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THETA FUNCTIONS ON THE THETA DIVISOR

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ABSTRACT. We show that the gradient and the Hessian of the Riemann theta function in dimension n can be combined to give a theta function of order $n+1$ and modular weight $(n+5)/2$ defined on the theta divisor. It can be seen that the zero locus of this theta function essentially gives the ramification locus of the Gaussian map. For Jacobians this leads to a description in terms of theta functions and their derivatives of the Weierstrass point locus on the associated Riemannian surface.

In the analytic theory of the Riemann theta function a natural place is taken by the study of the first and second order terms of its Taylor series expansion along the theta divisor. The first order term essentially gives the gradient, and hence the tangent bundle on the smooth locus, whereas the second order terms give rise to Hessians, which are widely recognized as carrying subtle geometric information along the singular locus of the theta divisor. For example, in the case of a Jacobian, these Hessians define quadrics containing the canonical image of the associated Riemann surface. Or, in the general case, one could investigate the properties of those principally polarized Abelian varieties that have a singular point of order two on their theta divisor, such that the Hessian of the theta function at that singular point has a certain given rank. This is a recent line of investigation begun by Grushevsky and Salvati Manni, with interesting connections to the Schottky problem [**6, 7**].

If one moves outside the singular locus of the theta divisor, it is not immediately clear whether the Hessian of the theta function continues to have some geometric significance. In this paper we prove that it does. More precisely, we show that a certain combination of the gradient and the Hessian gives rise to a well-defined theta function living on the theta divisor. We can compute its transformation behavior, i.e., its order and

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automorphy factor explicitly. As it turns out, the geometric meaning of our function is that on the smooth locus of the theta divisor it precisely gives the ramification locus of the Gaussian map, i.e., the map sending a smooth point to its tangent space, seen as a point in the dual of the projectivized tangent space of the origin of the ambient Abelian variety. We can study the situation in more detail for Jacobians, where our theory leads to a global description of the locus of Weierstrass points on the corresponding Riemann surface in terms of theta functions and their derivatives. By way of example, we state and prove an explicit formula for our theta function in genus two.

1. Definition and main theorem. Let $\mathbf{H}_n$ denote the Siegel upper half space of degree $n > 0$. On $\mathbf{C}^n \times \mathbf{H}_n$, we have the Riemann theta function

$$\theta = \theta(z, \tau) = \sum_{m \in \mathbf{Z}^n} e^{\pi i {}^t m \tau m + 2\pi i {}^t m z}.$$

Here and henceforth, vectors are column vectors and ^{t} denotes transpose. For any fixed τ, the function $\theta = \theta(z)$ on $\mathbf{C}^n$ satisfies the identity

$$\theta(z + \tau u + v) = e^{-\pi i {}^t u \tau u - 2\pi i {}^t u z} \theta(z), \tag{1.1}$$

for all z in $\mathbf{C}^n$ and for all u, v in $\mathbf{Z}^n$. Moreover, it has a symmetry property

$$\theta(-z) = \theta(z) \tag{1.2}$$

for all z in $\mathbf{C}^n$. Equation (1.1) implies that $\operatorname{div} \theta$ is well defined as a Cartier divisor on the complex torus $A = \mathbf{C}^n/(\mathbf{Z}^n + \tau \mathbf{Z}^n)$. We denote this divisor by Θ. In fact, θ gives rise to a global section of $O_A(\Theta)$, and we say that θ is a "theta function of order 1." By equation (1.2), the divisor Θ is symmetric: $[-1]_A^* \Theta = \Theta$.

For τ varying through $\mathbf{H}_n$, it turns out that $\theta = \theta(z, \tau)$ is a modular form of weight $1/2$. More precisely, consider the group $\Gamma_{1,2}$ of matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ in $\mathrm{Sp}\,(2n, \mathbf{Z})$ with a, b, c, d square matrices such that the diagonals of both ${}^t ac$ and ${}^t bd$ consist of even integers. We can let $\Gamma_{1,2}$ act on $\mathbf{C}^n \times \mathbf{H}_n$ via

$$(z, \tau) \longmapsto \left({}^t(c\tau + d)^{-1} z, (a\tau + b)(c\tau + d)^{-1} \right).$$

Under this action, the Riemann theta function transforms as

$$\theta({}^t(c\tau+d)^{-1}z,(a\tau+b)(c\tau+d)^{-1}) = \zeta_\gamma \det(c\tau+d)^{1/2} e^{\pi i\, {}^t z (c\tau+d)^{-1} c z}\theta(z,\tau) \tag{1.3}$$

for some eighth root of unity ζ_γ, cf., [**10**, page 189].

We write θ_i for the first order partial derivative $\partial\theta/\partial z_i$ and θ_{ij} for the second order partial derivative $\partial^2\theta/\partial z_i\partial z_j$. If (h_{ij}) is any square matrix, we denote by $(h_{ij})^c = (h^c_{ij})$ its cofactor matrix, i.e.,

$$h^c_{ij} = (-1)^{i+j}\det\,(h_{kl})_{\substack{k\neq i\\ l\neq j}}.$$

The function we want to study in this paper is then the following.

Definition 1.1. Let (θ_i) be the gradient of θ in the $\mathbf{C}^n$-direction, and let (θ_{ij}) be its Hessian. Then we put

$$\eta = \eta(z,\tau) = {}^t(\theta_i)(\theta_{ij})^c(\theta_j). \tag{1.4}$$

We want to consider this as a function on the vanishing locus $\theta^{-1}(0)$ of θ on $\mathbf{C}^n \times \mathbf{H}_n$.

Example 1.2. For $n=1$, we obtain

$$\eta = \left(\frac{d\theta}{dz}\right)^2$$

viewed as a function of (z,τ) in $\mathbf{C}\times\mathbf{H}$ with $z \equiv (1+\tau)/2 \bmod \mathbf{Z}+\tau\mathbf{Z}$. For $n=2$, we obtain

$$\eta = \theta_{11}\theta_2^2 - 2\theta_{12}\theta_1\theta_2 + \theta_{22}\theta_1^2,$$

which is already somewhat more complicated.

Our main result is that the function η transforms well with respect to both lattice translations and the action of the congruence symplectic group $\Gamma_{1,2}$.

Theorem 1.3. *The function $\eta = \eta(z,\tau)$ is a theta function of order $n+1$ and weight $(n+5)/2$ on the theta divisor. In other words, for any fixed τ in $\mathbf{H}_n$, the function η gives rise to a global section of the line bundle $O_\Theta(\Theta)^{\otimes n+1}$ on Θ in $A = \mathbf{C}^n/(\mathbf{Z}^n + \tau\mathbf{Z}^n)$. Furthermore, when viewed as a function of two variables (z,τ), the function η transforms under the action of $\Gamma_{1,2}$ with an automorphy factor* $\det(c\tau+d)^{(n+5)/2}$ *on $\theta^{-1}(0)$.*

It follows that, for any fixed τ, the zero locus of η is well-defined on Θ. This zero locus contains $\operatorname{Sing}\Theta$, the singular locus of Θ, as well as the set of 2-division points $\Theta \cap A[2]$ if $n \geq 2$. Moreover, this zero locus is stable under the involution $z \mapsto -z$ of Θ.

For τ varying through $\mathbf{H}_n$, it follows that the function η gives rise to a global section of a line bundle

$$L = O_\Theta(\Theta)^{\otimes n+1} \otimes \pi^* M$$

on the canonical symmetric theta divisor Θ of a universal principally polarized Abelian variety with level structure $\pi : \mathcal{U}_n \to \mathcal{A}_n^{(1,2)} = \Gamma_{1,2} \setminus \mathbf{H}_n$. Here M is a certain line bundle on $\mathcal{A}_n^{(1,2)}$. It follows from general principles that M is a power of λ, the determinant of the Hodge bundle on $\mathcal{A}_n^{(1,2)}$. By counting weights we find that $M \cong \lambda^{\otimes 2}$.

Example 1.4. When $n = 1$, the theorem states that $d\theta/dz$ is of order 1 and of modular weight $3/2$ on the theta divisor. Both statements can be checked directly from (1.1) and (1.3). Alternatively, the statement on the modular weight can be seen using Jacobi's derivative formula, cf. [**10**, page 64]. This formula says that

$$e^{\pi i\tau/4}\frac{d\theta}{dz}\left(\frac{1+\tau}{2}\right) = \pi i \theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1/2\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1/2\\0\end{bmatrix}(0,\tau),$$

where $\theta\begin{bmatrix}0\\0\end{bmatrix}$, $\theta\begin{bmatrix}0\\1/2\end{bmatrix}$ and $\theta\begin{bmatrix}1/2\\0\end{bmatrix}$ are the usual elliptic theta functions with even characteristic. Each of the three Thetanullwerte $\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)$, $\theta\begin{bmatrix}0\\1/2\end{bmatrix}(0,\tau)$ and $\theta\begin{bmatrix}1/2\\0\end{bmatrix}(0,\tau)$ is a modular form of weight $1/2$.

A proof of Theorem 1.3 will be given in Section 4.

Remark 1.5. It should be noted that Grushevsky and Salvati Manni propose in [**7**] a simpler definition

$$\eta = \eta(z, \tau) = \det \begin{pmatrix} \theta_{ij} & \theta_j \\ {}^t\theta_i & 0 \end{pmatrix}$$

for η. They also suggest there an alternative approach for a proof of Theorem 1.3.

2. Properties. In this section we collect some properties of η.

Proposition 2.1. *Assume that $\Theta = \operatorname{div}\theta$ on $\mathbf{C}^n/(\mathbf{Z}^n + \tau\mathbf{Z}^n)$ is nonsingular, and denote its canonical bundle by K_Θ. Then η gives rise to a global section $\widetilde{\eta}$ of $K_\Theta^{\otimes n+1}$, locally given by*

$$\widetilde{\eta}(z) = \eta(z) \cdot \left((-1)^{i-1} \frac{dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n}{\theta_i(z)} \right)^{\otimes n+1}$$

wherever $\theta_i(z)$ is nonzero.

Proof. We view η as a global section of the line bundle $O_\Theta(\Theta)^{\otimes n+1}$ on Θ. By the adjunction formula and the fact that the canonical bundle K_A of A is trivial, we have an identification

$$O_\Theta(\Theta) \cong (K_A \otimes O_A(\Theta))|_\Theta \cong K_\Theta.$$

This identification can be represented locally by the Poincaré residue map $K_A \otimes O_A(\Theta) \to K_\Theta$ given by

$$\frac{dz_1 \wedge \cdots \wedge dz_n}{\theta(z)} \longmapsto (-1)^{i-1} \frac{dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n}{\theta_i(z)}$$

wherever $\theta_i(z) \neq 0$. From this the corollary follows. □

By way of illustration, we make the multi-differential $\widetilde{\eta}$ more explicit in the case that $n = 2$. We need the notions of a Wronskian differential and of Weierstrass points, which we recall briefly. Let X be a compact Riemann surface of genus $n > 0$, and let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be any basis of $H^0(X, K_X)$. The Wronskian differential ω_ζ of ζ is then given as follows: let t be a local coordinate, and write $\omega_j(t) = f_j(t)\,dt$ with the f_j holomorphic. We put

$$\omega_\zeta(t) = \det\left(\frac{1}{(i-1)!}\frac{d^{i-1}f_j(t)}{dt^{i-1}}\right)\cdot (dt)^{\otimes n(n+1)/2}.$$

This local definition gives, in fact, rise to a global section of the line bundle $K_X^{\otimes n(n+1)/2}$. It can be proven that this section is nonzero. If the basis ζ is changed, the differential ω_ζ changes by a nonzero scalar. Hence the divisor $W = \operatorname{div}\omega_\zeta$ is independent of the choice of ζ. We call this divisor the classical divisor of Weierstrass points of X. It has degree $n^3 - n$.

Example 2.2. Assume that $A = \mathbf{C}^2/(\mathbf{Z}^2 + \tau\mathbf{Z}^2)$ is an indecomposable Abelian surface with theta divisor $\Theta = \operatorname{div}\theta$. Then Θ is a compact Riemann surface of genus 2, and there is a canonical identification $H^0(A, \Omega^1) \cong H^0(\Theta, K_\Theta)$. Let $\zeta = (\zeta_1(z), \zeta_2(z))$ be the basis of $H^0(\Theta, K_\Theta)$ corresponding under this identification to the standard basis (dz_1, dz_2) of $H^0(A, \Omega^1)$. We claim that $\widetilde{\eta}(z)$ is equal to the Wronskian differential ω_ζ of ζ. This amounts to a small computation: choose an open subset of Θ where z_1 is a local coordinate. In this local coordinate we can write $\zeta_1(z) = dz_1$, $\zeta_2(z) = z_2'(z)\cdot dz_1$, with $'$ denoting derivative with respect to z_1, so that

$$\omega_\zeta(z) = \det\begin{pmatrix} 1 & z_2'(z) \\ 0 & z_2''(z)\end{pmatrix}\cdot (dz_1)^{\otimes 3} = z_2''(z)\cdot (dz_1)^{\otimes 3}.$$

Now, since for z on Θ we have

$$\theta_1(z)\,dz_1 + \theta_2(z)\,dz_2 = 0,$$

the formula

$$z_2'(z) = -\frac{\theta_1(z)}{\theta_2(z)}$$

holds, leading to

$$\begin{aligned}
z_2''(z) &= -\frac{\theta_1'(z)\theta_2(z) - \theta_2'(z)\theta_1(z)}{\theta_2(z)^2} \\
&= -\frac{(\theta_{11}(z) + \theta_{12}(z) \cdot -(\theta_1(z)/\theta_2(z)))\,\theta_2(z)}{\theta_2(z)^2} \\
&\quad - \frac{(\theta_{12}(z) + \theta_{22}(z) \cdot -(\theta_1(z)/\theta_2(z)))\,\theta_1(z)}{\theta_2(z)^2} \\
&= -\frac{\theta_{11}(z)\theta_2^2(z) - 2\theta_{12}(z)\theta_1(z)\theta_2(z) + \theta_{22}(z)\theta_1^2(z)}{\theta_2(z)^3} \\
&= -\frac{\eta(z)}{\theta_2(z)^3}.
\end{aligned}$$

We find, indeed,

$$\omega_\zeta(z) = -\frac{\eta(z)}{\theta_2(z)^3} \cdot (dz_1)^{\otimes 3} = \widetilde{\eta}(z).$$

A similar computation can be done on the locus where z_2 is a local coordinate, giving $\omega_\zeta(z) = \widetilde{\eta}(z)$ globally on Θ, as required. Note that our identification $\omega_\zeta(z) = \widetilde{\eta}(z)$ gives, as a corollary, that $\operatorname{div}\eta = W$, the divisor of Weierstrass points of Θ. In Section 5 we will prove a generalization of this result.

It is interesting to know when η is identically zero on the theta divisor.

Proposition 2.3. *If A is a decomposable principally polarized Abelian variety, then η is identically zero on the theta divisor.*

Proof. Let us suppose that $A = A_1 \times A_2$ with A_1 given by a matrix τ_1 in $\mathbf{H}_k$ and A_2 given by a matrix τ_2 in $\mathbf{H}_{n-k}$ where k is an integer with $0 < k < n$. We can write $\theta(z) = F(z_1, \ldots, z_k)G(z_{k+1}, \ldots, z_n)$ where F, G are the Riemann theta functions for A_1 and A_2, respectively. Let $\Theta_1 \subset A_1$ be the divisor of F, and let $\Theta_2 \subset A_2$ be the divisor of G. By symmetry, it suffices to prove that η is zero on $\Theta_1 \times A_2 \subset \Theta = \operatorname{div}\theta$. On $\mathbf{C}^n$ we have

$${}^t(\theta_i) = (F_i G,\, F G_i)$$

and

$$(\theta_{ij}) = \begin{pmatrix} F_{ij}G & F_iG_j \\ F_jG_i & FG_{ij} \end{pmatrix}.$$

The subset $\Theta_1 \times A_2 \subset \Theta$ is given by the vanishing of F; there we find

$${}^t(\theta_i) = (F_iG, 0), \quad (\theta_{ij}) = \begin{pmatrix} F_{ij}G & F_iG_j \\ F_jG_i & 0 \end{pmatrix}.$$

Note that both F_iG_j and F_jG_i, being a product of a vector and a covector, have rank ≤ 1. Hence, a minor at (i,j) with $1 \leq i, j \leq k$ in (θ_{ij}) has rank $\leq 1 + (k-1) = k$. If $k < n-1$, the determinant of this minor vanishes, and the cofactor matrix of (θ_{ij}) has the shape

$$(\theta_{ij})^c = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}.$$

We find then that

$$\eta = {}^t(\theta_i)(\theta_{ij})^c(\theta_j) = (*, 0) \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} \begin{pmatrix} * \\ 0 \end{pmatrix} = 0.$$

If $k = n-1$, the last row of (θ_{ij}) and the vector ${}^t(\theta_i)$ are linearly dependent. If $1 \leq i \leq n-1$, we see that the ith entry of $(\theta_{ij})^c(\theta_j)$ is just the determinant of the matrix obtained from (θ_{ij}) by removing the ith row and adding ${}^t(\theta_i)$ in its place. But this matrix then contains two linearly dependent vectors, and its determinant vanishes. So we obtain

$$\eta = {}^t(\theta_i)(\theta_{ij})^c(\theta_j) = (*, 0) \begin{pmatrix} 0 \\ * \end{pmatrix} = 0,$$

in this case as well. $\square$

3. Interpretation. The contents of the present section are based on kind suggestions made by Professor Ciro Ciliberto.

As we observed in Section 1, for any complex principally polarized Abelian variety the zero locus of η is well-defined on the theta divisor. Given the simple description of η, one expects that this zero locus has some intrinsic geometric interpretation. This indeed turns out to be the case.

Theorem 3.1. *Let (A, Θ) be a complex principally polarized Abelian variety. On the smooth locus Θ^s of Θ, the zero locus of η is precisely the ramification locus of the Gauss map*

$$\Gamma : \Theta^s \longrightarrow \mathbf{P}(T_0A)^*$$

sending a point on Θ^s to its tangent space, translated to a subspace of T_0A.

Proof. It follows from formula (1.4) that a point x on Θ^s is in the zero locus of η if and only if the quadric Q in $\mathbf{P}(T_0A)$ defined by the Hessian is tangent to the projectivized tangent hyperplane $\mathbf{P}(T_x\Theta)$ defined by the gradient. The latter condition is equivalent to the condition that Q when restricted to $\mathbf{P}(T_x\Theta)$ becomes degenerate. Now note that Q when viewed as a linear map can be identified with the tangent map $d\Gamma : T_x\Theta \to T_{\Gamma(x)}(\mathbf{P}(T_0A)^*) = (T_x\Theta)^*$ of Γ. The locus where this map is degenerate is precisely the ramification locus of Γ. □

Using the above interpretation, a converse to Proposition 2.3 can be readily proved.

Corollary 3.2. *We have that η is identically zero on the theta divisor if and only if A is a decomposable Abelian variety.*

Proof. We need to prove that if Θ is irreducible, then the Gaussian map on Θ has a proper ramification locus. But, according to [**9**, Corollary 9.11] the Gaussian map is generically finite and dominant in this case, and the result follows. □

Remark 3.3. It follows from Theorem 1.3 that in the indecomposable case the divisor of η belongs to the linear system defined by $(n+1)\Theta$ on Θ. This fact can also be explained as follows. For simplicity, let us assume that Θ is nonsingular. The Hurwitz formula applied to the Gaussian map $\Gamma : \Theta \to \mathbf{P} = \mathbf{P}(T_0A)^*$ gives that $K_\Theta = \Gamma^* K_{\mathbf{P}} + R$, where R is the ramification locus of Γ. By definition, $\Gamma^* O_{\mathbf{P}}(1) \cong K_\Theta$, and hence $\Gamma^* K_{\mathbf{P}} \cong K_\Theta^{\otimes -n}$. We find that R is in the linear system belonging to $K_\Theta^{\otimes n+1}$. As we have seen at the end of the previous section, this is the same as $O_\Theta(\Theta)^{\otimes n+1}$. Alternatively, this remark shows that the equations $\theta = \eta = 0$ define the right scheme structure on R.

Remark 3.4. The description of the vanishing locus of η given in this section is also mentioned in the paper [**7**] of Grushevsky and Salvati Manni. In that paper an interesting application is given of the form η to the study of certain codimension-2 cycles in the moduli space of principally polarized Abelian varieties. In particular, one finds there a moduli interpretation of a certain cycle R_g introduced by Debarre in [**2**], Section 4.

4. Proof of the main theorem. In this section we give a proof of Theorem 1.3. We start with the statement on the order of η. We fix an element τ in $\mathbf{H}_n$. The case $n = 1$ being discussed already in Example 1.4, we assume here that $n \geq 2$. Recall from equation (1.1) that

$$\theta(z + \tau u + v) = p(z,u)\theta(z) \tag{4.1}$$

for all z in $\mathbf{C}^n$ and all u, v in $\mathbf{Z}^n$, where

$$p(z,u) = e^{-\pi i {}^t u \tau u - 2\pi i {}^t u z}.$$

We need to prove that

$$\eta(z + \tau u + v) = p(z,u)^{n+1}\eta(z)$$

for all z in $\mathbf{C}^n$ with $\theta(z) = 0$. Denote by p_i the first order partial derivative $\partial p/\partial z_i$. From (4.1) we have, for z with $\theta(z) = 0$,

$$\theta_i(z + \tau u + v) = p(z,u)\theta_i(z)$$

and

$$\theta_{ij}(z + \tau u + v) = p(z,u)\theta_{ij}(z) + p_i(z,u)\theta_j(z) + p_j(z,u)\theta_i(z).$$

We are done, therefore, if we can prove, formally in some domain R containing the symbols $p, p_i, \theta_i, \theta_{ij}$, the identity

$${}^t(p\theta_i)(p\theta_{ij} + p_i\theta_j + p_j\theta_i)^c(p\theta_j) = p^{n+1}\, {}^t(\theta_i)(\theta_{ij})^c(\theta_j)$$

or, equivalently, the identity

$$^t(\theta_i)(p\theta_{ij}+p_i\theta_j+p_j\theta_i)^c(\theta_j)={}^t(\theta_i)(p\theta_{ij})^c(\theta_j).$$

At this point we introduce some notation. Let $h=(h_{i_kj_l})$ be an R-valued matrix with rows and columns indexed by finite-length ordered integer tuples $I=(\dots,i_k,\dots)$ and $J=(\dots,j_l,\dots)$, respectively. If $I'\subset I$ and $J'\subset J$ are proper subtuples, we denote by $h^{I'}_{J'}$ the submatrix obtained from h by deleting the rows indexed by I' and the columns indexed by J'. If h is a square R-valued matrix with both the row index set $I=(i_1,\dots,i_m)$ and the column index set $J=(j_1,\dots,j_m)$ subtuples of $(1,2,\dots,n)$, we define $\eta(h)$ to be the element

$$\eta(h)={}^t(\theta_{i_k})(h_{i_kj_l})^c(\theta_{j_l})$$

of R. It can be written more elaborately as

$$\eta(h)=\sum_{k=1}^m\sum_{l=1}^m(-1)^{k+l}\theta_{i_k}\theta_{j_l}\det h^{\{i_k\}}_{\{j_l\}}.$$

On the other hand, the identity we need to prove can be written more compactly as

$$\eta(p\theta_{ij}+p_i\theta_j+p_j\theta_i)=\eta(p\theta_{ij}).$$

The function η can be defined recursively.

Lemma 4.1. *Let $h=(h_{i_kj_l})$ be a square R-valued matrix of size $m\geq 2$ with both the row index set $I=(i_1,\dots,i_m)$ and the column index set $J=(j_1,\dots,j_m)$ subtuples of $(1,2,\dots,n)$. Then the identity*

$$(m-1)\eta(h)=\sum_{k=1}^m\sum_{l=1}^m(-1)^{k+l}h_{i_kj_l}\eta\left(h^{\{i_k\}}_{\{j_l\}}\right)$$

holds.

Proof. The double sum on the righthand side can be expanded as

$$\begin{aligned}
&\sum_{(k,l)}(-1)^{k+l}h_{i_kj_l}\eta\left(h^{\{i_k\}}_{\{j_l\}}\right)\\
&\qquad=\sum_{(k,l)}(-1)^{k+l}h_{i_kj_l}\sum_{(k',l')\neq(k,l)}(-1)^{k'+l'}\theta_{i_{k'}}\theta_{j_{l'}}\det h^{\{i_k,i_{k'}\}}_{\{j_l,j_{l'}\}}\\
&\qquad=\sum_{(k',l')}(-1)^{k'+l'}\theta_{i_{k'}}\theta_{j_{l'}}\sum_{(k,l)\neq(k',l')}(-1)^{k+l}h_{i_kj_l}\det h^{\{i_{k'},i_k\}}_{\{j_{l'},j_l\}}.
\end{aligned}$$

Expanding the determinant of $h^{\{i_{k'}\}}_{\{j_{l'}\}}$ along each of its rows, one finds

$$(m-1)\det h^{\{i_{k'}\}}_{\{j_{l'}\}} = \sum_{(k,l)\neq(k',l')} (-1)^{k+l} h_{i_k j_l} \det h^{\{i_{k'},i_k\}}_{\{j_{l'},j_l\}}.$$

Combining both formulas, one gets

$$\begin{aligned}\sum_{(k,l)}(-1)^{k+l} h_{i_k j_l}\eta\left(h^{\{i_k\}}_{\{j_l\}}\right) &= (m-1)\sum_{(k',l')}(-1)^{k'+l'}\theta_{i_{k'}}\theta_{j_{l'}}\det h^{\{i_{k'}\}}_{\{j_{l'}\}}\\ &= (m-1)\eta(h),\end{aligned}$$

as required. □

Denote by $a=(a_{ij})$ the n-by-n R-valued matrix

$$(a_{ij}) = (p_i\theta_j + p_j\theta_i).$$

It has the property that the ηs of all its square submatrices vanish.

Lemma 4.2. *Let h be any square submatrix of a of size ≥ 2. Then $\eta(h)=0$.*

Proof. By Lemma 4.1, the statement follows by induction once we prove the special case that h is a square submatrix of size 2. In this case h has the shape

$$\begin{pmatrix} p_{i_1}\theta_{j_1}+p_{j_1}\theta_{i_1} & p_{i_1}\theta_{j_2}+p_{j_2}\theta_{i_1}\\ p_{i_2}\theta_{j_1}+p_{j_1}\theta_{i_2} & p_{i_2}\theta_{j_2}+p_{j_2}\theta_{i_2}\end{pmatrix},$$

and $\eta(h)$ has an expansion

$$\begin{aligned}\eta(h) = {}&\theta_{i_1}\theta_{j_1}\left(p_{i_2}\theta_{j_2}+p_{j_2}\theta_{i_2}\right) - \theta_{i_1}\theta_{j_2}\left(p_{i_2}\theta_{j_1}+p_{j_1}\theta_{i_2}\right)\\ &-\theta_{i_2}\theta_{j_1}\left(p_{i_1}\theta_{j_2}+p_{j_2}\theta_{i_1}\right)+\theta_{i_2}\theta_{j_2}\left(p_{i_1}\theta_{j_1}+p_{j_1}\theta_{i_1}\right).\end{aligned}$$

This is identically equal to zero. □

Denote by $b=(b_{ij})$ the n-by-n R-valued matrix

$$(b_{ij}) = (p\theta_{ij}),$$

and by $\widetilde{b} = (\widetilde{b}_{ij})$ the n-by-n R-valued matrix

$$(\widetilde{b}_{ij}) = (p\theta_{ij} + p_i\theta_j + p_j\theta_i).$$

We need to prove that $\eta(\widetilde{b}) = \eta(b)$. We can expand $\eta(\widetilde{b}) - \eta(b)$ as

$$\eta(\widetilde{b}) - \eta(b) = \sum_{m=0}^{n-2} p^m \sum_{(I,J)} \varepsilon_{I,J}\theta_{i_1j_1}\cdots\theta_{i_mj_m}\eta(a_J^I). \tag{4.2}$$

Here the second sum is over all pairs (I,J) of subtuples of $(1,2\dots,n)$ of length m, and $\varepsilon_{I,J}$ is a sign. In order to see this, expand in the cofactor matrix $\tilde{b}^c$ of $\tilde{b}$ each minor as a sum of $(n-1)!$ terms. A product $p^m\theta_{i_1j_1}\cdots\theta_{i_mj_m}$ occurs as a factor in such a term exactly at all entries (k,l) of $\tilde{b}^c$ for which (k,l) is not in $I\times J$. If (k,l) is such an entry, at that entry the product $p^m\theta_{i_1j_1}\cdots\theta_{i_mj_m}$ is multiplied, up to a sign $\varepsilon_{I,J}$ depending only on I,J, by det $\left(a_{J\cup\{l\}}^{I\cup\{k\}}\right)$. This determinant is understood to be equal to 1 if $m = n-1$. It follows that the entry (k,l) contributes to $\eta(\tilde{b})$ with a term

$$p^m\theta_{i_1j_1}\cdots\theta_{i_mj_m}\varepsilon_{I,J}(-1)^{k+l}\theta_k\theta_l\det\left(a_{J\cup\{l\}}^{I\cup\{k\}}\right).$$

Summing over all possible (k,l) we obtain, if $m<n-1$, a contribution $p^m\theta_{i_1j_1}\cdots\theta_{i_mj_m}\varepsilon_{I,J}\eta(a_J^I)$, and if $m=n-1$ the contribution $\eta(b)$.

By Lemma 4.2, every $\eta(a_J^I)$ with I,J of size smaller than $n-1$ is zero. Therefore, all terms in the summation on the righthand side in (4.2) vanish. This proves the first half of Theorem 1.3.

In order to prove the statement on modular weight, we recall from equation (1.3) that

$$\begin{aligned}\theta({}^t(c\tau+d)^{-1}z,(a\tau+b)(c\tau+d)^{-1}) \\ = \zeta_\gamma \det(c\tau+d)^{1/2}q(z,\gamma,\tau)\theta(z,\tau)\end{aligned} \tag{4.3}$$

for all z in $\mathbf{H}_n$ and all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ in $\Gamma_{1,2}$, where

$$q(z,\gamma,\tau) = e^{\pi i {}^t z(c\tau+d)^{-1}cz}$$

and where ζ_γ is an eighth root of unity. We claim that

$$\begin{aligned}(4.4)\quad & \eta({}^t(c\tau+d)^{-1}z,(a\tau+b)(c\tau+d)^{-1}) \\ &\qquad = \det(c\tau+d)^{(n+5)/2}\zeta_\gamma^{n+1}q(z,\gamma,\tau)^{n+1}\eta(z,\tau)\end{aligned}$$

for all (z,τ) satisfying $\theta(z,\tau)=0$. This is just a calculation. For (z,τ) with $\theta(z,\tau)=0$ we have by (4.3)

$$\begin{aligned}&\big(\theta_i({}^t(c\tau+d)^{-1}z,(a\tau+b)(c\tau+d)^{-1})\big) \\ &\qquad = {}^t(c\tau+d)\zeta_\gamma\det(c\tau+d)^{1/2}q(z,\gamma,\tau)\left(\theta_i(z,\tau)\right).\end{aligned}$$

Furthermore, for such a (z,τ) we have

$$\begin{aligned}&\big(\theta_{ij}({}^t(c\tau+d)^{-1}z,(a\tau+b)(c\tau+d)^{-1})\big) \\ &\qquad = {}^t(c\tau+d)\zeta_\gamma\det(c\tau+d)^{1/2}\left(q(z,\gamma,\tau)\theta_{ij}(z,\tau)\right. \\ &\qquad\quad \left.+q_i(z,\gamma,\tau)\theta_j(z,\tau)+q_j(z,\gamma,\tau)\theta_i(z,\tau)\right)(c\tau+d).\end{aligned}$$

We can write, at least for the purpose of this proof,

$$\eta = \det(\theta_{ij})^t(\theta_i)(\theta_{ij})^{-1}(\theta_j). \tag{4.5}$$

This gives that $\eta({}^t(c\tau+d)^{-1}z,(a\tau+b)(c\tau+d)^{-1})$ is equal to

$$\begin{aligned}&\det\Big({}^t(c\tau+d)\zeta_\gamma\det(c\tau+d)^{1/2}\left(q\theta_{ij}(z,\tau)+q_i\theta_j(z,\tau)\right. \\ &\qquad\qquad \left.+q_j\theta_i(z,\tau)\right)(c\tau+d)\Big) \\ &\quad\cdot\det(c\tau+d)^{1/2}\,\zeta_\gamma\,{}^t\left(q\theta_i(z,\tau)\right)(c\tau+d) \\ &\quad\cdot(c\tau+d)^{-1}\left(q\theta_{ij}(z,\tau)+q_i\theta_j(z,\tau)\right. \\ &\qquad\quad \left.+q_j\theta_i(z,\tau)\right)^{-1}\zeta_\gamma^{-1}\det(c\tau+d)^{-1/2}\,{}^t(c\tau+d)^{-1} \\ &\quad\cdot{}^t(c\tau+d)\zeta_\gamma\det(c\tau+d)^{1/2}\,\left(q\theta_j(z,\tau)\right).\end{aligned}$$

This simplifies to

$$\begin{aligned}&\det(c\tau+d)^{2+(n/2)+(1/2)}\,\zeta_\gamma^{n+1} \\ &\quad\cdot{}^t\left(q\theta_i(z,\tau)\right)\left(q\theta_{ij}(z,\tau)+q_i\theta_j(z,\tau)+q_j\theta_i(z,\tau)\right)^c\left(q\theta_j(z,\tau)\right),\end{aligned}$$

which, in turn, is equal to

$$\det(c\tau+d)^{(n+5)/2}\zeta_\gamma^{n+1}q^{n+1}\eta(z,\tau)$$

by the same methods as above when we dealt with the order of η. This completes the proof of Theorem 1.3.

5. Jacobians. The purpose of this section is to study η for Jacobians.

Assume that $(A,\Theta) = (\mathbf{C}^n/(\mathbf{Z}^n+\tau\mathbf{Z}^n), \operatorname{div}\theta)$, with $n \geq 2$, is the Jacobian belonging to a compact Riemann surface X marked with a symplectic basis $\mathcal{B} = (A_1,\dots,A_n,B_1,\dots,B_n)$ of homology. This situation determines uniquely a basis $(\zeta_1,\dots,\zeta_n) = \zeta = \zeta_{\mathcal{B}}$ of $H^0(X,K_X)$ such that

$$\int_{A_i}\zeta_j = \delta_{ij}, \quad \int_{B_i}\zeta_j = \tau_{ij}.$$

The isomorphism of $\mathbf{C}$-vector spaces $H^0(X,K_X)^* \overset{\approx}{\to} \mathbf{C}^n$ given by ζ gives an isomorphism of complex tori

$$H^0(X,K_X)^*/H_1(X,\mathbf{Z}) \overset{\approx}{\to} \mathbf{C}^n/(\mathbf{Z}^n+\tau\mathbf{Z}^n) = A.$$

By a theorem of Abel-Jacobi, the natural map

$$AJ : \operatorname{Pic}^0 X \longrightarrow \mathbf{C}^n/(\mathbf{Z}^n+\tau\mathbf{Z}^n),$$
$$\sum(P_i-Q_i) \longmapsto \sum\int_{Q_i}^{P_i} {}^t(\zeta_1,\dots,\zeta_n)$$

is a bijection. By a theorem of Riemann, there is a unique element $\Delta = \Delta_{\mathcal{B}}$ of $\operatorname{Pic}^{1-n}X$ such that under the composition of bijections

$$\operatorname{Pic}^{n-1}X \xrightarrow{t_\Delta} \operatorname{Pic}^0 X \xrightarrow{AJ} \mathbf{C}^n/(\mathbf{Z}^n+\tau\mathbf{Z}^n) = A,$$

the set $\Theta_0 = \{[D] \in \operatorname{Pic}^{n-1}X : h^0(D) > 0\}$ is identified with $\Theta = \operatorname{div}\theta$ on A. From now on, we will take this identification of (A,Θ) with $(\operatorname{Pic}^{n-1}X, \Theta_0)$ for granted.

We have a natural surjection $\Sigma : X^{(n-1)} \to \Theta$ which is an isomorphism above Θ^s. If we let this isomorphism be followed by the Gaussian map $\Gamma : \Theta^s \to \mathbf{P}(T_0 A)^* = \mathbf{P}(H^0(X, K_X))$, we get the map sending a nonspecial divisor D of degree $n-1$ on X to the linear span of its points on the canonical image in $\mathbf{P}(H^0(X, K_X)^*)$. By Riemann-Roch, this span is indeed a hyperplane.

We denote by

$$\kappa : X \longrightarrow \Theta \subset A$$

the map sending x to the class of $(n-1) \cdot x$. It is natural to study the pullback of η along κ. We claim the following result:

Theorem 5.1. *The section $\kappa^*\eta$ is not identically zero. The divisor of $\kappa^*\eta$ is equal to $(n-1)W$, where W is the divisor of Weierstrass points of X.*

We proceed in a few steps, starting with some notation. For any (m, i) in $\mathbf{Z} \times \mathbf{Z}_{\geq 0}$, we put

$$f_{m,i}(x) = \left(\frac{d^i}{dy^i} \theta((n-1-m)x + my) \right) \Bigg|_{y=x},$$

interpreted as a global section of the bundle $\kappa^* O_A(\Theta) \otimes K_X^{\otimes i}$ of differential i-forms with coefficients in $\kappa^* O_A(\Theta)$. For example, for $i = 0$, we get

$$f_{m,0}(x) = 0$$

identically; for $i = 1$, we get

$$f_{m,1}(x) = m \sum_{j=1}^{n} \theta_j(\kappa(x)) \zeta_j(x);$$

and, for $i = 2$:

$$f_{m,2}(x) = m^2 \sum_{j,k=1}^{n} \theta_{jk}(\kappa(x)) \zeta_j(x) \zeta_k(x) + m \sum_{j=1}^{n} \theta_j(\kappa(x)) \zeta_j'(x).$$

For $m = 0, \ldots, n-1$, all $f_{m,i}$ are identically equal to zero. Of particular interest for us will be the section

$$F(x) = \frac{1}{n!} f_{-1,n}(x)$$

of $\kappa^* O_A(\Theta) \otimes K_X^{\otimes n}$. It turns out that $\operatorname{div} F = W$.

Lemma 5.2 (cf. [**8**, Corollary 3]). *The section F of $\kappa^* O_A(\Theta) \otimes K_X^{\otimes n}$ is not identically zero, and we have* $\operatorname{div} F = W$.

Proof. Consider the map $\Phi : X \times X \to \operatorname{Pic}^{n-1} X$ given by $(x, y) \mapsto nx - y$. From [**3**, page 31] we obtain that $\Phi^* \Theta = W \times X + n \cdot \Delta_X$. Restricting to the diagonal, we get

$$\kappa^* \Theta = \Phi^* \Theta|_{\Delta_X} = (W \times X + n \cdot \Delta_X)\,|_{\Delta_X},$$

and by the adjunction formula it follows that

$$\kappa^* O_A(\Theta) \xrightarrow{\approx} O_X(W) \otimes K_X^{\otimes -n}$$

via $F(x) \cdot (dx)^{\otimes -n} \mapsto 1_W \otimes (dx)^{\otimes -n}$. Here 1_W denotes the tautological section of $O_X(W)$. $\square$

Lemma 5.3. *We have $\kappa^* \eta(x) = 0$ if and only if x is a Weierstrass point. In particular, $\kappa^* \eta$ is not identically zero.*

Proof. According to [**2**, page 691] the ramification locus of the Gaussian map is precisely given by the set of divisors $D + x$ with D effective of degree $g - 2$ and x a point of X such that $D + 2x$ is dominated by a canonical divisor, i.e., such that $K_X - D - 2x$ is linearly equivalent to an effective divisor. Thus, to say that $\eta(\kappa(x)) = 0$ means precisely that $h^0(K_X - n \cdot x) > 0$ or equivalently, by Riemann-Roch, that $h^0(n \cdot x) > 1$. But this means precisely that x is a Weierstrass point. $\square$

Proof of Theorem 5.1. It follows from Section 1 that $\kappa^* \eta$ is a global section of $\kappa^* O_A(\Theta)^{\otimes n+1} \otimes \lambda^{\otimes 2}$ where λ is the trivial bundle $\det H^0(X, K_X) \otimes O_X$. From Lemma 5.3 we know that $\kappa^* \eta$ is nonzero. It is stated in Lemma 5.2 that F is a nonzero global section of $\kappa^* O_A(\Theta) \otimes K_X^{\otimes n}$. As was observed by Arakelov (cf. [**1**, Lemma 3.3]), the bundle

$$K_X^{\otimes n(n+1)/2} \otimes \lambda^{\otimes -1}$$

has a nonzero section given by

$$\xi_1 \wedge \cdots \wedge \xi_n \longmapsto \frac{\xi_1 \wedge \cdots \wedge \xi_n}{\zeta_1 \wedge \cdots \wedge \zeta_n} \cdot \omega_\zeta$$

with ω_ζ the Wronskian differential on ζ. Combining, we find that $\omega_\zeta^2 \otimes \kappa^*\eta \otimes F^{\otimes -(n+1)}$ is a nonzero global section of O_X. Hence, it is a nonzero constant. We find

$$\operatorname{div} \kappa^*\eta = (n+1)\operatorname{div} F - 2\operatorname{div} \omega_\zeta = (n+1)W - 2W = (n-1)W$$

as required. □

Remark 5.4. An elaborate computation shows that actually

$$\omega_\zeta^2 \otimes \kappa^*\eta = F^{\otimes n+1}.$$

Let us prove this relation in the case that $n = 2$. So we look at indecomposable (A, Θ) with $A = \mathbf{C}^2/(\mathbf{Z}^2 + \tau\mathbf{Z}^2)$ and $\Theta = X = \operatorname{div} \theta$. For z on $\Theta = X$ put

$$P = \theta_{11}(z)\zeta_1(z)^{\otimes 2} + 2\theta_{12}(z)\zeta_1(z) \otimes \zeta_2(z) + \theta_{22}(z)\zeta_2(z)^{\otimes 2}$$

and

$$Q = \theta_1(z)\zeta_1'(z) + \theta_2(z)\zeta_2'(z).$$

Then according to what we have said before Lemma 5.2 we have

$$P + Q = 0, \quad P - Q = 2F.$$

We conclude that

$$F = P = \theta_{11}(z)\zeta_1(z)^{\otimes 2} + 2\theta_{12}(z)\zeta_1(z) \otimes \zeta_2(z) + \theta_{22}(z)\zeta_2(z)^{\otimes 2},$$

and writing as before,

$$\zeta_1(z) = dz_1, \quad \zeta_2(z) = dz_2 = z_2'(z)\, dz_1 = -\frac{\theta_1(z)}{\theta_2(z)}\, dz_1,$$

we get

$$\begin{aligned} F(z) &= \theta_{11}(z)\zeta_1(z)^{\otimes 2} + 2\theta_{12}(z)\zeta_1(z)\otimes\zeta_2(z) + \theta_{22}(z)\zeta_2(z)^{\otimes 2} \\ &= \big(\theta_{11}(z) + 2\theta_{12}(z)z_2'(z) + \theta_{22}(z)(z_2'(z))^2\big)\cdot(dz_1)^{\otimes 2} \\ &= \frac{\theta_{11}(z)\theta_2(z)^2 - 2\theta_{12}\theta_1(z)\theta_2(z) + \theta_{22}(z)\theta_1(z)^2}{\theta_2(z)^2}\cdot(dz_1)^{\otimes 2} \\ &= \frac{\eta(z)}{\theta_2(z)^2}\cdot(dz_1)^{\otimes 2}. \end{aligned}$$

We have seen in Example 2.2 that $\omega_\zeta(z) = -(\eta(z)/\theta_2(z)^3)(dz_1)^{\otimes 3}$. Combining, we find

$$\omega_\zeta^2 \otimes \kappa^*\eta = F^{\otimes 3}$$

as required. We note in passing that our formula (for general n) leads to an alternative description of one of the analytic invariants studied in [**3**].

6. Explicit formula. In the case that $n = 2$ it is possible to give a closed formula for η using a more familiar theta function. This formula can be viewed as a generalization of Jacobi's derivative formula, cf. Example 1.4, which gives η in the case that $n = 1$ as a product of even Thetanullwerte. Recall that for a, b column vectors of dimension n with entries in $\{0, 1/2\}$ we have on $\mathbf{C}^n \times \mathbf{H}_n$ the theta function with *characteristic* $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ given by

$$\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\tau) = \sum_{m\in\mathbf{Z}^n} e^{\pi i^t(m+a)\tau(m+a) + 2\pi i^t(m+a)(z+b)}.$$

The choice $a = b = 0$ gives the Riemann theta function, and it follows from the definition that $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](-z,\tau) = e^{4\pi i^t ab}\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\tau)$. We call $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ an even or odd theta characteristic depending on whether $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z,\tau)$ is an even or odd function of z. If $n = 2$, there are ten even theta characteristics, and six odd ones. The product $\prod_{\varepsilon\text{ even}}\theta[\varepsilon](0,\tau)^2$ of Thetanullwerte is a modular form of weight 10 and level 1 and can be related to the discriminant of a hyperelliptic equation.

Theorem 6.1. *For (z,τ) in $\mathbf{C}^2 \times \mathbf{H}_2$ with $\theta(z,\tau)=0$, the formula*

$$\eta(z,\tau)^3 = \pm\pi^{12} \prod_{\varepsilon \text{ even}} \theta[\varepsilon](0,\tau)^2 \cdot \theta(3z,\tau)$$

holds.

Proof. It suffices to prove the formula for τ corresponding to an indecomposable Abelian surface. We write $A = \mathbf{C}^2/(\mathbf{Z}^2 + \tau\mathbf{Z}^2)$ and $\Theta = \operatorname{div}\theta$ and assume that Θ is irreducible. By Theorem 1.3, the section $\eta(z)$ is a theta function of order 3 on Θ, and by Example 2.2 or Theorem 5.1 it has zeroes exactly at $\Theta \cap A[2]$, the Weierstrass points of Θ, all of multiplicity 1. On the other hand, the function $\theta(3z)$ gives rise to a global section of $O_A(\Theta)^{\otimes 9}$, i.e., is a theta function of order 9 on A, and has zeroes on Θ exactly at $\Theta \cap A[2]$, with multiplicity 3. It follows that $\eta(z)^3 = c \cdot \theta(3z)$ on the zero locus of $\theta(z)$ in $\mathbf{C}^2$, where c is a constant only depending on τ. In order to compute c, we recall from Remark 5.4 that $\omega_\zeta(z)^2\eta(z) = F(z)^3$, with ζ the basis of $H^0(\Theta, K_\Theta)$ given by (dz_1, dz_2), so that

$$c = \frac{F(z)^9}{\omega_\zeta(z)^6\theta(3z)} \tag{6.1}$$

for z on $\Theta \setminus (\Theta \cap A[2])$. We find c by letting z approach a point Q of $\Theta \cap A[2]$, along Θ, and computing Taylor expansions of the numerator and denominator in (6.1). Note that the leading coefficient of a Taylor expansion of $F(z)$ around Q is the same as the leading coefficient of a Taylor expansion of $\theta(2z)|_\Theta$ around Q. We start with the standard Euclidean coordinates z_1, z_2, but now translated suitably so as to have them both vanish at Q on A. According to [**5**, formula (1.6)] there exist a constant b_3 and an invertible 2-by-2 matrix μ such that in the coordinates $(u_1, u_2) = (z_1, z_2)^t\mu$ one has a Taylor expansion

$$\theta(z) = \theta(\mu^{-1}u) = \gamma\, e^{G(u)} \left(u_1 + \frac{1}{24} b_3 u_1^3 - \frac{1}{12} u_2^3 + \text{higher order terms} \right), \tag{6.2}$$

with γ some nonzero constant and with $G(u)$ some holomorphic function that vanishes at $u=0$. This gives u_2 as a local coordinate around Q on Θ, as well as an expansion

$$u_1 = \frac{1}{12} u_2^3 + \text{higher order terms} \tag{6.3}$$

locally around Q on Θ. The way u_1, u_2 are obtained is as follows. One may identify (Θ, Q) with a hyperelliptic curve (X, ∞) given by a hyperelliptic equation $y^2 = f(x)$ with f a monic separable polynomial of degree 5. We have then around Q a local coordinate t such that $x = t^{-2} + \text{h.o.t.}$ and $y = -t^{-5} + \text{h.o.t.}$ A computation yields that

$$\int_\infty \frac{dx}{y} = \frac{2}{3}t^3 + \text{h.o.t.}, \qquad \int_\infty \frac{xdx}{y} = 2t + \text{h.o.t.}$$

so that, putting

$$u_1 = \int_\infty \frac{dx}{y}, \quad u_2 = \int_\infty \frac{xdx}{y}$$

gives the required relation (6.3). The matrix μ corresponds then to a change of basis of holomorphic differentials on Θ from $\zeta = (dz_1, dz_2)$ to $\zeta' = (dx/y, xdx/y) = (du_1, du_2)$. From (6.2) and (6.3), one computes

$$\begin{aligned}\theta(2z)|_\Theta &= \gamma\, e^{G(2u)}\left(2u_1 - \frac{1}{12}(2u_2)^3 + \text{h.o.t.}\right)\\ &= -\frac{1}{2}\gamma\, e^{G(2u)} \cdot u_2^3 + \text{h.o.t.}\end{aligned}$$

and

$$\begin{aligned}\theta(3z)|_\Theta &= \gamma\, e^{G(3u)}\left(3u_1 - \frac{1}{12}(3u_2)^3 + \text{h.o.t.}\right)\\ &= -2\gamma\, e^{G(3u)} \cdot u_2^3 + \text{h.o.t.}\end{aligned}$$

As to the Wronskian of ζ, we have

$$\omega_\zeta = (\det \mu)^{-1}\omega_{\zeta'}.$$

Writing out $\omega_{\zeta'}$ with respect to u_2 gives

$$\begin{aligned}\omega_{\zeta'}(u_2) &= \det\begin{pmatrix} u_1' & 1 \\ u_1'' & 0\end{pmatrix}(du_2)^{\otimes 3}\\ &= -u_1''\,(du_2)^{\otimes 3}\\ &= \left(-\frac{1}{2}u_2 + \text{h.o.t.}\right)(du_2)^{\otimes 3}.\end{aligned}$$

We find

$$c = \frac{F(z)^9}{\omega_\zeta(z)^6\theta(3z)} = \frac{2^{-9}\gamma^9}{2^{-6}(\det\mu)^{-6}\cdot 2\gamma} = 2^{-4}(\det\mu)^6\gamma^8.$$

By [**5**, Theorem 2.11] one has

$$\gamma^8 = \pm 2^4 \pi^{12} (\det \mu)^{-6} \prod_{\varepsilon \text{ even}} \theta[\varepsilon](0,\tau)^2.$$

Substituting this in our formula for c we get the result. □

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