ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 6, 2009

ON e-POWER b-HAPPY NUMBERS

XIA ZHOU AND TIANXIN CAI

ABSTRACT. Let e and b be positive integers; an e-power b-happy number is a positive integer, a such that $S_{e,\ b}^r(a) = 1$ for some $r \geq 0$. Here $S_{e,\ b}(a)$ is the sum of the eth powers of digits of a in base b and $S_{e,\ b}^r(a) = S_{e,\ b}(S_{e,\ b}^{r-1}(a))$. Let

$$\mathcal{A} = \{p \text{ prime} : p \mid (b-1) \text{ and } (p-1) \mid (e-1)\}, \quad P = \prod_{p \in \mathcal{A}} p.$$

In this paper, we prove that arbitrarily long sequences of P-consecutive e-power b-happy numbers exist for any e, b.

1. Introduction. For $a \in \mathbf{Z}^+$, we define $S_2(a)$ as the sum of the squares the decimal digits of a. For $a \in \mathbf{Z}^+$, let $S_2^0(a) = a$, and for $r \geq 1$, let $S_2^r(a) = S_2(S_2^{r-1}(a))$. A happy number is a positive integer a such that $S_2^r(a) = 1$ for some $r \geq 0$. In [**4**], Guy asked whether there exist sequences of consecutive happy numbers of arbitrary length. In 2000, El-Sedy and Siksek [**1**] gave an affirmative answer to this question.

Let e and b be positive integers. In 2001, Grundman and Teeple [**2**] first defined the so-called e-power b-happy number, i.e., they named positive integer a an e-power b-happy number if $S_{e,\ b}^r(a) = 1$ for some $r \geq 0$: here $S_{e,\ b}(a)$ is the sum of the eth powers of the digits of a in base b and $S_{e,\ b}^r(a) = S_{e,\ b}(S_{e,\ b}^{r-1}(a))$.

Let

$$\mathcal{A} = \{p \text{ prime} : p \mid (b-1) \text{ and } (p-1) \mid (e-1)\}, \qquad P = \prod_{p \in \mathcal{A}} p.$$

If $P > 1$ for some e and b, there are no consecutive e-power b-happy numbers. In fact, for any $p \in \mathcal{A}$,

$$S_{e,\ b}\bigg(\sum_{j=0}^{k} a_j \times b^j\bigg) \equiv \sum_{j=0}^{k} a_j^e \equiv \sum_{j=0}^{k} a_j \equiv \sum_{j=0}^{k} a_j \times b^j \pmod{p}.$$

2000 AMS *Mathematics subject classification.* Primary 11A07, 11A63.
Keywords and phrases. Happy numbers, cycles of numbers, digital problems.
This research was supported by the National Natural Science Foundation of China No. 10871169.
Received by the editors on November 9, 2006, and in revised form on June 13, 2007.
DOI:10.1216/RMJ-2009-39-6-2073

That is to say, $S_{e,\ b}(n) \equiv n \pmod{P}$ for every n. So n is an e-power b-happy number only if $n \equiv 1 \pmod{P}$. In [**3**], a d-consecutive sequence is defined to be an arithmetic sequence with constant difference d. It is natural to ask the following question.

Question. Do there exist sequences of P-consecutive e-power b-happy numbers of arbitrary length for any $e,\ b$?

Some partial answers to this question have been obtained. In [**3**], Grundman and Teeple showed that, when $e = 2,\ b \geq 2$; $e \geq 2,\ b = 2$; $e = 3,\ 2 \leq b \leq 13$ or $b = 2^r + 1,\ 3 \times 2^r + 1$, the answer is yes. Recently, Pan [**5**] gave an affirmative answer to this question when $P = 1$. In this paper, we give an affirmative answer to the above question completely.

Main theorem. *There exist sequences of P-consecutive e-power b-happy numbers of arbitrary length.*

2. Some preliminary properties.

Definition 2.1. Given e and b, let $a = \sum_{i=0}^{n} a_i b^i$ with $0 \leq a_i \leq b-1$. We define the function $S_{e,\ b} : \mathbf{N} \to \mathbf{N}$ by

$$S_{e,\ b}(a) = \sum_{i=0}^{n} a_i^e.$$

A positive integer x is then said to be e-power b-happy if $S_{e,\ b}^r(x) = 1$ for some $r \geq 0$ (sometimes, we simply say x is happy; otherwise, we say it is unhappy).

As usual, we define $S_{e,\ b}^0(x) = x$.

Lemma 2.1. *If $S_{e,\ b}^r(y) = x$ where $r \geq 0$ and x is happy, then y is also happy.*

Proof. Obvious. □

Lemma 2.2. *For any positive integer x the set $S_{e,\ b}^{-1}(x) = \{y \in \mathbf{N} : S_{e,\ b}(y) = x\}$ is nonempty.*

Proof. We simply observe that the number $y = \sum_{i=0}^{x-1} b^i$ has x digits with each one 1, and so $S_{e,\ b}(y) = x$. $\square$

Lemma 2.3. *Suppose that x, y and t are positive integers such that $b^t > y$. Then $S_{e,\ b}(b^t\ x + y) = S_{e,\ b}(x) + S_{e,\ b}(y)$.*

Proof. This follows from the fact that the digits of $b^t\ x + y$ are the digits of x and the digits of y with possibly some zeros in between. $\square$

Lemma 2.4. *For any positive integers $a_1, a_2, \dots , a_n$ and r, y, there exists an integer Y such that $S^r_{e,\ b}(Y + a_i) = y + S^r_{e,\ b}(a_i)$ for $1 \le i \le n$.*

Proof. Since the set $\{S^j_{e,\ b}(a_i) :\ 1 \le i \le n,\ 0 \le j \le r\}$ is finite, there exists a positive integer t such that $b^t > \max\{S^j_{e,\ b}(a_i) :\ 1 \le i \le n,\ 0 \le j \le r\}$. From Lemma 2.2, there exists an integer l_1 such that $S_{e,\ b}(l_1) = y$. Now, for each $k \ge 2$, choose l_k such that $S_{e,\ b}(l_k) = b^t l_{k-1}$. From Lemma 2.3, we have

$$\begin{aligned} S^r_{e,\ b}(b^t l_r + a_i) &= S^{r-1}_{e,\ b}(S_{e,\ b}(b^t l_r + a_i)) = S^{r-1}_{e,\ b}(b^t l_{r-1} + S_{e,\ b}(a_i)) \\ &= S^{r-2}_{e,\ b}(S_{e,\ b}(b^t l_{r-1} + S_{e,\ b}(a_i))) \\ &= S^{r-2}_{e,\ b}(b^t l_{r-2} + S^2_{e,\ b}(a_i)) \\ &= \cdots = S_{e,\ b}(b^t l_1 + S^{r-1}_{e,\ b}(a_i)) = y + S^r_{e,\ b}(a_i). \end{aligned}$$

Taking $Y = b^t l_r$ proves the lemma. $\square$

The above four lemmas are similar to those in [**1**].

Lemma 2.5. *There exists a positive integer M such that $S_{e,\ b}(a) < a$ for all $a > M$.*

Proof. This is clear since $S_{e,\ b}(a) \le (b-1)^e \log_b a$. $\square$

Lemma 2.6. *Let a be a positive integer. Applying the function $S_{e,\ b}$ repeatedly to a, we eventually reach a cycle with finite length or arrive at 1. Moreover, the number of cycles of $S_{e,\ b}$ is finite.*

Proof. This follows trivially from Lemma 2.5. □

From Lemma 2.6, we know that, for any e and b, there exists a finite set $\mathcal{B}$, such that:

(1) for any positive integer n, $S^r_{e,\ b}(n) \in \mathcal{B}$ for some integer $r \geq 0$.

(2) for any $d \in \mathcal{B}$, $S^l_{e,\ b}(d) = d$ for some integer $l \geq 0$.

Choose the subset $\mathcal{D}_{e,\ b} = \{x \in \mathcal{B} \mid x \equiv 1 \pmod{P}\}$. Since for any integer n, $S_{e,\ b}(n) \equiv n \pmod{P}$, we see that $n \equiv 1 \pmod{P}$ if and only if $S^r_{e,\ b}(n) \in \mathcal{D}_{e,\ b}$, for some $r \geq 0$.

3. Some further properties.

In order to prove the main theorem, we need some more lemmas.

Lemma 3.1. *Given e and b, if for any $d \in \mathcal{D}_{e,\ b}$, there exists a positive integer y such that both $y+1$ and $y+d$ are e-power b-happy numbers, then there exists a sequence $\{l, l+P, \ldots, l+Pm-P\}$ of e-power b-happy numbers of any length m.*

Proof. Let $\mathcal{D}_{e,\ b} = \{1, d_1, d_2, \ldots, d_k\}$. We prove the lemma by induction on the length m.

Suppose first that $m = 2$. There exists an r such $S^r_{e,\ b}(P+1) \in \mathcal{D}_{e,\ b}$; thus, $S^r_{e,\ b}(P+1) = 1$ or d_i for some $1 \leq i \leq k$. By assumption, there exists y_i such that both y_i+1 and y_i+d_i are e-power b-happy numbers. According to Lemma 2.4, there exist Y, $S^r_{e,\ b}(Y+1) = y_i + S^r_{e,\ b}(1) = y_i+1$, and $S^r_{e,\ b}(Y+P+1) = y_i + S^r_{e,\ b}(P+1) = y_i+d_i$ or y_i+1. Let $l = Y+1$. It follows from Lemma 2.1 that both l and $l+P$ are happy.

Now we assume that the lemma holds for $m = u-1$; that is, there exists an l' such that $l', \ldots, l'+(u-2)P$ are all happy. We consider two cases. If $l'+(u-1)P$ is happy, take $l = l'$ and the proof is complete. So suppose $l'+(u-1)P$ is unhappy. Note that $l'+(u-1)P \equiv l' \equiv 1 \pmod{P}$. Thus, there exists an r such that $S^r_{e,\ b}(l'+(u-1)P) = d_j$ for some $1 \leq j \leq k$. According to the properties of $\mathcal{D}_{e,\ b}$, there exists a positive number v so that

$$S^{r+v}_{e,\ b}(l'+(u-1)P) = S^r_{e,\ b}(l'+(u-1)P) = d_j.$$

Meanwhile, there exists an R such that $S^R_{e,\ b}(l'+(i-1)P) = 1$ for any

$1 \le i \le u-1$, since $l', \ldots, l'+(u-2)P$ are all happy. Let $K \equiv r \pmod{v}$ satisfy $K > R$. By Lemma 2.4, there exists a positive integer Y such that

$$S_{e,\ b}^{K}(Y+l'+(i-1)P) = y_j + S_{e,\ b}^{K}(l'+(i-1)P) = y_j + 1,$$

for $1 \le i \le u-1$, and

$$\begin{aligned} S_{e,\ b}^{K}(Y+l'+(u-1)P) &= y_j + S_{e,\ b}^{K}(l'+(u-1)P) \\ &= y_j + S_{e,\ b}^{r}(l'+(u-1)P) = y_j + d_j. \end{aligned}$$

That is to say, $Y+l'+(i-1)P$ $(1 \le i \le u)$ are all happy. Taking $l = Y + l'$, then $l, \ldots, l+(u-1)P$ are all happy. This completes the proof of Lemma 3.1. □

In [**5**], Pan provided a method to find the number y, such that $y+1$ and $y+d$ are e-power b-happy numbers for any $d \in \mathcal{D}_{e,\ b}$ when $P = 1$. Now we modify his method and find such a number y for any P.

Lemma 3.2. *Suppose that for any integer $a \equiv 1 \pmod{P}$, there exists a happy number h such that $h \equiv\ a \pmod{(b-1)^e}$. Then, for any $d \equiv 1 \pmod{P}$, there exists a positive integer l such that $l+1$ and $l+d$ are also happy.*

Proof. Assume $d = 1 + xP$ and choose a positive integer s such that $b^s > xP$. Let $x^* = b^s - xP$. Then $x^* \equiv 1 \pmod{P}$. Since $S_{e,\ b}(x^*) \equiv 1 \pmod{P}$, we have a happy number h such that $h \equiv S_{e,\ b}(x^*) \pmod{(b-1)^e}$. Suppose $h > S_{e,\ b}(x^*)$ (if $h \le S_{e,\ b}(x^*)$ there exists a t such that $b^t > S_{e,\ b}(x^*)$, and then we can replace h by hb^t) and write $h = k(b-1)^e + S_{e,\ b}(x^*)$. Taking

$$l = x^* - 1 + \sum_{j=0}^{k-1}(b-1)b^{s+j},$$

we have

$$S_{e,\ b}(l+1) = k(b-1)^e + S_{e,\ b}(x^*) = h,$$

and

$$S_{e,\ b}(l+d) = S_{e,\ b}(b^{s+k}) = 1.$$

Therefore, both $l+1$ and $l+d$ are happy. □

Lemma 3.3. *If, for any integer* $a \equiv 1 \pmod{P}$, *there exists a happy number* h *such that* $h \equiv a \pmod{b-1}$, *then we can find a happy number* h', $h' \equiv a \pmod{(b-1)^e}$.

Proof. Let $s = \varphi((b-1)^e)$. We have $b^s \equiv 1 \pmod{(b-1)^e}$. Choose h happy such that $h \equiv a \pmod{b-1}$. Replacing h by hb^{ms} for a suitable m, we may suppose that both that h is happy and that $h > (b-1)^e$. Then

$$h \equiv a + k(b-1) \pmod{(b-1)^e}, \quad 0 \le k \le \ (b-1)^{e-1} - 1.$$

Taking

$$h' = \sum_{i=1}^{(b-1)^e - k} b^{is+1} + \sum_{j=(b-1)^e-k+1}^{h} b^{2js},$$

then

$$h' \equiv ((b-1)^e - k)b + (h - (b-1)^e + k) \equiv h - k(b-1) \equiv a \pmod{(b-1)^e},$$

and $S_{e,\ b}(h') = h$, hence h' is happy. □

Lemma 3.4. *If, for any integer* $a \equiv 1 \pmod{P}$, *there exists a happy number* h *such that*

$$h \equiv S_{e,\ b}(l) \pmod{b-1},$$

for some $l \equiv a \pmod{b-1}$, *then we can find a happy number* h' *such that*

$$h' \equiv a \pmod{b-1}.$$

Proof. We choose $h > S_{e,\ b}(l)$, $h \equiv S_{e,\ b}(l) \pmod{b-1}$ and $s \ge 1$ such that $b^s > l$. Taking

$$h' = \sum_{j=1}^{h - S_{e,\ b}(l)} b^{s+j} + l,$$

then

$$h' \equiv h - S_{e,\ b}(l) + l \equiv a \pmod{b-1},$$

and

$$S_{e,\ b}(h') = h,$$

hence h' is happy. This completes the proof of Lemma 3.4. □

Now we give the proof of the main theorem.

4. Proof of the main theorem. Let $b-1 = \prod_{i=1}^{s} p_i^{\alpha_i} \prod_{j=1}^{r} q_j^{\beta_j}$ be the standard factorization, where $p_i \in \mathcal{A}$, $1 \le i \le s$, and $q_j \notin \mathcal{A}$, $1 \le j \le r$. Noting that q_j must be odd, we can find a primitive root g_j of $q_j^{\beta_j}$ for $1 \le j \le r$. For any $a \equiv 1 \pmod{P}$, taking $L(a)$ such that

$$L(a) \equiv \begin{cases} a - p_i + p_i^e \pmod{p_i^{\alpha_i}} & \text{if } a \not\equiv 1 \pmod{p_i^{\alpha_i}}, \\ 1 \pmod{p_i^{\alpha_i}} & \text{if } a \equiv 1 \pmod{p_i^{\alpha_i}}, \end{cases} \qquad 1 \le i \le s,$$

and

$$L(a) \equiv \begin{cases} a - g_j + g_j^e \pmod{q_i^{\beta_j}} & \text{if } a \not\equiv 1 \pmod{q_j^{\beta_j}}, \\ 1 \pmod{q_i^{\beta_j}} & \text{if } a \equiv 1 \pmod{q_j^{\beta_j}}, \end{cases} \qquad 1 \le j \le r.$$

Let $r_a = \min\{r \mid L^r(a) \equiv 1 \pmod{b-1}\}$, where L^r denotes the rth iterate of L. Since $a \equiv 1 \pmod{P}$, we have $a \equiv 1 \pmod{p_i}$. Noting that $p_i^2 \nmid (p_i - p_i^e)$ and $e \not\equiv 1 \pmod{q_j - 1}$, we have $(g_j - g_j^e, q_j) = 1$. By the definition of $L(a)$, r_a exists.

From Lemmas 3.1–3.3, we only need to prove that for every $a \equiv 1 \pmod{P}$, there exists a happy number h satisfying $h \equiv a \pmod{b-1}$.

If $r_a = 0$, then $a \equiv 1 \pmod{b-1}$; the above assertion holds. If $r_a = m$, we assume the assertion holds for any integer a' with $r_{a'} < m$. Since $r_{L(a)} = r_a - 1$, by inductive hypothesis, there exists a happy number h' such that

$$h' \equiv L(a) \pmod{b-1}.$$

Let g and n be positive integers such that

$$g \equiv \begin{cases} p_i \pmod{p_i^{\alpha_i}} & \text{if } a \not\equiv 1 \pmod{p_i^{\alpha_i}}, \\ 1 \pmod{p_i^{\alpha_i}} & \text{if } a \equiv 1 \pmod{p_i^{\alpha_i}}, \end{cases} \quad 1 \leq i \leq s,$$

$$g \equiv \begin{cases} g_j \pmod{q_i^{\beta_j}} & \text{if } a \not\equiv 1 \pmod{q_j^{\beta_j}}, \\ 1 \pmod{q_i^{\beta_j}} & \text{if } a \equiv 1 \pmod{q_j^{\beta_j}}, \end{cases} \quad 1 \leq j \leq r;$$

and

$$n \equiv \begin{cases} a - p_i \pmod{p_i^{\alpha_i}} & \text{if } a \not\equiv 1 \pmod{p_i^{\alpha_i}}, \\ 0 \pmod{p_i^{\alpha_i}} & \text{if } a \equiv 1 \pmod{p_i^{\alpha_i}}, \end{cases} \quad 1 \leq i \leq s,$$

$$n \equiv \begin{cases} a - g_j \pmod{q_i^{\beta_j}} & \text{if } a \not\equiv 1 \pmod{q_j^{\beta_j}}, \\ 0 \pmod{q_i^{\beta_j}} & \text{if } a \equiv 1 \pmod{q_j^{\beta_j}}, \end{cases} \quad 1 \leq j \leq r.$$

Taking

$$l = \sum_{i=1}^{b-1+n} b^i + g,$$

$$S_{e,\ b}(l) \equiv n + g^e \equiv L(a) \equiv h' \pmod{b-1},$$

and

$$l \equiv n + g \equiv a \pmod{b-1},$$

from Lemma 3.4, we can find a happy number h, such that $h \equiv a \pmod{b-1}$. By induction, we are done.

Acknowledgments. The authors thank the referee for his careful reading of the manuscript and making it much more readable.

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Box 1151, Zhejiang University, Hangzhou, 310027, P.R. China
Email address: xiazhou0821@hotmail.com

Department of Mathematics, Zhejiang University, Hangzhou, 310027, P.R. China
Email address: caitianxin@hotmail.com