ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 1, 2010

GLEASON-KAHANE-ZELASKO TYPE THEOREMS FOR COMPLEX RIESZ ALGEBRAS

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Dedicated to Professor Melvin 'Mel' Henriksen
on the occasion of his 80th birthday

ABSTRACT. Let $\mathfrak{A}$ be a complex f-algebra with a unit element e. It is shown that a linear functional f on $\mathfrak{A}$ is a lattice homomorphism with $f(e) = 1$ if and only if $f(\mathfrak{a}) \in \sigma(\mathfrak{a})$ for all $\mathfrak{a} \in \mathfrak{A}$. More generally, let $\mathfrak{A}$ be a complex Riesz algebra with a positive unit element e. It turns out that the principal band $\mathfrak{B}_e$ in $\mathfrak{A}$ generated by e is a projection band in $\mathfrak{A}$. Moreover, a linear functional f on $\mathfrak{A}$ is a lattice homomorphism with $f(e) = 1$ if and only if $f(\mathfrak{a}) \in \sigma(P_e(\mathfrak{a}))$ for all $\mathfrak{a} \in \mathfrak{A}$, where P_e denotes the band projection of $\mathfrak{A}$ onto $\mathfrak{B}_e$. It follows that if E is a Dedekind complete complex Riesz space then a linear functional f on $L^r(E)$ is an identity preserving lattice homomorphism if and only if for each $T \in L^r(E)$ the scalar $f(T)$ is a spectral value in $L(E)$ of the diagonal component $D(T)$ of T.

1. Introduction. At the end of the 1960s, Zelasko [**20**] proved one of the most famous characterizations of a complex-valued algebra homomorphism on a complex Banach algebra $\mathfrak{A}$ with a unit element. Namely, a nonzero linear functional f on $\mathfrak{A}$ is an algebra homomorphism on $\mathfrak{A}$ if and only if $f(\mathfrak{a}) \in \sigma(\mathfrak{a})$ for all $\mathfrak{a} \in \mathfrak{A}$, where $\sigma(\mathfrak{a})$ denotes the spectrum of $\mathfrak{a}$ in $\mathfrak{A}$. The commutative version of this remarkable result was obtained earlier by Gleason [**6**] and, independently, by Kahane and Zelasko [**14**]. Henceforth, this result is known as the Gleason-Kahane-Zelasko theorem in the vast literature on the subject. In this regard, Jarosz [**13**] gave an interesting historical account which can be consulted for more bibliographic information concerning the Gleason-Kahane-Zelasko theorem. It is well known that, to a quite large extent,

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2010 AMS *Mathematics subject classification.* Primary 06F25,46A40, 46A40, 47A10, 47B65.
Keywords and phrases. Band projection, complex Riesz algebra, Dedekind complete complex Riesz space, diagonal operator, lattice homomorphism, regular operator, spectrum.
The second author gratefully acknowledges support from the NSF-Grant INT-0423522.
Received by the editors on April 13, 2007, and in revised form on August 3, 2007.
DOI:10.1216/RMJ-2010-40-1-1

algebra homomorphisms play the same role in the theory of Banach algebras that lattice homomorphisms do in the theory of f-algebras. Thus, it is to be expected that complex-valued lattice homomorphisms on unital complex f-algebras will have a similar characterization. This indeed turns out to be the case. Our first result asserts that if $\mathfrak{A}$ is a complex f-algebra with a unit element e and f is a linear functional on $\mathfrak{A}$, then f is a lattice homomorphism with $f(e) = 1$ if and only if $f(\mathfrak{a}) \in \sigma(\mathfrak{a})$ for all $\mathfrak{a} \in \mathfrak{A}$. Here also $\sigma(\mathfrak{a})$ indicates the spectrum of $\mathfrak{a}$.

Quite recently, Huijsmans [**8**] observed that if $\mathfrak{A}$ is a complex Banach lattice algebra with a positive unit element e, then the principal band $\mathfrak{B}_e$ in $\mathfrak{A}$ generated by e is a projection band. He then bet on the Gleason-Kahane-Zelasko theorem to prove that if f is a linear functional on $\mathfrak{A}$ then f is a lattice homomorphism with $f(e) = 1$ if and only if $f(\mathfrak{a}) \in \sigma(P_e(\mathfrak{a}))$ for all $\mathfrak{a} \in \mathfrak{A}$, where P_e denotes the band projection of $\mathfrak{A}$ onto $\mathfrak{B}_e$. In this note, we use our aforementioned main theorem to generalize Huijsmans's theorem to arbitrary complex Riesz algebras with positive unit elements. Explicitly, we consider a complex Riesz algebra $\mathfrak{A}$ with a positive unit element e as introduced by Huijsmans [**9, 11**] himself, and we show that the principal band $\mathfrak{B}_e$ in $\mathfrak{A}$ generated by e is again a projection band. Then we prove that a linear functional f on $\mathfrak{A}$ is a lattice homomorphism with $f(e) = 1$ if and only if for every $\mathfrak{a} \in \mathfrak{A}$ the scalar $f(\mathfrak{a})$ is a spectral value of the projection of $\mathfrak{a}$ onto $\mathfrak{B}_e$. As a consequence, we show that if E is a Dedekind complete complex Riesz space and $L^r(E)$ is the complex Riesz algebra of all regular linear operators on E, then a linear functional f on $L^r(E)$ is an identity preserving lattice homomorphism on $L^r(E)$ if and only if for each $T \in L^r(E)$ the scalar $f(T)$ is a spectral value in $L(E)$ of the diagonal component $D(T)$ of T introduced by Schep [**18**]. Here, $L(E)$ is the algebra of all linear operators on E. This extends another earlier result by Huijsmans [**8**], who discussed the case where E is in addition a Banach lattice.

Finally, we point out that in each section we summarize enough necessary background material to keep this note reasonably self contained. In this connection, the classical book [**16**] by Meyer-Nieberg is adopted as the unique source of unexplained terminology and notation.

2. Complex lattice homomorphisms on f-algebras.

A real Riesz space A is called a *Riesz algebra* (or a *lattice-ordered algebra*) if A

is simultaneously an associative algebra over the real field $\mathbf{R}$ such that

$$|ab| \leq |a|\,|b| \quad \text{for all } a, b \in A.$$

Clearly, the positive cone A^+ of the Riesz algebra A is closed under multiplication. The Riesz algebra A is called an *f-algebra* if

$$|ac| \wedge |b| = |ca| \wedge |b| = 0 \quad \text{for all } a, b, c \in A \text{ with } |a| \wedge |b| = 0.$$

It is readily verified that the Riesz algebra A is an f-algebra if and only if $a, b, c \in A^+$ and $a \wedge b = 0$ imply $(ac) \wedge b = (ca) \wedge b = 0$. An Archimedean f-algebra is commutative and has positive squares. In particular, if A has a unit element e then e is positive. The reader can consult [**11, 19**] for more information about real Riesz algebras and real f-algebras.

Now, let $\mathfrak{A}$ be a complex Riesz space, that is, $\mathfrak{A}$ is the complexification $A + iA$ of a uniformly complete real Riesz space A [**16**]. Thus, each element $\mathfrak{a} \in \mathfrak{A}$ has a unique decomposition $\mathfrak{a} = \mathrm{Re}\,(\mathfrak{a}) + i\mathrm{Im}\,(\mathfrak{a})$ with $\mathrm{Re}\,(\mathfrak{a}), \mathrm{Im}\,(\mathfrak{a}) \in A$. Recall that the modulus of an element $\mathfrak{a}$ of $\mathfrak{A}$ is the positive element $|\mathfrak{a}|$ of A given by the formula

$$|\mathfrak{a}| = \sup \{\mathrm{Re}\ (\exp(ix)\,\mathfrak{a}) : 0 \leq x < 2\pi\}.$$

Obviously, if $\mathfrak{a} \in \mathfrak{A}$ then

$$|\mathrm{Re}\ (\mathfrak{a})| \leq |\mathfrak{a}| \text{ and } |\mathrm{Im}\ (\mathfrak{a})| \leq |\mathfrak{a}|.$$

As for the real case, $\mathfrak{A}$ is called a *Riesz algebra* (or a *lattice-ordered algebra*) if $\mathfrak{A}$ is in addition an associative algebra over the complex field $\mathbf{C}$ such that

$$|\mathfrak{a}b| \leq |\mathfrak{a}|\,|\mathfrak{b}| \quad \text{for all } \mathfrak{a}, b \in \mathfrak{A}.$$

It is easily seen that if $\mathfrak{A}$ is a complex Riesz algebra then A is a real Riesz algebra with respect to the multiplication inherited from $\mathfrak{A}$. Conversely, if A is a real Riesz algebra, then its multiplication extends uniquely to a multiplication in $\mathfrak{A}$ so that $\mathfrak{A}$ becomes a complex Riesz algebra. The proof of this converse is far from being trivial and can be found in [**9**]. If the complex Riesz algebra $\mathfrak{A}$ has a unit element e, then e must be a member of A but need not be positive [**8**]. The spectrum of an element

$\mathfrak{a}$ of the complex Riesz algebra $\mathfrak{A}$ is denoted by $\sigma(\mathfrak{a})$, as usual. We refer to [**9, 11**] for more background on complex Riesz algebras.

The above definition of real f-algebras can be extended in an obvious way to the complex case. Precisely, the complex Riesz algebra $\mathfrak{A} = A + iA$ is called an *f-algebra* if

$$|\mathfrak{a}\mathfrak{c}| \wedge |\mathfrak{b}| = |\mathfrak{c}\mathfrak{a}| \wedge |\mathfrak{b}| = 0 \quad \text{for all } \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathfrak{A} \text{ with } |\mathfrak{a}| \wedge |\mathfrak{b}| = 0.$$

Actually, the complex Riesz algebra $\mathfrak{A}$ is an f-algebra if and only if the underlying real Riesz algebra A is an f-algebra [**3**]. Hence, any complex f-algebra $\mathfrak{A}$ is commutative since the real f-algebra A is assumed to be uniformly complete and then Archimedean. Moreover, the unit element e of a complex f-algebra is always positive. On the other hand, the equality

$$|\mathfrak{a}\mathfrak{b}| = |\mathfrak{a}|\,|\mathfrak{b}| \quad \text{for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$$

holds as soon as $\mathfrak{A}$ is a complex f-algebra. Also, if $\mathfrak{a}$ is an invertible element in the complex f-algebra $\mathfrak{A}$ with a unit element e, then so is its modulus $|\mathfrak{a}|$. The converse also holds, that is, if $|\mathfrak{a}|$ has an inverse in $\mathfrak{A}$, then so has $\mathfrak{a}$. In particular, if $\mathfrak{a} \in \mathfrak{A}$ and $|\mathfrak{a}| \geq e$ then $\mathfrak{a}$ has an inverse in $\mathfrak{A}$ [**4**]. Complex f-algebras are studied extensively in [**3, 11**].

At this point, let $\mathfrak{A}$ be a complex Riesz space and A the uniformly complete real Riesz space such that $\mathfrak{A} = A + iA$. A *linear functional* on $\mathfrak{A}$ is a linear map of $\mathfrak{A}$ to $\mathbf{C}$. The linear functional f on $\mathfrak{A}$ is said be *real* if f maps A into $\mathbf{R}$. Hence, the linear functional f on $\mathfrak{A}$ is real if and only the restriction of f to A is a linear map of A to $\mathbf{R}$. The linear functional f on $\mathfrak{A}$ is said to be *positive* if f is increasing, that is, $f(a) \leq f(b)$ in $\mathbf{R}$ whenever $a \leq b$ in A. Clearly, the linear functional f on $\mathfrak{A}$ is positive if and only if f sends A^+ to positive real numbers. Hence, any positive linear functional on $\mathfrak{A}$ is real. The linear functional f on $\mathfrak{A}$ is called a *lattice homomorphism* if

$$f(|\mathfrak{a}|) = |f(\mathfrak{a})| \quad \text{for all } \mathfrak{a} \in \mathfrak{A}.$$

Obviously, any lattice homomorphism on $\mathfrak{A}$ is positive and then real. In fact, the linear functional f on $\mathfrak{A}$ is a lattice homomorphism if and only if f is real and the restriction of f to A is a real lattice homomorphism on A. So, the linear functional f on $\mathfrak{A}$ is a lattice homomorphism on A if and only if f is positive and

$$f(a)\,f(b) = 0 \quad \text{for all } a, b \in A \text{ with } a \wedge b = 0.$$

Finally, if $\mathfrak{A}$ is a complex Riesz algebra then a linear functional on $\mathfrak{A}$ is called an *algebra homomorphism* if

$$f(\mathfrak{a}b) = f(\mathfrak{a}) f(\mathfrak{b}) \quad \text{for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}.$$

More about lattice homomorphisms on complex Riesz spaces can be found in [**16, 17**].

We have gathered now all the ingredients for the main result of this note.

Theorem 2.1. *Let $\mathfrak{A}$ be a complex f-algebra with a unit element e, and let f be a linear functional on $\mathfrak{A}$. Then the following are equivalent.*

(i) *f is a lattice homomorphism with $f(e) = 1$.*

(ii) *f is a nonzero algebra homomorphism.*

(iii) *$f(\mathfrak{a}) \in \sigma(\mathfrak{a})$ for all $\mathfrak{a} \in \mathfrak{A}$.*

Proof. (i) $\Rightarrow$ (ii). The proof of this implication follows straightforwardly from its correspondent real version, which can be found in [**12**].

(ii) $\Rightarrow$ (iii). Routine.

(iii) $\Rightarrow$ (i). The equality $f(e) = 1$ is straightforward. We first prove that f is positive (and then real). Let z be a complex number and $a \in A^{+}$. Hence,

$$|a - ze| = |(a - \mathrm{Re}\,(z)\,e) + i\mathrm{Im}\,(z)\,e| \geq (\max\{-\mathrm{Re}\,z, |\mathrm{Im}\,z|\})\,e.$$

Thus, if $(\mathrm{Im}\,(z) \neq 0)$ or $(\mathrm{Im}\,(z) = 0$ and $\mathrm{Re}\,z < 0)$, then $a - ze$ is invertible in $\mathfrak{A}$. It follows quickly that all spectral values of a are positive real numbers. In view of (iii), we derive that $f(a)$ is again a positive real number. This means that the linear functional f is positive, as required.

Finally, we establish that f is a lattice homomorphism. Since f is positive, it suffices to prove that $f(a)f(b) = 0$ for all $a, b \in A$ with $a \wedge b = 0$. Arguing by contradiction, assume that there exist $a, b \in A$ such that $a \wedge b = 0$ and $f(a)f(b) \neq 0$. Define

$$u = \frac{1}{f(a)}a \quad \text{and} \quad v = \frac{1}{f(b)}b.$$

Obviously, $f(u) = f(v) = 1$. On the other hand, from $a \wedge b = 0$ it follows that $u \wedge v = 0$. Therefore,

$$e = e - (u \wedge v) = (e - u) \vee (e - v) \leq |e - u + i(e - v)|.$$

We derive that $e - u + i(e - v)$ is invertible in $\mathfrak{A}$. Consequently, we get $0 \notin \sigma(e - u + i(e - v))$ and, by (iii), $f(e - u + i(e - v)) \neq 0$. But this contradicts the calculation

$$f(e - u + i(e - v)) = f(e) - f(u) + i(f(e) - f(v)) = 0.$$

The proof of the theorem is complete. □

We end this section with the following remarks.

Remark 2.1. Let $C(X)$ indicate the set of all complex-valued continuous functions on a topological space X. Clearly, $C(X)$ is a complex f-algebra with respect to the pointwise operations and ordering. Besides, the constant one function on X is a unit element in $C(X)$. If X is pseudocompact, i.e., each function in $C(X)$ is bounded [**5**], then $C(X)$ is a complex Banach algebra under the uniform-norm. From the Gleason-Kahane-Zelasko theorem, it follows directly that a linear functional f on $C(X)$ is a nonzero algebra homomorphism if and only if

$$f(\mathfrak{a}) \in \mathfrak{a}(X) = \{\mathfrak{a}(x) : x \in X\}.$$

However, Theorem 2.1 shows that this characterization of nonzero complex-valued algebras homomorphisms on $C(X)$ holds actually for any topological space X.

3. A generalization of a theorem by Huijsmans. Let $\mathfrak{A}$ be a complex Riesz space, and let A denote the underlying uniformly complete real Riesz space. Recall that a vector subspace $\mathfrak{B}$ of $\mathfrak{A}$ is called an *ideal* if $\mathfrak{a} \in \mathfrak{B}$ whenever $\mathfrak{a} \in \mathfrak{A}$ and $|\mathfrak{a}| \leq |\mathfrak{b}|$ for some $\mathfrak{b} \in \mathfrak{B}$. An *ideal* $\mathfrak{B}$ of $\mathfrak{A}$ is called a *band* in $\mathfrak{A}$ if $\mathfrak{B} \cap A$ is a band in A. Also, the band $\mathfrak{B}$ in $\mathfrak{A}$ is said to be a *projection band* if $\mathfrak{B} \cap A$ is a projection band in A. The band $\mathfrak{B}$ in $\mathfrak{A}$ is a projection band if and only if $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{B}^{\perp}$, where $\mathfrak{B}^{\perp}$ is the disjoint complement of $\mathfrak{B}$ in $\mathfrak{A}$ given by

$$\mathfrak{B}^{\perp} = \{\mathfrak{a} \in \mathfrak{A} : |\mathfrak{a}| \wedge |\mathfrak{b}| = 0 \text{ for all } \mathfrak{b} \in \mathfrak{B}.\}$$

Clearly, the set of all bands in $\mathfrak{A}$ is closed under arbitrary intersection, so that the *principal band* $\mathfrak{B}_e$ in $\mathfrak{A}$ generated by an element $e \in \mathfrak{A}$ in $\mathfrak{A}$ can be defined as the smallest band in $\mathfrak{A}$ that contains e. We refer the reader to [**16, 17, 19**] for further information about bands in complex Riesz spaces.

Now assume $\mathfrak{A}$ to be a complex Riesz algebra with a positive unit element e. A subset $\mathfrak{B}$ of $\mathfrak{A}$ is called an *f-subalgebra* of $\mathfrak{A}$ if $\mathfrak{B}$ is a Riesz subspace of $\mathfrak{A}$ and an f-algebra with respect to the operations and ordering inherited from $\mathfrak{A}$. Furthermore, the f-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is said to be *full* if $\mathfrak{B}$ is inversion closed, that is, if $\mathfrak{a} \in \mathfrak{B}$ has an inverse $\mathfrak{a}^{-1}$ in $\mathfrak{A}$ then $\mathfrak{a}^{-1} \in \mathfrak{B}$. It turns out that the principal band $\mathfrak{B}_e$ in $\mathfrak{A}$ generated by e is a full f-subalgebra of $\mathfrak{A}$. The details follow next.

Theorem 3.1. *Let $\mathfrak{A}$ be a complex Riesz algebra with a positive unit element e. Then the following hold.*

(i) *$\mathfrak{B}_e$ is a projection band in $\mathfrak{A}$.*

(ii) *$\mathfrak{B}_e$ is a full f-subalgebra of $\mathfrak{A}$.*

Proof. First, recall that the underlying real Riesz algebra A is uniformly complete and has e as a positive unit element. Let B_e be the principal band in A generated by e. It is readily verified that $\mathfrak{B}_e$ coincides with $B_e + iB_e$.

(i) By definition, $\mathfrak{B}_e$ is a projection band in $\mathfrak{A}$ if B_e is a projection band in A, which has been proved by Basly, Huijsmans, de Pagter, and Triki [**1**, Theorem 4] (see also [**10**]).

(ii) Lavric [**15**, Theorems 1 and 2] proved that B_e is a full f-subalgebra of A. This means that B_e is an f-algebra with respect to the ordering inherited from A and contains the inverse of every element of B_e which is invertible in A. It follows directly that $\mathfrak{B}_e$ is an f-subalgebra of $\mathfrak{A}$. It remains to show that $\mathfrak{B}_e$ is full. Hence, let $\mathfrak{a} \in \mathfrak{B}_e$ and assume that $\mathfrak{a}$ has an inverse $\mathfrak{a}^{-1}$ in $\mathfrak{A}$. We claim that $\mathfrak{a}^{-1} \in \mathfrak{B}_e$. To this end, put $\overline{\mathfrak{a}} = \mathrm{Re}\,(\mathfrak{a}) - i\mathrm{Im}\,(\mathfrak{a})$ and remark that $\overline{\mathfrak{a}} \in \mathfrak{B}_e$. A simple calculation yields that $\overline{\mathfrak{a}}$ has $\overline{\mathfrak{a}}^{-1} = \mathrm{Re}\,(\mathfrak{a}^{-1}) - i\mathrm{Im}\,(\mathfrak{a}^{-1})$ as an inverse in $\mathfrak{A}$. Since $\mathfrak{B}_e$ is an f-subalgebra of $\mathfrak{A}$, the equalities $|\mathfrak{a}|^2 = \mathfrak{a}\overline{\mathfrak{a}} = \mathrm{Re}\,(\mathfrak{a})^2 + \mathrm{Im}\,(\mathfrak{a})^2$ hold in $\mathfrak{B}_e$, where we use a result by Beukers, Huijsmans, and de Pagter [**3**, Section 5]. We derive that $|\mathfrak{a}|^2$

has an inverse $|\mathfrak{a}|^{-2}$ in $\mathfrak{A}$. Observe that

$$e = |\mathfrak{a}|^2 |\mathfrak{a}|^{-2} = |\mathfrak{a}|^2 \operatorname{Re}\left(|\mathfrak{a}|^{-2}\right) + i |\mathfrak{a}|^2 \operatorname{Im}\left(|\mathfrak{a}|^{-2}\right)$$

and so

$$\operatorname{Im}\left(|\mathfrak{a}|^{-2}\right) = |\mathfrak{a}|^{-2}\left(|\mathfrak{a}|^2 \operatorname{Im}\left(|\mathfrak{a}|^{-2}\right)\right) = 0.$$

Hence, $|\mathfrak{a}|^{-2} \in A$. On the other hand, $\mathfrak{a} \in \mathfrak{B}_e$ and thus $|\mathfrak{a}|^2 \in B_e$. But then $|\mathfrak{a}|^{-2} \in B_e$ because B_e is a full f-subalgebra of A. As $\mathfrak{B}_e$ is an f-subalgebra of $\mathfrak{A}$, we obtain

$$\mathfrak{a}^{-1} = \mathfrak{a}^{-1} |\mathfrak{a}|^2 |\mathfrak{a}|^{-2} = \overline{\mathfrak{a}} |\mathfrak{a}|^{-2} \in \mathfrak{B}_e$$

and we are done. □

Let $\mathfrak{A}$ be a complex Riesz algebra with a positive unit element e. By Theorem 3.1 (i), one may define the projection operator P_e of $\mathfrak{A}$ onto $\mathfrak{B}_e$, which is usually called a *band* (or *order*) *projection* because $\mathfrak{B}_e$ is a projection band in $\mathfrak{A}$. In particular,

$$|P_e(\mathfrak{a})| \wedge |\mathfrak{a} - P_e(\mathfrak{a})| = 0 \quad \text{for all } \mathfrak{a} \in \mathfrak{A},$$

and hence,

$$|\mathfrak{a}| = |\mathfrak{a} - P_e(\mathfrak{a})| + |P_e(\mathfrak{a})| \quad \text{for all } \mathfrak{a} \in \mathfrak{A}$$

(see [**19**, Section 91] for these properties).

We are in a position at this point to characterize identity preserving complex-valued lattice homomorphisms on $\mathfrak{A}$.

Theorem 3.2. *Let $\mathfrak{A}$ be a complex Riesz algebra with a positive unit element e and f a linear functional on $\mathfrak{A}$. Then the following are equivalent.*

(i) *f is a lattice homomorphism with $f(e) = 1$.*

(ii) *$f(\mathfrak{a}) \in \sigma(P_e(\mathfrak{a}))$ for all $\mathfrak{a} \in \mathfrak{A}$.*

Proof. (i) ⇒ (ii). Let $\mathfrak{a} \in \mathfrak{B}_e^{\perp}$ and observe that $|\mathfrak{a}| \wedge e = 0$ leads to

$$|f(\mathfrak{a})| \wedge 1 = f(|\mathfrak{a}|) \wedge f(e) = 0.$$

Therefore, $f(\mathfrak{a}) = 0$. It follows that

$$f(\mathfrak{a}) = f(P_e(\mathfrak{a})) \quad \text{for all } \mathfrak{a} \in \mathfrak{A}.$$

By Theorem 2.1, if $\mathfrak{a} \in \mathfrak{A}$, then $f(\mathfrak{a}) = f(P_e(\mathfrak{a})) \in \sigma_e(P_e(\mathfrak{a}))$, where $\sigma_e(P_e(\mathfrak{a}))$ denotes the spectrum of $P_e(\mathfrak{a})$ in the f-subalgebra $\mathfrak{B}_e$ of $\mathfrak{A}$, see Theorem 3.1 (i). But the f-subalgebra $\mathfrak{B}_e$ is full in $\mathfrak{A}$, where we use Theorem 3.1 (ii). We derive that $\sigma_e(P_e(\mathfrak{a})) = \sigma(P_e(\mathfrak{a}))$ and then $f(\mathfrak{a}) \in \sigma(P_e(\mathfrak{a}))$, which is the desired result.

(ii) $\Rightarrow$ (i). First of all, notice that

$$f(e) \in \sigma(P_e(e)) = \sigma(e) = \{1\},$$

so $f(e) = 1$. Furthermore, if $\mathfrak{a} \in \mathfrak{B}_e$ then $f(\mathfrak{a}) \in \sigma(\mathfrak{a})$. Hence, Theorem 2.1 yields that f is a lattice homomorphism as a linear functional on $\mathfrak{B}_e$. On the other hand, if $\mathfrak{a} \in \mathfrak{B}_e^{\mathrm{d}}$ then

$$f(\mathfrak{a}) \in \sigma(P_e(\mathfrak{a})) = \{0\}.$$

Consequently, if $\mathfrak{a} \in \mathfrak{A}$, then

$$\begin{aligned} f(|\mathfrak{a}|) &= f(|P_e(\mathfrak{a})| + |\mathfrak{a} - P_e(\mathfrak{a})|) \\ &= f(|P_e(\mathfrak{a})|) = |f(P_e(\mathfrak{a}))| \\ &= |f(P_e(\mathfrak{a})) + f(\mathfrak{a} - P_e(\mathfrak{a}))| = |f(\mathfrak{a})|. \end{aligned}$$

Hence, f is a lattice homomorphism and the proof is finished. □

Now, let E be a Dedekind complete complex Riesz space, and let $L^r(E)$ denote the Dedekind complete complex Riesz space of all regular linear operators on E. Clearly, $L^r(E)$ is a complex Riesz algebra with respect to linear operators composition. The principal band in $L^r(E)$ generated by the identity operator I on E coincides with the set of all orthomorphisms $\mathrm{Orth}(E)$ on E. Recall, by the way, that $\pi \in L^r(E)$ is called an *orthomorphism* on E if

$$|\pi(\mathfrak{a})| \wedge |\mathfrak{b}| = 0 \quad \text{for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{A} \text{ with } |\mathfrak{a}| \wedge |\mathfrak{b}| = 0.$$

Hence, any orthomorphism π on E preserves disjointness, meaning that

$$|\pi(\mathfrak{a})| \wedge |\pi(\mathfrak{b})| = 0 \quad \text{for all } \mathfrak{a}, \mathfrak{b} \in \mathfrak{A} \text{ with } |\mathfrak{a}| \wedge |\mathfrak{b}| = 0.$$

Therefore, if $\pi \in \operatorname{Orth}(E)$, then

$$|\pi(\mathfrak{a})| = |\pi|(|\mathfrak{a}|) = |\pi(|\mathfrak{a}|)| = ||\pi|(\mathfrak{a})| \quad \text{for all } \mathfrak{a} \in \mathfrak{A},$$

where we use a result by Grobler and Huijsmans [**7**] (see also [**2**]). In view of Theorem 3.1, $\operatorname{Orth}(E)$ is a projection band in $L^r(E)$ and a full f-subalgebra of $L^r(E)$. But these properties of $\operatorname{Orth}(E)$ can be obtained alternatively and quite easily from their corresponding real versions [**16**]. The band projection of $L^r(E)$ onto $\operatorname{Orth}(E)$ is called the *diagonal operator* and denoted by D rather than by P_I. The terminology and notation were chosen by Schep [**18**] because if E is of finite dimension then $\operatorname{Orth}(E)$ can be identified with the collection of all diagonal matrices. The reader is encouraged to consult [**4**] for further information on orthomorphisms on complex Riesz spaces.

The complex algebra of all linear operators is denoted by $L(E)$, as usual. Clearly, $L^r(E)$ is a subalgebra of $L(E)$. The spectrum of an operator $T \in L(E)$ is denoted by $\sigma(T)$, that is,

$$\sigma(T) = \{z \in \mathbf{C} : T - zI \text{ is not invertible in } L(E)\}.$$

The order-spectrum of $T \in L^r(E)$ is denoted by $\sigma_o(T)$ and it is given by

$$\sigma_o(T) = \{z \in \mathbf{C} : T - zI \text{ is not invertible in } L^r(E)\}$$

(see [**16**]).

We arrive to the last result of this note.

Corollary 3.1. *Let E be a Dedekind complete complex Riesz space and f a linear functional on $L^r(E)$. Then the following are equivalent.*

(i) *f is lattice homomorphism with $f(I) = 1$.*

(ii) *$f(T) \in \sigma(D(T))$ for all $T \in L^r(E)$.*

Proof. According to Theorem 3.2, we have only to show that

$$\sigma_o(\pi) = \sigma(\pi) \quad \text{for all } \pi \in \operatorname{Orth}(E).$$

Obviously, if $\pi \in \operatorname{Orth}(E)$ then $\sigma(\pi) \subset \sigma_o(\pi)$. For the converse inclusion, it suffices to prove that the f-subalgebra $\operatorname{Orth}(E)$ of $L^r(E)$

is full as a subalgebra of $L(E)$. To do this, let $\pi \in \mathrm{Orth}\,(E)$ and assume that π has an inverse π^{-1} in $L(E)$. If $\mathfrak{a} \in E$ such that $|\pi|(\mathfrak{a}) = 0$, then

$$0 = \left||\pi|\,(\mathfrak{a})\right| = \left|\pi\,(\mathfrak{a})\right|.$$

Therefore, $\pi(\mathfrak{a}) = 0$, so $\mathfrak{a} = 0$ because π is injective. Hence $|\pi|$ is injective. Moreover, if a is a positive element of E, then

$$|\pi|\left(\left|\pi^{-1}\,(a)\right|\right) = \left|\pi\left(\pi^{-1}\,(a)\right)\right| = a.$$

It follows quickly that $|\pi|$ is surjective, so $|\pi|$ has an inverse $|\pi|^{-1}$ in $L(E)$. Furthermore, if a is a positive element of E, then

$$|\pi|^{-1}\,(a) = |\pi|^{-1}\left(\left|\pi\left(\pi^{-1}\,(a)\right)\right|\right) = |\pi|^{-1}\left(|\pi|\left(\left|\pi^{-1}\,(a)\right|\right)\right) = \left|\pi^{-1}\,(a)\right|.$$

This yields that $|\pi|^{-1}$ is a positive operator on E. In summary, $|\pi|$ is a positive orthomorphism on E and its inverse is a positive operator on E. Using [**16**, Theorem 3.1.10], we derive that $|\pi|^{-1}$ is an orthomorphism on E and so is $|\pi|^{-2}$. On the other hand, it is clear that

$$\overline{\pi} = \mathrm{Re}\,(\pi) - i\mathrm{Im}\,(\pi) \in \mathrm{Orth}\,(E) \quad \text{and} \quad |\pi|^2 = \pi \circ \overline{\pi}.$$

Hence,

$$\pi^{-1} = \overline{\pi} \circ |\pi|^{-2} \in \mathrm{Orth}\,(E),$$

and we are done.

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