{q}{p}\int_0^t u^{(q/p)-1} du\right\}^{1/q} \le \left\{\frac{q}{p}\int_0^t u^{(q/p)-1} f^*(u)^q \, du\right\}^{1/q} \le \|f\|_{p,q}.$$

Taking the supremum on both sides, we derive (2.1). We use (2.1) to obtain

$$\|f\|_{p,q_2} = \left\{\frac{q_2}{q_1} \cdot \frac{q_1}{p}\int_0^\infty \left(t^{1/p} f^*(t)\right)^{q_2-q_1} t^{(q_1/p)-1} f^*(t)^{q_1} \, dt\right\}^{1/q_2}$$
$$\le \left(\frac{q_2}{q_1}\right)^{1/q_2} \|f\|_{p,\infty}^{(q_2-q_1)/q_2} \|f\|_{p,q_1}^{q_1/q_2} \le \left(\frac{q_2}{q_1}\right)^{1/q_2} \|f\|_{p,q_1},$$

which is (2.2). For $t < \mu(\Omega)$, $f^*(t) \le \|f\|_{p_1,\infty} t^{-1/p_1}$, and hence

$$\begin{aligned} \|f\|_{p_2,q_2} &\le \left\{ \frac{q_2}{p_2} \int_0^{\mu(\Omega)} t^{q_2((1/p_2)-(1/p_1))-1}\, dt \right\}^{1/q_2} \|f\|_{p_1,\infty} \\ &\le \left(\frac{p_1}{p_1 - p_2} \right)^{1/q_2} \mu(\Omega)^{(1/p_2)-(1/p_1)} \|f\|_{p_1,\infty}, \end{aligned}$$

which, using (2.1), implies (2.3). We obtain (2.4) using (2.2) in combination with

$$\|f\|_{p_2,q_2} \le \left(\frac{p_1}{p_2} \right)^{1/q_2} \mu(\Omega)^{(1/p_2)-(1/p_1)} \|f\|_{p_1,q_2}, \tag{2.6}$$

which follows immediately from (1.1). For $\mu(\Omega) < \infty$ we have (2.5) using

$$\|f\|_{p_2,q_2} \le \left\{ \frac{q_2}{p_2} \int_0^{\mu(\Omega)} t^{(q_2/p_2)-1}\, dt \right\}^{1/q_2} f^*(0) = \mu(\Omega)^{1/p_2} \|f\|_\infty. \tag{2.7}$$ □

Theorem 2.3. *Suppose that* $f \in \mathcal{A}_\sigma(\gamma)$, *i.e., satisfying* (1.5), *and that* $0 < p, p_i, q, q_i \le \infty$. *Then*:

$$\|f\|_\infty \le \gamma \sigma^{1/p} \|f\|_{p,q} \quad \text{when } p < \infty; \tag{2.8}$$

$$\|f\|_{p_2,q_2} \le \left(\gamma^{q_2} + \frac{p_1}{p_2 - p_1} \right)^{1/q_2} \sigma^{(1/p_1)-(1/p_2)} \|f\|_{p_1,q_1} \tag{2.9}$$

when $p_1 < p_2 < \infty$ *and* $q_2 < q_1$;

$$\|f\|_{p_2,q_2} \le \left(\frac{q_2}{q_1} \right)^{1/q_2} \left(\gamma^{q_1} + \frac{p_1}{p_2} \right)^{1/q_1} \sigma^{(1/p_1)-(1/p_2)} \|f\|_{p_1,q_1} \tag{2.10}$$

when $p_1 < p_2 < \infty$ *and* $q_1 \le q_2 < \infty$;

$$\|f\|_{p_2,\infty} \le \left(\gamma^{q_1} + \frac{p_1}{p_2} \right)^{1/q_1} \sigma^{(1/p_1)-(1/p_2)} \|f\|_{p_1,q_1} \tag{2.11}$$

when $p_1 < p_2 < \infty$ *and* $q_1 < \infty$;

(2.12) $$\|f\|_{p_2,\infty} \le \sigma^{(1/p_1)-(1/p_2)} \|f\|_{p_1,\infty} \quad \text{when } p_1 < p_2 < \infty;$$

(2.13) $$\|f\|_{p,q_2} \le c(\gamma, \mu(\Omega), p, q_1, q_2)(\ln(\sigma+1))^{(1/q_2)-(1/q_1)} \|f\|_{p,q_1}$$

when $\mu(\Omega) < \infty$, $p < \infty$ *and* $q_2 < q_1$.

Proof. Using (1.5), we obtain

(2.14) $$\|f\|_{p,\infty} = \sup_{0<t<\mu(\Omega)} t^{1/p} f^*(t) \ge \sigma^{-1/p} f^*(\sigma^{-1}) \ge \gamma^{-1} \sigma^{-1/p} \|f\|_\infty,$$

which, together with (2.1), implies (2.8). We now represent $\|f\|_{p_2,q_2}$ by

(2.15) $$\begin{aligned} \|f\|_{p_2,q_2}^{q_2} &= \frac{q_2}{p_2} \int_0^{\sigma^{-1}} u^{(q_2/p_2)-1} f^*(u)^{q_2}\, du \\ &\quad + \frac{q_2}{p_2} \int_{\sigma^{-1}}^{\mu(\Omega)} u^{(q_2/p_2)-1} f^*(u)^{q_2}\, du \\ &=: I_1 + I_2. \end{aligned}$$

To estimate I_1, we write

$$I_1 \le \|f\|_\infty^{q_2} \frac{q_2}{p_2} \int_0^{\sigma^{-1}} u^{(q_2/p_2)-1}\, du = \sigma^{-q_2/p_2} \|f\|_\infty^{q_2},$$

and using (2.8), we have

(2.16) $$I_1 \le \left\{ \gamma \sigma^{(1/p_1)-(1/p_2)} \|f\|_{p_1,q_1} \right\}^{q_2}.$$

To obtain (2.9), we estimate I_2 by

$$\begin{aligned} I_2 &\le \Big(\sup_{\sigma^{-1}<t\le\mu(\Omega)} t^{1/p_1} f^*(t) \Big)^{q_2} \cdot \frac{q_2}{p_2} \int_{\sigma^{-1}}^{\mu(\Omega)} t^{(q_2/p_2)-(q_2/p_1)-1} dt \\ &\le \|f\|_{p_1,\infty}^{q_2} \cdot \frac{p_1}{p_2-p_1} \sigma^{((1/p_1)-(1/p_2))q_2} \le \ (\text{by } (2.1)) \\ &\le \|f\|_{p_1,q_1}^{q_2} \cdot \frac{p_1}{p_2-p_1} \sigma^{((1/p_1)-(1/p_2))q_2}, \end{aligned}$$

which, combined with (2.16), implies (2.9). For $\sigma^{-1} \le t < \mu(\Omega)$, we have $t^{(q/p_2)-1} \le \sigma^{((1/p_1)-(1/p_2))q} t^{(q/p_1)-1}$ and hence

$$\begin{aligned} I_2 &\le \frac{p_1}{p_2}\sigma^{((1/p_1)-(1/p_2))q} \cdot \frac{q}{p_1}\int_{\sigma^{-1}}^{\mu(\Omega)} t^{(q/p_1)-1} f^*(t)^q \, dt \\ &= \frac{p_1}{p_2}\sigma^{((1/p_1)-(1/p_2))q}\|f\|_{p_1,q}^q, \end{aligned}$$

which, combined with (2.16), implies

$$\|f\|_{p_2,q} \le \left(\gamma^q + \frac{p_1}{p_2}\right)^{1/q} \sigma^{((1/p_1)-(1/p_2))}\|f\|_{p_1,q}. \tag{2.17}$$

Setting $q = q_1 \le q_2$, and using (2.2) as well as (2.17), we obtain (2.10). We now use (2.1) with $p = p_2$ and $q = q_1$ and then (2.17) with $q = q_1$ to obtain (2.11). We split the supremum defining $\|f\|_{p_2,\infty}$, writing

$$\|f\|_{p_2,\infty} = \max\Big(\sup_{t \le \sigma^{-1}} t^{1/p_2} f^*(t), \sup_{t \ge \sigma^{-1}} t^{1/p_2} f^*(t)\Big) \equiv \max(J_1, J_2).$$

Using (2.8), we have

$$J_1 \le \sigma^{-1/p_2} f^*(0) = \sigma^{-1/p_2}\|f\|_\infty \le \sigma^{(1/p_1)-(1/p_2)}\|f\|_{p_1,\infty},$$

and as $p_1 < p_2$,

$$J_2 \le \sigma^{(1/p_1)-(1/p_2)} \sup_{\sigma^{-1}<t<\mu(\Omega)} t^{1/p_1} f^*(t) \le \sigma^{(1/p_1)-(1/p_2)}\|f\|_{p_1,\infty},$$

which establishes (2.12). To establish (2.13), we estimate I_1 and I_2 given in (2.15) with $p_2 = p$. We follow (2.16) to estimate I_1 by

$$I_1 \le \|f\|_\infty^{q_2}\sigma^{-q_2/p} \le \gamma^{q_2}\|f\|_{p,q_1}^{q_2}.$$

Using the Hölder inequality, we estimate I_2 by

$$\begin{aligned} I_2 &= \frac{q_2}{p}\int_{\sigma^{-1}}^{\mu(\Omega)} \Big\{t^{(q_1/p)-1} f^*(t)^{q_1}\Big\}^{q_2/q_1} t^{-1+(q_2/q_1)} \, dt \\ &\le \frac{q_2}{p}\bigg\{\int_{\sigma^{-1}}^{\mu(\Omega)} t^{(q_1/p)-1} f^*(t)^{q_1} \, dt\bigg\}^{q_2/q_1} \cdot \bigg\{\int_{\sigma^{-1}}^{\mu(\Omega)} \frac{dt}{t}\bigg\}^{1-(q_2/q_1)} \\ &\le \Big\{q_2^{(1/q_2)} q_1^{-(1/q_1)} p^{(1/q_1)-(1/q_2)} (\ln\mu(\Omega) + \ln\sigma)^{(1/q_2)-(1/q_1)}\|f\|_{p_1,q_1}\Big\}^{q_2}, \end{aligned}$$

which, together with the estimate of I_1, implies (2.14). □

Remark 2.4. The condition $q_2 < q_1$ in (2.9) is not needed and is just an indication that when $q_2 \geq q_1$, better results hold and are given in (2.10), (2.11) and (2.12).

3. Applications using Remez-type inequalities. The most versatile and hence desirable way to achieve an inequality like (1.5) is via a Remez-type inequality given by

$$\|f\|_{L_\infty(\Omega)} \leq e^{A\nu|B|^{1/\beta}} \|f\|_{L_\infty(\Omega\setminus B)}, \quad f \in \mathcal{N}_\nu = \mathcal{N}_\nu(\beta), \tag{3.1}$$

for any measurable set B satisfying $\mu(B) \equiv |B| < M$ with some positive M, A and β. The Remez inequality (3.1) implies for $\alpha > 0$

$$e^{-A\alpha} f^*\left(\left(\frac{\alpha}{\nu}\right)^\beta\right) \geq \|f\|_{L_\infty(\Omega)} \quad \text{for } f \in \mathcal{N}_\nu; \tag{3.2}$$

that is, (1.5) holds with $\gamma = e^{-A\alpha}$, $\sigma = (\nu/\alpha)^\beta$ and $\mathcal{A}_{(\nu/\alpha)^\beta} = \mathcal{N}_\nu$. In Section 6 we will discuss the superiority of (3.1) over the other method described in Sections 4 and 5 (when (3.1) is not available or not valid).

As an immediate corollary of Theorems 2.1 and 2.3 we have:

Theorem 3.1. *Suppose that $\mu(\Omega) < \infty$, $0 < p, p_i, q_i \leq \infty$, and that $f \in \mathcal{N}_\nu = \mathcal{N}_\nu(\beta)$ satisfies* (3.1). *Then*

$$\|f\|_{p_2,q_2} \leq c_1(A, p_i, q_i)\nu^{\beta((1/p_1)-(1/p_2))_+} \|f\|_{p_1,q_1} \quad \textit{when } p_1 \neq p_2 \tag{3.3}$$

and

$$\|f\|_{p,q_1} \leq c_2(A, \beta, p, q_i)(\ln(\nu+1))^{((1/q_2)-(1/q_1))_+} \|f\|_{p,q_2} \tag{3.4}$$

where

$$c_1(A, p_i, q_i) = O\left(\left(\frac{p_1}{p_1 - p_2}\right)^{1/q_2}\right)$$

as $p_2 - p_1 \to 0+$ when $q_2 < q_1$ and $c_1(A, p_i, q_i) = O(1)$ when $q_1 \leq q_2$ and/or $p_2 < p_1$.

Remark 3.2. The constant β of (3.1) influences the power in (3.3), but only the constant c_2 in (3.4), see (2.13). The dependence of the constants c_1 on A, p_i and q_i and of c_2 on A, β, p and q_i stems from the inequalities (2.1) to (2.13) with $\sigma = (\nu/\alpha)^\beta$, $\gamma = e^{-A\alpha}$ and a good choice of α (with $\alpha = 1$, the default choice).

We present three applications in this section.

A. Trigonometric polynomials on T^d of degree ν in each variable.

Theorem 3.3. *For $f \in \mathcal{N}_\nu(d)$, where $\mathcal{N}_\nu(d)$ is the class of trigonometric polynomials of degree ν in each of the d variables,* (3.3) *and* (3.4) *hold with $\Omega = T^d$, $\beta = d$ and μ the Lebesgue measure on T^d.*

To prove Theorem 3.3 we need only establish the following Remez-type inequality which we could not locate in the literature and therefore present here.

Theorem 3.4. *Suppose that T_ν is a trigonometric polynomial of degree $\le \nu$ in each variable and that $B \subset T^d = [-\pi, \pi]^d$ is any Lebesgue measurable set satisfying $\mu(B) \equiv |B| < (\pi/2)^d$. Then*

$$\|T_\nu\|_{L_\infty(T^d)} \le e^{4d\nu|B|^{1/d}} \|T_\nu\|_{L_\infty(T^d \setminus B)}. \tag{3.5}$$

Proof. We prove (3.5) using induction on d. For $d = 1$ (3.5) is well known, see [**1**, Theorem 5.1.2, page 230]. We assume (3.5) for $d - 1$. Using $\chi_B(x_1, \dots, x_d)$, the characteristic function of B, we define $g(x_1)$ by

$$g(x_1) := \int_{[-\pi,\pi]^{d-1}} \chi_B(x_1, x_2, \dots, x_d)\, dx_2 \cdots dx_d.$$

We set $B_1 = \{x_1 \in [-\pi, \pi] : g(x_1) \le |B|^{(d-1)/d}\}$ and $B(x_1) = \{(x_2, \dots, x_d) : (x_1, \dots, x_d) \in B\}$ and observe that for $x_1 \in B_1$ we have $|B(x_1)| \le |B|^{(d-1)/d}$. The induction hypothesis implies now for each $x_1 \in B_1$

$$\|T_\nu\|_{L_\infty(\{x_1\}\times[-\pi,\pi]^{d-1})} \le e^{4(d-1)\nu|B|^{1/d}} \|T_\nu\|_{L_\infty((\{x_1\}\times[-\pi,\pi]^{d-1})\setminus(\{x_1\}\times B(x_1)))},$$

and as

$$\bigcup_{x_1 \in B_1} (\{x_1\} \times [-\pi,\pi]^{d-1}) \setminus (\{x_1\} \times B(x_1)) \subset [-\pi,\pi]^d \setminus B,$$

we have

$$\|T_\nu\|_{L_\infty(B_1 \times [-\pi,\pi]^d)} \le e^{4(d-1)\nu|B|^{1/d}} \|T_\nu\|_{L_\infty([-\pi,\pi]^d \setminus B)}. \tag{3.6}$$

Using the definitions of $g(x_1)$ and of B_1, we have

$$\begin{aligned} |B| = \int_{-\pi}^{\pi} g(x_1)\,dx_1 &\ge \int_{[-\pi,\pi]\setminus B_1} g(x_1)\,dx_1 \\ &\ge \int_{[-\pi,\pi]\setminus B_1} |B|^{(d-1)/d}\,dx_1 = |[-\pi,\pi]\setminus B_1| \cdot |B|^{(d-1)/d}, \end{aligned}$$

and hence $|[-\pi,\pi]\setminus B_1| \le |B|^{1/d} < \pi/2$. For any fixed $(x_2,\ldots,x_d)$ we now use (3.5) with $d = 1$ to obtain

$$\|T_\nu\|_{L_\infty([-\pi,\pi]^d)} \le e^{4\nu|B|^{1/d}} \|T_\nu\|_{L_\infty(B_1 \times [-\pi,\pi]^{d-1})},$$

which, combined with (3.6), implies (3.5). □

B. Spherical polynomials on the unit sphere S^{d-1}.

The Nikol'skii result for these spherical polynomials is given in the following theorem:

Theorem 3.5. *For $f \in \mathcal{N}_\nu$, $\mathcal{N}_\nu$ the class of polynomials of degree $\le \nu$ on S^{d-1} (the unit sphere in $\mathbf{R}^d$), (3.3) and (3.4) hold with $\Omega = S^{d-1}$, $\beta = d-1$ and μ the Lebesgue measure on S^{d-1}.*

Proof. To prove Theorem 3.5 we recall that under the conditions of Theorem 3.5 and for a measurable set $B \subset S^{d-1}$ satisfying $\mu(B) = |B| \le 4/5$, Dai proved (see [**2**, (6.1)]) that

$$\|P_\nu\|_{L_\infty(S^{d-1})} \le e^{c\nu|B|^{1/(d-1)}} \|P_\nu\|_{L_\infty(S^{d-1}\setminus B)} \tag{3.7}$$

where $c > 0$ is independent of P_ν, ν and B. Using Theorem 3.1 with Ω, β and $\mathcal{N}_\beta$ of our theorem, we complete the proof. □

C. Algebraic polynomials on $[-1,1]$.

Theorem 3.6. *For $f \in \mathcal{N}_\nu$, $\mathcal{N}_\nu$ the class of algebraic polynomials of degree $\le \nu$ on $[-1,1]$, (3.3) and (3.4) hold with $\Omega = [-1,1]$, $\beta = 2$ and μ the Lebesgue measure on $[-1,1]$.*

Proof. We use the well-known Remez inequality (see [**1**, pages 228–230]), which establishes for $B \subset [-1,1]$ satisfying $|B| < 1/2$,

$$\|P_\nu\|_{L_\infty[-1,1]} \le e^{4\nu|B|^{1/2}} \|P_\nu\|_{L_\infty([-1,1]\setminus B)} \tag{3.8}$$

(in a somewhat different form). Theorem 3.1 can now be used with (3.8) taking the place of (3.1) to prove Theorem 3.6. □

4. Applications using Bernstein-type inequalities assuming $\mu(\Omega) < \infty$.

D. Algebraic polynomials on a class of domains in $\mathbf{R}^d$.

We define first the class of domains that will be dealt with.

Definition 4.1. For fixed θ, a and d satisfying $0 < \theta \le \pi/2$, $a > 0$ and d is an integer, a bounded closed domain Ω, $\Omega \subset \mathbf{R}^d$ is in the class $\mathcal{C}(\theta, a, d)$ if for any $\mathbf{x} \in \Omega$ there exists a $\mathbf{y} \in S^{d-1}$ ($\mathbf{y}$ depends on $\mathbf{x}$) for which

$$\{\mathbf{x} + \lambda\mathbf{z} : 0 \le \lambda \le a, \mathbf{z} \in S^{d-1}, \mathbf{z}\cdot\mathbf{y} \ge \cos\theta\} \subset \Omega.$$

For $\Omega \in \mathcal{C}(\theta, a, d)$ we now prove the following theorem:

Theorem 4.2. *For P_ν, any polynomial of total degree $\le \nu$, and Ω in the class $\mathcal{C}(\theta, a, d)$*

$$2P_\nu^*\Big(\frac{c(\theta,a,d)}{\nu^{2d}}\Big) \ge P_\nu^*(0). \tag{4.1}$$

Proof. We choose $\mathbf{x_0} \in \Omega$ such that $|P_\nu(\mathbf{x_0})| = \|P_\nu\|_{L_\infty(\Omega)}$ and write for $\mathbf{z} \in S^{d-1}$

$$P_\nu(\mathbf{x_0}+\lambda\mathbf{z}) = P_\nu(\mathbf{x_0}) + \lambda\frac{\partial}{\partial\mathbf{z}}P_\nu(\mathbf{x_0}+\eta\mathbf{z}) \quad \text{for some } \eta \text{ between } 0 \text{ and } \lambda.$$

The Markov inequality yields

$$\sup_{0\le\eta\le a}\left|\frac{\partial}{\partial\mathbf{z}}P_\nu(\mathbf{x_0}+\eta\mathbf{z})\right| \le \frac{2\nu^2}{a}\sup_{0\le\eta\le a}\left|P_\nu(\mathbf{x_0}+\eta\mathbf{z})\right| \le \frac{2\nu^2}{a}\|P_\nu\|_{L_\infty(\Omega)}.$$

Hence, for $S=\{\mathbf{x_0}+\lambda\mathbf{z} : 0\le\lambda<(a/4\nu^2), \mathbf{z}\cdot\mathbf{y}\ge\cos\theta, \mathbf{z}\in S^{d-1}\}$, where $\mathbf{y}$ is given in Definition 4.1 for $\mathbf{x}=\mathbf{x_0}$, one has

$$|P_\nu(\mathbf{x_0}+\lambda\mathbf{z})| \ge \|P_\nu\|_{L_\infty(\Omega)} - \frac{1}{2}\|P_\nu\|_{L_\infty(\Omega)} = \frac{1}{2}\|P_\nu\|_{L_\infty(\Omega)}.$$

We now have $|S|=c(\theta,a,d)\nu^{-2d}$ and thus (4.1). □

Theorem 4.2 implies that P_ν, the polynomials of total degree $\le\nu$ on Ω, is in the class $\mathcal{A}_{\nu^{2d}/c}(2)$ (see (1.5)), and this implies the following Nikol'skii-type inequality.

Theorem 4.3. *For P_ν, any polynomial of total degree $\le\nu$, and Ω in the class $\mathcal{C}(\theta,a,d)$, (3.3) and (3.4) hold with $\beta=2d$.*

Remark 4.4. In Theorem 4.3 we still have $c_1(A,p_i,q_i)=O((p_1/(p_1-p_2))^{1/q_2})$ as $p_2-p_1\to 0+$ when $q_2<q_1$, and $c_1(A,p_i,q_i)=O(1)$ in most other cases, but as $p_1\to 0$, c_1 may diverge.

E. Splines on $[0,1]$.

The class of spline functions $S_{\nu,k}$ is given by $0=x_0<x_1<\cdots<x_m=1$, $f(x)=P_i(x)$ on $(x_i,x_{i+1})=I_i$ where $P_i(x)$ is a polynomial of degree $\le k$, for $1\le i\le m-1$ and $\nu^{-1}=\min_{0\le i\le m-1}x_{i+1}-x_i$. We note that here we did not assume anything about $\max_{0\le i\le m-1}x_{i+1}-x_i$ or the continuity of f in $S_{\nu,k}$ as assumptions of this nature are imposed for the sake of approximation by splines, not for the Nikol'skii-type inequality for splines.

Theorem 4.4. *For $f\in S_{\nu,k}$ we have*

$$2f^*\left(\frac{1}{4\nu k^2}\right)\ge f^*(0). \tag{4.2}$$

Proof. For $y \in [0,1]$ satisfying $|f(y)| = \|f\|_{L_\infty[0,1]}$ we write $f(y+\lambda) = f(y) + \lambda f'(y+\eta)$ for $y+\eta$ between y and $y+\lambda$ where $y, y+\lambda \in I_i$ and where $|f'(y+\eta)| \le \|f'\|_{L_\infty(I_i)} \le 2\nu k^2 \|f\|_{L_\infty(I_i)} \le 2\nu k^2 \|f\|_{L_\infty[0,1]}$. As $y \in I_i$ for some i, we can choose λ_0, $|\lambda_0| = 1/(4\nu k^2)$ such that $y + \lambda_0 \in I_i$. For $|\lambda| \le |\lambda_0|$ and $\lambda \cdot \lambda_0 > 0$ we now have $|f(y+\lambda)| \ge |f(y)| - (1/2)\|f\|_{L_\infty[0,1]}$ which implies (4.2). □

As a result of (4.2), which means that (1.5) holds with $\gamma = 2$ and $\sigma = 4\nu k^2$ for any $f \in S_{\nu,k}$, we have the following Nikol'skii-type inequalities.

Theorem 4.5. *For $f \in S_{\nu,k}$ (3.3) and (3.4) hold with $\beta = 1$.*

5. Applications in case $\mu(\Omega) = \infty$. In this section we present two applications when $\mu(\Omega) = \infty$. We use Bernstein-type inequalities, and an estimate following (3.4) is not achieved (see also (2.13)).

F. Entire functions of exponential type.

Definition 5.1. The functions of exponential type R, on $\mathbf{R}^d$, $\mathcal{N}_R$, are those given by

$$f(\mathbf{x}) = \int_{|\mathbf{y}| \le R} e^{i\mathbf{x}\cdot\mathbf{y}}\, dm(\mathbf{y}), \tag{5.1}$$

where $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{y} = (y_1, \dots, y_d)$, $|\mathbf{y}|^2 = y_1^2 + \cdots + y_d^2$ and $dm(\mathbf{y})$ is a measure satisfying

$$\int_{|\mathbf{y}| \le R} |\, dm(\mathbf{y})| \le K$$

for some K.

For any $\xi \in S^{d-1}$ and f satisfying (5.1),

$$\left\| \frac{\partial}{\partial \xi} f \right\|_{L_\infty(\mathbf{R}^d)} \le cR \|f\|_{L_\infty(\mathbf{R}^d)}, \tag{5.2}$$

which follows for instance from

$$\|\Delta f\|_{L_\infty(\mathbf{R}^d)} \le c_1 R^2 \|f\|_{L_\infty(\mathbf{R}^d)}, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$$

given in [**3**, Theorem 2.1] and the Kolmogorov inequality (see [**7**])

$$\left\| \frac{\partial}{\partial \xi} f \right\|_{L_\infty(\mathbf{R}^d)} \le \sqrt{2} \|\Delta f\|^{1/2}_{L_\infty(\mathbf{R}^d)} \|f\|^{1/2}_{L_\infty(\mathbf{R}^d)}.$$

We now choose $\mathbf{x_0}$, for which $\|f\|_{L_\infty(\mathbf{R}^d)} = |f(\mathbf{x_0})|$, and as

$$f(\mathbf{x_0} + \lambda \mathbf{y}) = f(\mathbf{x_0}) + \lambda \frac{\partial}{\partial \mathbf{y}} f(\mathbf{x_0} + \eta \mathbf{y})$$
$$\text{for } \mathbf{y} \in S^{d-1} \text{ and } 0 < \eta = \eta(\mathbf{y}, \lambda) < \lambda,$$

we have

$$|f(\mathbf{x_0} + \lambda \mathbf{y})| \ge \|f\|_\infty - \frac{\|f\|_\infty}{2} = \frac{1}{2} \|f\|_\infty$$

for $\lambda \le 1/(2R)$ and $\mathbf{y} \in S^{d-1}$. Therefore,

$$2 f^* \left(\frac{M}{R^d} \right) \ge f^*(0)$$

where M is the volume of the unit ball in $\mathbf{R}^d$ divided by 2^d. As a corollary of the above, we have:

Theorem 5.2. *For f satisfying* (5.1) *and* $0 < p_1 < p_2 < \infty$, $0 < q_i \le \infty$,

$$\|f\|_{L_{p_2, q_2}(\mathbf{R}^d)} \le c R^{d((1/p_1) - (1/p_2))} \|f\|_{L_{p_1, q_1}(\mathbf{R}^d)} \tag{5.3}$$

where $c = c(p_i, q_i, d)$.

G. Polynomials on $\mathbf{R}$ with Freud weights.

We deal here with the most prominent of Freud weights, $W_\alpha(x) \equiv e^{-|x|^\alpha}$ with $\alpha > 1$. For polynomials P_n of degree $\le n$, one has

$$\|P_n W_\alpha\|_{L_\infty[-a_n, a_n]} = \|P_n W_\alpha\|_{L_\infty(\mathbf{R})} \tag{5.4}$$

and (see [**5**, Theorem 7.4])

$$\|(P_n W_\alpha)'\|_{L_\infty[-a_n, a_n]} \le c n^{1-(1/\alpha)} \|P_n W_\alpha\|_{L_\infty(\mathbf{R})} \tag{5.5}$$

where a_n are the Mhaskar-Rakhmanov-Saff numbers satisfying $a_n \approx n^{1-(1/\alpha)}$. In fact, a better estimate is given in [**5**] near $\pm a_n$. For x_0 for which $|P_n(x_0)W_\alpha(x_0)| = \|P_nW_\alpha\|_{L_\infty[-a_n,a_n]}$, we use the Taylor formula (following earlier computations) to establish

$$(5.6)\qquad 2f^*\left(\frac{1}{2cn^{1-(1/\alpha)}}\right) \geq f^*(0) \quad \text{for } f = P_nW_\alpha,$$

which is (1.5) with $\gamma = 2$ and $\sigma = 1/(2cn^{1-(1/\alpha)})$. Therefore, we have:

Theorem 5.3. *For P_n, any polynomial of degree $\leq n$, and $W_\alpha(x) = e^{-|x|^\alpha}$ with $\alpha > 1$, one has*

$$(5.7)\quad \|P_nW_\alpha\|_{L_{p_2,q_2}(\mathbf{R})} \leq c_1 n^{(1-(1/\alpha))((1/p_1)-(1/p_2))}\|P_nW_\alpha\|_{L_{p_1,q_1}(\mathbf{R})}$$

for $0 < p_1 < p_2 < \infty$ (where $c_1 = c_1(p_i, q_i, \alpha)$ following Theorem 2.3).

6. Further remarks.

Remark 6.1. The Remez inequality leads to $\gamma = e^{-A\alpha}$ and $\sigma = (\nu/\alpha)^\beta$ (see (3.2)) with our choice of α, and this usually leads to better results via Theorem 2.3 than the method used in Sections 4 and 5 where the possible γ is in a much tighter range.

Remark 6.2. In particular, the constant c_1 in (3.3) tends to zero as $p_1 \to 0$ and p_2 is fixed when we use the Remez inequality. This, however, does not always occur when using the Bernstein-type inequality as in Sections 4 and 5. When $p_i = q_i$, the common method (see for instance [**4**, Section 6]) sometimes yields the same consequence.

Remark 6.3. In the classic Nikol'skii inequality $\|T_n\|_{L_\infty(T)} \leq c(p)n^{1/p}\|T_n\|_{L_p(T)}$, $c(p) \leq 1.032$. To show that, we write without loss of generality $\|T_n\|_{L_\infty(T)} = T_n(0) = 1$, and as $\|T_n''\|_{L_\infty} \leq n^2$, we have $T_n(x) \geq 1 - (n^2x^2)/2$ for $|x| \leq \sqrt{2}/n$. We now calculate the L_p norm of T_n to obtain

$$c(p) \leq \left(\sqrt{2}\int_0^1 (1-u)^p \frac{du}{\sqrt{u}}\right)^{-1/p} = \left(\frac{\sqrt{2\pi}\Gamma(p+1)}{\Gamma(p+(3/2))}\right)^{-1/p},$$

which, using numerical computations, implies $c(p) < 1.032$. We note that, while one is tempted to try and show $c(p) \le 1$, this is not so. Suppose $\|T_n\|_\infty = 1$ and $c(p) \le 1$. Then

$$1 = \|T_n\|_{L_\infty}^p \le n \int_{-\pi}^{\pi} |T_n(x)|^p \, dx,$$

and, as for $T_n \not\equiv \pm 1$, $|T_n(x)| < 1$ for almost all x, and hence the right-hand side of the last equation tends to zero as $p \to \infty$ which contradicts $c(p) \le 1$. We note that even the classical estimate

$$\|T_n\|_\infty \le \left(\frac{2n+1}{2\pi}\right)^{1/p} \|T_n\|_p$$

for $p \le 2$ implies $c(p) = o(1)$ as $p \to 0$.

Remark 6.4. In most classical cases the Nikol'skii classes are nested. This condition was not necessary for the proofs here, and while in most applications here the classes $\mathcal{N}_\nu$ are nested, in the application E (Theorems 4.5 and 4.6) which concerns splines on $[0, 1]$, they are not.

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