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THE METHOD OF UPPER AND LOWER SOLUTIONS FOR SECOND ORDER DIFFERENTIAL INCLUSIONS WITH INTEGRAL BOUNDARY CONDITIONS

MOUFFAK BENCHOHRA, SAMIRA HAMANI AND JUAN J. NIETO

ABSTRACT. In this paper, we prove the existence of solutions of second order differential inclusion with integral boundary conditions. We rely on the nonlinear alternative of Leray-Schauder combined with the lower and upper solutions method.

1. Introduction. This paper is concerned with the existence of solutions of second order differential inclusion with integral boundary conditions. We consider the following second order differential inclusion with integral boundary conditions:

$$x''(t) + \lambda x'(t) \in F(t, x(t)), \text{ almost everywhere } t \in [0, 1], \tag{1}$$

$$x(0) = a, \tag{2}$$

$$x(1) = \int_0^1 g(x(s))\, ds, \tag{3}$$

where $F : [0, 1] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a compact valued multi-valued map, $\mathcal{P}(\mathbf{R})$ is the family of all subsets of $\mathbf{R}$, $\lambda > 0$, $a \in \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ is continuous. Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point and nonlocal boundary value problems as special cases. Integral boundary conditions appear in population dynamic [**11**] and cellular systems [**1**]. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers by Gallardo [**21, 22**], and the

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references therein. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors, for instance, [**5, 12–14, 18, 19, 26–28, 30, 33, 37**], and the references therein. The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems for differential equations with nonlinear conditions. This method has been used only in the context of single-valued differential equations. In this regard, we refer the reader to the monograph by Heikkila and Lakshmikantham [**24**] and Ladde et al. [**31**], and to the recent papers by De Coster and Habets [**15, 16**], Jiang et al. [**29**] and Nieto [**34, 35**]. Recently this method has been used for initial and nonlinear boundary conditions for differential inclusions in the papers [**6, 8–10, 23, 36**] ; see also the papers [**2, 7**]. In this paper, we shall present an existence result of solutions for the problem (1)–(3). Our proof relies on the nonlinear alternative of Leray-Schauder combined with the lower and upper solutions method. These results extend to the multi-valued case those considered in the literature.

2. Preliminaries. In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $AC^1([0,1],\mathbf{R})$ be the space of differential functions $x:[0,1]\to\mathbf{R}$, whose first derivative, x', is absolutely continuous. The property

$$x\le \overline{x}, \quad \Longleftrightarrow x(t)\le\overline{x}(t), \quad \text{for all } t\in[0,1]$$

defines a partial ordering in $AC^1([0,1],\mathbf{R})$. If $\alpha,\ \beta\in AC^1([0,1],\mathbf{R})$ and $\alpha\le\beta$, we let

$$[\alpha,\beta]=\{x\in AC^1([0,1],\mathbf{R}):\alpha\le x\le\beta\}.$$

$C([0,1],\mathbf{R})$ is the Banach space of all continuous functions from $[0,1]$ into $\mathbf{R}$ with the norm

$$\|x\|_\infty=\sup\{|x(t)|:0\le t\le 1\},$$

and we let $L^1([0,1],\mathbf{R})$ denote the Banach space of measurable functions $x:[0,1]\to\mathbf{R}$ which are Lebesgue integrable norm

$$\|x\|_{L^1}=\int_0^1|x(t)|\,dt \quad \text{for all } x\in L^1([0,1],\mathbf{R}).$$

Let $(X, |\cdot|)$ be a normed space, $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$. A multi-valued map $G : X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$, i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$. G is called upper semi-continuous (usc) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty closed subset of X and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is usc if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is an $x \in X$ such that $x \in G(x)$. The fixed point set of the multi-valued operator G will be denoted by $\operatorname{Fix} G$. A multi-valued map $G : [0,1] \to P_{cl}(\mathbf{R})$ is said to be measurable if for every $x \in \mathbf{R}$, the function

$$t \longmapsto d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

is measurable. For more details on multi-valued maps, see the books of Aubin and Cellina [**3**], Aubin and Frankowska [**4**], Deimling [**17**] and Hu and Papageorgiou [**25**].

Definition 2.1. A multi-valued map $F : [0,1] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is said to be L^1-Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbf{R}$;

(ii) $x \mapsto F(t,x)$ is upper semi-continuous for almost all $t \in [0,1]$;

(iii) for each $q > 0$, there exists a $\varphi_q \in L^1([0,1], \mathbf{R}_+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_q(t) \quad \text{for all } |x| \le q \text{ and for a.e. } t \in [0,1].$$

Definition 2.2. A function $x \in AC^1((0,1), \mathbf{R})$ is said to be a solution of (1)–(3) if $x''(t) + \lambda x'(t) \in F(t, x(t))$ almost everywhere on $[0,1]$ and the function x satisfies conditions (2) and (3).

For each $x \in C([0,1], \mathbf{R})$, define the set of selections of F by

$$S_{F,x} = \{v \in L^1([0,1], \mathbf{R}) : v(t) \in F(t, x(t)) \text{ a.e. } t \in [0,1]\}.$$

Definition 2.3. A function $\alpha \in AC^1((0,1), \mathbf{R})$ is said to be a lower solution of (1)–(3) if there exists a function $v_1 \in L^1([0,1], \mathbf{R})$ with $v_1(t) \in F(t, \alpha(t))$ for almost every $t \in [0,1]$ such that $\alpha''(t) + \lambda\alpha'(t) \geq v_1(t)$ almost everywhere on $[0,1]$, $\alpha(0) \leq a$ and $\alpha(1) \leq \int_0^1 g(\alpha(s))\,ds$. Similarly, a function $\beta \in AC^1((0,1), \mathbf{R})$ is said to be an upper solution of (1)–(3) if there exists a function $v_2 \in L^1([0,1], \mathbf{R})$ with $v_2(t) \in F(t, \beta(t))$ for almost every $t \in [0,1]$ such that $\beta''(t) + \lambda\beta'(t) \leq v_2(t)$ almost everywhere on $[0,1]$, $\beta(0) \geq a$ and $\beta(1) \geq \int_0^1 g(\beta(s))\,ds$.

The following lemma is crucial in the proof of our main theorem.

Lemma 2.1 [**32**]. *Let X be a Banach space. Let $F : [0,1] \times X \to P_{cp,c}(X)$ be an L^1-Carathéodory multi-valued map, and let Γ be a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$. Then the operator*

$$\begin{aligned} \Gamma \circ S_F : C([0,1], X) &\longrightarrow P_{cp,c}(C([0,1], X)), \\ x &\longmapsto (\Gamma \circ S_F)(x) := \Gamma(S_{F,x}) \end{aligned}$$

is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.

We need the following auxiliary result. Its proof uses a standard argument.

Lemma 2.2. *For any $\sigma(t)$, $\rho(t) \in C([0,1], \mathbf{R})$, the nonhomogeneous linear problem*

$$\begin{aligned} x''(t) + \lambda x'(t) &= \sigma(t), \quad a.e. \ t \in [0,1], \\ x(0) = a, &\qquad x(1) = b, \end{aligned}$$

has a unique solution $x \in AC^1((0,1), \mathbf{R})$ given by

$$x(t) = P(t) + \int_0^1 G(t,s)\sigma(s)\,ds,$$

where

$$P(t) = \frac{1}{e^{-\lambda} - 1}\left[a(e^{-\lambda} - e^{-\lambda t}) + b(e^{-\lambda t} - 1)\right]$$

is the unique solution of the problem

$$\begin{gathered} x''(t) + \lambda x'(t) = 0, \ \ a.e. \ t \in [0,1], \\ x(0) = a, \qquad x(1) = b \end{gathered}$$

and $\int_0^1 G(t,s)\sigma(s)\,ds$ *is the unique solution of the problem*

$$\begin{gathered} x''(t) + \lambda x'(t) = \sigma(t), \ \ a.e. \ t \in [0,1], \\ x(0) = 0, \qquad x(1) = 0. \end{gathered}$$

Here G is the Green's function associated to the corresponding homogeneous problem, given by

$$G(t,s) = \frac{1}{(1 - e^{-\lambda})}\begin{cases} \frac{(e^{-\lambda t}-1)(e^{-\lambda s}-e^{-\lambda})}{e^{-\lambda s}} & 0 \le t \le s \le 1 \\ \frac{(e^{-\lambda s}-1)(e^{-\lambda t}-e^{-\lambda})}{e^{-\lambda s}} & 0 \le s \le t \le 1. \end{cases} \tag{4}$$

3. Main result. In this section, we are concerned with the existence of solutions for problem (1)–(3). We first list the following hypotheses:

(H1) The function $F : [0,1] \times \mathbf{R} \to P_{cp,c}(\mathbf{R})$ is L^1-Carathéodory;

(H2) There exist α and $\beta \in AC^1([0,1], \mathbf{R})$, lower and upper solutions for problem (1)–(3) such that $\alpha(t) \le \beta(t)$ for each $t \in [0,1]$;

(H3) g is a continuous and nondecreasing function.

Theorem 3.1. *Suppose that hypotheses* (H1)–(H3) *are satisfied. Then problem* (1)–(3) *has at least one solution such that*

$$\alpha(t) \le x(t) \le \beta(t), \quad \textit{for all } t \in [0,1].$$

Proof. Consider the modified problem

$$x''(t) + \lambda x'(t) \in F_1(t, x(t)), \ \text{almost every } t \in [0,1], \tag{5}$$

$$x(0) = a, \tag{6}$$

$$x(1) = \int_0^1 g_1(x(s))\,ds, \tag{7}$$

where

$$F_1(t,x(t)) = \begin{cases} F(t,\alpha(t)) + \dfrac{x(t)-\alpha(t)}{1+|x(t)-\alpha(t)|} & x(t) < \alpha(t) \\ F(t,x(t)) & \alpha(t) \le x(t) \le \beta(t) \\ F(t,\beta(t)) + \dfrac{x(t)-\beta(t)}{1+|x(t)-\beta(t)|} & x(t) > \beta(t), \end{cases}$$

and

$$g_1(x(t)) = \begin{cases} g(\alpha(t)) & x(t) < \alpha(t) \\ g(x(t)) & \alpha(t) \le x(t) \le \beta(t) \\ g(\beta(t)) & x(t) > \beta(t). \end{cases}$$

A solution to (5)–(7) is a fixed point of the operator $N : C([0,1],\mathbf{R}) \to \mathcal{P}(C([0,1],\mathbf{R}))$ defined by

$$N(x) = \{h \in C([0,1],\mathbf{R}) : h(t) = P_x(t) + \int_0^1 G(t,s)v(s)\,ds,\ v \in \widetilde{S}^1_{F_1(x)},$$

where

$$\widetilde{S}^1_{F_1(x)} = \{v \in S^1_{F_1(x)} : v(t) \ge v_1(t) \text{ on } A_1 \text{ and } v(t) \le v_2(t) \text{ on } A_2\},$$

$$S^1_{F_1(x)} = \{v \in L^1([0,1],\mathbf{R}) : v(t) \in F_1(t,x(t)) \text{ for } t \in [0,1]\},$$

$$A_1 = \{t \in [0,1] : x(t) < \alpha(t) \le \beta(t)\},$$
$$A_2 = \{t \in [0,1] : \alpha(t) \le \beta(t) < x(t)\},$$

and

$$P_x(t) = \frac{1}{e^{-\lambda}-1}\left[a(e^{-\lambda} - e^{-\lambda t}) + (e^{-\lambda t} - 1)\int_0^1 g_1(x(s))\,ds\right]$$

is the unique solution of the problem

$$x''(t) + \lambda x'(t) = 0, \text{ almost everywhere } t \in [0,1],$$

$$x(0) = a, \qquad x(1) = \int_0^1 g_1(x(s))\,ds$$

and $G(t,s)$ is the Green's function given by (4).

Remark 3.1. (i) For each $x \in C([0,1],\mathbf{R}))$, the set $S^1_{F_1(x)}$ is nonempty, see [**32**].

(ii) For each $x \in C([0,1],\mathbf{R}))$, the set $\widetilde{S}^1_{F_1(x)}$ is nonempty.

In fact, (i) implies that there exists a $v \in S^1_{F_1(x)}$, so we set

$$w = v_1\chi_{A_1} + v_2\chi_{A_2} + v\chi_{A_3},$$

where

$$A_3 = \{t \in [0,1] : \alpha(t) \leq x(t) \leq \beta(t)\}.$$

Then, by decomposability, $w \in \widetilde{S}^1_{F_1(x)}$.

Remark 3.2. Clearly, from Lemma 2.2, the fixed points of N are solutions to (1)–(3).

Remark 3.3. Notice that F_1 is an L^1-Carathéodory multi-valued map with compact convex values and that there exists a $\phi \in L^1([0,1],\mathbf{R})$ such that

$$\|F_1(t,x)\| \leq \phi(t) + 1, \quad \text{for each } x \in \mathbf{R} \text{ and } t \in [0,1].$$

In order to apply the nonlinear alternative of Leray-Schauder type, we first show that N is completely continuous with convex values. The proof will be given in several steps.

Step 1. $N(x)$ is convex for each $x \in C([0,1],\mathbf{R})$.

Indeed, if h_1 and h_2 belong to $N(x)$, then there exist $w_1, w_2 \in \widetilde{S}_{F_1(x)}$ such that for each $t \in [0,1]$ we have

$$h_i(t) = P_x(t) + \int_0^1 G(t,s)w_i(s)\,ds, \quad i = 1,2.$$

Let $0 \leq d \leq 1$. Then, for each $t \in [0,1]$ we have

$$(dh_1 + (1-d)h_2)(t) = P_x(t) + \int_0^1 G(t,s)[dw_1(s) + (1-d)w_2(s)]\,ds.$$

Since $\widetilde{S}_{F_1(x)}$ is convex (because F_1 has convex values), then

$$dh_1 + (1-d)h_2 \in N(x).$$

Step 2. N maps bounded sets into bounded sets in $C([0,1],\mathbf{R})$.

It suffices to show that, for each $q>0$, there exists a positive constant ℓ such that, for each $x \in B_q = \{x \in C([0,1],\mathbf{R}) : \|x\|_\infty \le q\}$, we have

$$\|N(x)\|_{\mathcal{P}} := \sup\{\|h\|_\infty : h \in N(x)\} \le \ell.$$

Let $x \in B_q$ and $h \in N(x)$; then there exists a $v \in \widetilde{S}_{F_1(x)}$ such that

$$h(t) = P_x(t) + \int_0^1 G(t,s)v(s)\,ds.$$

By (H1), for each $t \in [0,1]$ we have

$$\begin{aligned} |h(t)| &\le |P_x(t)| + \int_0^1 |G(t,s)||v(s)|\,ds \\ &\le \overline{p} + G_0 \int_0^1 \varphi_q(s)\,ds, \end{aligned}$$

where

$$G_0 = \sup_{(t,s)\in[0,1]\times[0,1]} |G(t,s)|,$$

and

$$\overline{p} = |a| + \frac{1}{e^{-\lambda}-1}\left(\max\left(g\left(\min_{t\in[0,1]}|\alpha(t)|\right), g\left(\max_{t\in[0,1]}|\beta(t)|\right)\right)\right).$$

Step 3. N maps bounded sets into equicontinuous sets of $C([0,1],\mathbf{R})$.

Let $r_1, r_2 \in [0,1]$, $r_1 < r_2$ and B_q be a bounded set of $C([0,1],\mathbf{R})$ as in Step 2 and $x \in B_q$. For each $h \in N(x)$ there exists $v \in \widetilde{S}_1(x)$ such that for each $t \in [0,1]$, we have

$$\begin{aligned} |h(r_2) - h(r_1)| &\le |P_x(r_2) - P_x(r_1)| + \int_0^1 |G(r_2,s) - G(r_1,s)||v(s)|\,ds \\ &\le |P_x(r_2) - P_x(r_1)| + \int_0^1 |G(r_2,s) - G(r_1,s)|\varphi_q(s)\,ds. \end{aligned}$$

Since $t \to P(t)$ and $t \to G(t,s)$ for each $s \in [0,1]$ are continuous on $[0,1]$, thus uniformly continuous on $[0,1]$, the righthand side of the above inequality tends to zero as $r_2 - r_1 \to 0$. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : C([0,1], \mathbf{R}) \to \mathcal{P}(C([0,1], \mathbf{R}))$ is completely continuous.

Step 4. N has a closed graph.

Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(x_*)$. $h_n \in N(x_n)$ means that there exists a $v_n \in \widetilde{S}_{F_1(x_n)}$ such that for each $t \in [0,1]$

$$h_n(t) = P_{x_n}(t) + \int_0^1 G(t,s)v_n(s)\,ds,$$

where

$$P_{x_n}(t) = \frac{1}{e^{-\lambda} - 1}\left[a(e^{-\lambda} - e^{-\lambda t}) + (e^{-\lambda t} - 1)\int_0^1 g_1(x_n(s))\,ds\right].$$

We must show that there exists an $h_* \in \widetilde{S}_{F_1(x_*)}$ such that for each $t \in [0,1]$

$$h_*(t) = P_{x_*}(t) + \int_0^1 G(t,s)v_*(s)\,ds,$$

where

$$P_{x_*}(t) = \frac{1}{e^{-\lambda} - 1}\left[a(e^{-\lambda} - e^{-\lambda t}) + (e^{-\lambda t} - 1)\int_0^1 g_1(x_*(s))\,ds\right].$$

Clearly we have

$$\|(h_n - P_{x_n}) - (h_* - P_{x_*})\|_\infty \longrightarrow 0 \text{ as } n \to \infty.$$

Consider the continuous linear operator

$$\Gamma : L^1([0,1], \mathbf{R}) \longrightarrow C([0,1], \mathbf{R})$$

defined by

$$v \longmapsto (\Gamma v)(t) = \int_0^1 G(t,s)v(s)\,ds.$$

From Lemma 2.1, it follows that $\Gamma \circ \widetilde{S}_{F_1(x)}$ is a closed graph operator. Moreover, we have

$$\Big(h_n(t) - P_{x_n}(t)\Big) \in \Gamma(\widetilde{S}_{F_1(x_n)}).$$

Since $x_n \to x_*$, it follows from Lemma 2.1 that

$$h_*(t) = P_{x_*}(t) + \int_0^1 G(t,s)v_*(s)\,ds$$

for some $v_* \in \widetilde{S}_{F_1(x_*)}$.

Step 5. A priori bounds on solutions.

Let $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. Then there exists a $v \in \widetilde{S}_{F_1,x}$ such that, for each $t \in [0,1]$,

$$x(t) = \lambda\Big[P_x(t) + \int_0^1 G(t,s)v(s)\,ds\Big].$$

By Remark 3.3, for each $t \in [0,1]$, we have

$$\begin{aligned}|x(t)| &\leq |P_x(t)| + \int_0^1 |G(t,s)||v(s)|\,ds\\ &\leq \overline{p} + \frac{1}{1-e^{-\lambda}}\int_0^1 \phi(s)\,ds + \frac{1}{1-e^{-\lambda}}.\end{aligned}$$

Thus,

$$\|x\|_\infty \leq \overline{p} + \frac{1}{1-e^{-\lambda}}\int_0^1 \phi(s)\,ds + \frac{1}{1-e^{-\lambda}} := \ell_*.$$

Set

$$U = \{x \in C([0,1],\mathbf{R}) : \|x\|_\infty < \ell_* + 1\}.$$

The operator $N : \overline{U} \to \mathcal{P}(C([0,1],\mathbf{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [**20**], we deduce that N has a fixed point x in $\overline{U}$ which is a solution of problem (5)–(7).

Step 6. The solution x of (5)–(7) satisfies

$$\alpha(t) \leq x(t) \leq \beta(t) \quad \text{for all } t \in [0,1].$$

Let x be a solution to (5)–(7). We prove that

$$\alpha(t) \leq x(t) \quad \text{for all } t \in [0,1].$$

Suppose this is not the case. Let $r(t) = \alpha(t) - x(t)$; then the function r has positive maximum some $t_0 \in [0,1]$, such that $r(t_0) = \max r(t)$. Thus,

$$r(t_0) > 0, \qquad r'(t_0) = 0, \qquad r''(t_0) \leq 0.$$

If $t_0 \in (0,1)$, then there exists a $c_1 \in [0,1]$ such that

$$r(t) > 0, \quad \text{for all } t \in [c_1, t_0].$$

In view of the definition of F_1, we have

$$x''(t) + \lambda x'(t) \in F(t, \alpha(t)) + \frac{x(t) - \alpha(t)}{1 + |x(t) - \alpha(t)|}, \text{ a.e. } t \in [c_1, t_0].$$

Thus, there exists a $v(t) \in F(t, \alpha(t))$ almost everywhere $t \in [c_1, t_0]$ with $v(t) \geq v_1(t)$ almost everywhere $t \in [c_1, t_0]$ such that

$$x''(t) + \lambda x'(t) = v(t) + \frac{x(t) - \alpha(t)}{1 + |x(t) - \alpha(t)|}, \text{ almost everywhere } t \in [c_1, t_0].$$

Using the fact that α is a lower solution to (5)–(7), we have

$$\alpha''(t) - x''(t) = \alpha''(t) + \lambda\alpha'(t) - v(t) - \frac{x(t) - \alpha(t)}{1 + |x(t) - \alpha(t)|} > 0,$$
$$\text{almost everywhere } t \in [c_1, t_0].$$

Hence,

$$0 \geq \alpha''(t_0) - x''(t_0) > 0,$$

which is a contradiction. If $t_0 = 0$ and by definition of the lower solution, we have

$$0 < r(0) = \alpha(0) - x(0) = \alpha(0) - a \leq 0,$$

which is a contradiction. If $t_0 = 1$,

$$0 < r(1) = \alpha(1) - \int_0^1 g_1(x(s))\, ds.$$

By definition of the lower solution, we have

$$\begin{aligned} 0 &< \int_0^1 g(\alpha(s))\, ds - \int_0^1 g_1(x(s))\, ds \\ &= \int_0^1 [g(\alpha(s)) - g_1(x(s))]\, ds \\ &\le \max_{t\in[0,1]} (g(\alpha(t)) - g_1(x(t))). \end{aligned}$$

Let $t_* \in [0,1]$ be such that

$$\max_{t\in[0,1]} (g(\alpha(t)) - g_1(x(t))) = g(\alpha(t_*)) - g_1(x(t_*)).$$

If $\alpha(t_*) > x(t_*)$, we have $g_1(x(t_*)) = g(\alpha(t_*)))$, which implies $0 < r(1) \le 0$; this is a contradiction.

If $x(t_*) > \beta(t_*)$, we have $g_1(x(t_*)) = g(\beta(t_*)))$, which implies $0 < r(1) \le 0$; another contradiction. Thus, $\alpha(t) \le x(t), t \in [0,1]$. Similarly, we can show that $x(t) \le \beta(t)$ for all $t \in [0,1]$. This shows that the problem (5)–(7) has a solution in the interval $[\alpha, \beta]$, which is a solution of the problem (1)–(3).

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Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000, Sidi Bel Abbès, Algérie
Email address: benchohra@yahoo.com

Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000, Sidi Bel Abbès, Algérie
Email address: hamani_samira@yahoo.fr

Departamento de Analisis Matematico, Facultad de Matematicas, Universidad de Santiago de Compostela, Santiago de Compostela, 15782, España
Email address: juanjose.nieto.roig@usc.es