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A DIRECT PROOF OF A THEOREM ON MONOTONICALLY NORMAL SPACES FOR GO-SPACES

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ABSTRACT. Let X be a GO-space. We give a direct proof of the following fact: If $\mathcal{O}$ is an open cover of X, then $\mathcal{O}$ has a σ-disjoint open, partial refinement $\mathcal{V}$ such that $X \setminus \cup \mathcal{V}$ is the union of a discrete family of stationary subsets of regular uncountable cardinals.

1. Introduction and terminology. We say that an open cover $\mathcal{O}$ of a topological space X has the $(\star)$ *property*, if $\mathcal{O}$ has a σ-disjoint open partial refinement $\mathcal{V}$ such that $X \setminus \cup \mathcal{V}$ is the union of a discrete family of closed subspaces which are homeomorphic to some stationary subset of a regular uncountable cardinal. A topological space X is said to have the $(\star)$*property*, if each open cover $\mathcal{O}$ of X has the $(\star)$ property.

In [**1**], Balogh and Rudin have proved that monotonically normal spaces have the $(\star)$ property, and in [**3**] it was proved that any GO-space is monotonically normal. So, GO-spaces have the $(\star)$ property. In this paper we give a direct and simple proof of this fact using "the method of coherent collections" described in [**4**].

A subset C of a linearly ordered set $(X, \leq)$ is called *convex* if

$$\{x \in X : a \leq x \leq b\} \subseteq C$$

for each $a, b \in C$ with $a \leq b$. Let $(X, \mathcal{T})$ be a topological space and $\leq$ a linear order on X. Recall that $(X, \mathcal{T}, \leq)$ is called a *generalized ordered space* (or GO-*space*), if $\mathcal{T}$ contains usual open interval topology on X and has a base consisting of convex subsets.

Let $\mathcal{B}$ be a collection of subsets of X and $A \subseteq X$. Then we write

$$st(A, \mathcal{B}) = \bigcup \{B \in \mathcal{B} : A \cap B \neq \varnothing\}.$$

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In particular, we write $st(x, \mathcal{B})$ instead of $st(\{x\}, \mathcal{B})$. For each positive integer $n > 1$, $st^n(x, \mathcal{B})$ denotes $st^{n-1}(st(x, \mathcal{B}), \mathcal{B})$ and

$$st^\infty(x, \mathcal{B}) = \bigcup_{n=1}^{\infty} st^n(x, \mathcal{B}).$$

A subset C of a limit ordinal κ is called *cub* in κ if C is closed and unbounded in κ. A subset L of κ is called *stationary* in κ if each cub in κ meets with L.

Lim and $\overline{A}$ denote the class of limit ordinals and the closure of $A \subseteq X$, respectively. As usual, ω denotes the first infinite ordinal.

We refer to [**2, 4**] for unexplained terminology and notations.

2. Main result. The following lemmas are needed in the proof of the main theorem of this paper. The proof of the first lemma is routine.

Lemma 1. *Let X be a set, $\mathcal{B}$ a collection of subsets of X, x, $y \in X$, and n an integer with $n > 1$. Then $x \in st^n(y, \mathcal{B})$ if and only if there exists $B_1, B_2, \dots, B_n$ in $\mathcal{B}$ such that $y \in B_1$, $x \in B_n$, and for each i with $1 \leq i < n$, $B_i \cap B_{i+1} \neq \varnothing$.*

The proof of the following two lemmas immediately follow from the previous lemma.

Lemma 2. *Let X be a set and $\mathcal{B}$ a collection of subsets of X. Then, for each $x, y \in X$, either $st^\infty(x, \mathcal{B}) = st^\infty(y, \mathcal{B})$ or $st^\infty(x, \mathcal{B}) \cap st^\infty(y, \mathcal{B}) = \varnothing$.*

Lemma 3. *Let $(X, \leq)$ be a linear ordered set and $\mathcal{C}$ a collection of convex subsets of X. Then:*

(a) *For each $x \in X$, $st^\infty(x, \mathcal{C})$ is a convex subset of X.*

(b) *For each $x \in X$, there are $\alpha_x \leq \omega$, $\beta_x \leq \omega$, an increasing sequence $\{x_n : n \in \alpha_x\}$ in X and a decreasing sequence $\{y_n : n \in \beta_x\}$in X such that*

(b_1) $x_0 = y_0 = x$,

(b$_2$) $st^\infty(x,\mathcal{C}) = (\cup_{n\in\alpha_x} st(x_n,\mathcal{C})) \cup (\cup_{n\in\beta_x} st(y_n,\mathcal{C}))$,

(b$_3$) *For each* m, $n \in \alpha_x$ *with* $m < n$, $x_n \notin st(x_m,\mathcal{C})$ *and, for each* m, $n \in \beta_x$ *with* $m < n$, $y_n \notin st(y_m,\mathcal{C})$,

(b$_4$) *For each* n, $n+1 \in \alpha_x$, $st(x_n,\mathcal{C}) \cap st(x_{n+1},\mathcal{C}) \neq \varnothing$ *and for each* n, $n+1 \in \beta_x$, $st(y_n,\mathcal{C}) \cap st(y_{n+1},\mathcal{C}) \neq \varnothing$.

The above lemma can also be found in [**4**].

Now we are ready for the main theorem.

Theorem 1. *Let* X *be a* GO*-space. Then* X *has the* $(\star)$ *property.*

Proof. Let $\mathcal{O}$ be an open cover of X. Since the space X has a base consisting of convex subsets of X, we can assume that each element of $\mathcal{O}$ is a convex subset of X. From Lemma 2 it is sufficient to prove that $\mathcal{O}$ has the $(\star)$ property in the subspace $st^\infty(x,\mathcal{O})$, for each $x \in X$. Let $x \in X$. From Lemma 3, there are $\alpha \leq \omega$, $\beta \leq \omega$ and sequences $\{x_n : n \in \alpha\}$, $\{y_n : n \in \beta\}$ as in Lemma 3 (b). First we prove for part $st^\infty(x,\mathcal{O}) \cap [x,\rightarrow)$ of $st^\infty(x,\mathcal{O})$ using the increasing sequence $\{x_n : n \in \alpha\}$. By similar arguments, it is proved for part $st^\infty(x,\mathcal{O}) \cap (\leftarrow, x]$ of $st^\infty(x,\mathcal{O})$.

We consider the following cases.

Case i. $\alpha = \omega$. Since $st^\infty(x,\mathcal{O}) \cap [x,\rightarrow) = (\cup_{n\in\alpha} st(x_n,\mathcal{O})) \cap [x,\rightarrow)$ and for each $n \in \omega$, $st(x_n,\mathcal{O}) \cap st(x_{n+1},\mathcal{O}) \neq \varnothing$ and each element of $\mathcal{O}$ is convex, the set $st^\infty(x,\mathcal{O}) \cap [x,\rightarrow)$ can be covered by a countable subfamily of $\mathcal{O}$.

Case ii. $\alpha = m+1$, $m \in \omega$. The interval $[x, x_m)$ can be covered by a finite subfamily $\mathcal{O}_0$ of $\mathcal{O}$. Since $st^\infty(x,\mathcal{O}) \cap [x_m,\rightarrow) = st(x_m,\mathcal{O}) \cap [x_m,\rightarrow)$, there are two cases.

ii$_1$) If the set $st^\infty(x,\mathcal{O}) \cap [x_m,\rightarrow)$ has the last element p: In this case, since $p \in st(x_m,\mathcal{O})$ and each element of $\mathcal{O}$ is convex then the interval $[x_m, p)$ is contained by an element of $\mathcal{O}$. Hence, the set $[x, x_m) \cup [x_m, p)$ can be covered by a finite subfamily of $\mathcal{O}$.

ii$_2$) If the set $st^\infty(x,\mathcal{O}) \cap [x_m,\to)$ has no last element: In this case, there is an increasing and cofinal sequence $\{z_\rho : \rho < \kappa\}$ in $st^\infty(x,\mathcal{O}) \cap [x_m,\to)$ where κ is a regular cardinal. If κ is countable, then $\{z_\rho : \rho < \kappa\}$ is countable and so there exists a countable open refinement of $\mathcal{O}$. Hence, we can assume that κ is uncountable. Let

$$L = \Big\{\lambda \in \kappa \cap Lim : \sup\{z_\rho : \rho < \lambda\} \text{ exists in } X \\ \text{and } \sup\{z_\rho : \rho < \lambda\} \in \overline{\{z_\rho : \rho < \lambda\}}\Big\}.$$

Either L is stationary in κ or L is not stationary in κ.

If L is stationary in κ: Let $F = \{\sup\{z_\rho : \rho < \lambda\} : \lambda \in L\}$. It is easy to see that F is closed in the subspace $st^\infty(x,\mathcal{O}) \cap [x_m,\to)$, and hence F is closed in X. Let f be a function from L onto F such that $f(\lambda) = \sup\{z_\rho : \rho < \lambda\}$, for each $\lambda \in L$. Since each element of L is a limit ordinal, f is one to one. Now, we shall show that the function f is a homeomorphism. Let λ be any element of L, V a convex open subset of X such that $f(\lambda) \in V$. From the definition of L, we have $f(\lambda) \in \overline{\{z_\rho : \rho < \lambda\}}$, and hence we have a $\rho_0 < \lambda$ such that $z_{\rho_0} \in V$. Since V is convex, we have $[z_{\rho_0}, f(\lambda)] \subseteq V$. Also, since $f((\rho_0,\lambda) \cap L) \subseteq [z_{\rho_0}, f(\lambda)] \cap F$, the function f is continuous. We will show that the function f^{-1} is continuous. Let $p = \sup\{z_\rho : \rho < \lambda\} \in F$, where $\lambda \in L$ and W is an open subset of κ with $f^{-1}(p) = \lambda \in W$. There exists a $\gamma < \lambda$ such that $(\gamma,\lambda] \cap L \subseteq W$. Since λ is a limit ordinal, we have $p \in (z_\gamma, z_{\lambda+1})$, and since $f^{-1}((z_\gamma, z_{\lambda+1}) \cap F) \subseteq (\gamma,\lambda] \cap L$, the function f^{-1} is continuous. Hence the function f is a homeomorphism from L onto F.

Since the sequence $\{z_\rho : \rho < \kappa\}$ is cofinal in $st^\infty(x,\mathcal{O}) \cap [x_m,\to)$ and L is stationary in κ (hence it is unbounded in κ), then F is cofinal in $st^\infty(x,\mathcal{O}) \cap [x_m,\to)$. It is easy to see that each open subset of a GO-space can be written as a union of disjoint, open convex sets. In addition, since F is cofinal in $st^\infty(x,\mathcal{O}) \cap [x_m,\to)$, there exists a disjoint family $\mathcal{J} = \{J_t : t \in I\}$ which consists of open convex and bounded subsets of X such that

$$\left(st^\infty(x,\mathcal{O}) \bigcap (x_m,\to)\right) \Big\backslash F = \bigcup \{J_t : t \in I\}.$$

As J_t is bounded and there is no lower bound of $st^\infty(x,\mathcal{O}) \cap [x_m,\to)$, for each $t \in I$, there exists $y_t \in st^\infty(x,\mathcal{O}) \cap [x_m,\to)$ such that $J_t \subseteq [x_m, y_t]$.

The equality $st^{\infty}(x,\mathcal{O})\cap[x_m,\rightarrow) = st(x_m,\mathcal{O})\cap[x_m,\rightarrow)$, and convexity of each element of $\mathcal{O}$ leads us to the fact that an $O_t \in \mathcal{O}$ exists such that $[x_m, y_t] \subseteq O_t$. Then the family $\mathcal{J}$ is a disjoint open, partial refinement of $\mathcal{O}$. Let $\mathcal{V} = \mathcal{J}\cup\mathcal{O}_0$. It is clear that $\mathcal{V}$ is a σ-disjoint open, partial refinement of $\mathcal{O}$ and $(st^{\infty}(x,\mathcal{O})\cap[x,\rightarrow))\setminus\cup\mathcal{V} = F$.

If L is not stationary in κ: There exists a cub set A of κ such that $L\cap A = \varnothing$. Also, the set $C = A\cap Lim$ is a cub set in κ. Let us define a function g from κ onto C such that $g(\rho) = \min\{C\setminus\{g(i) : i\in\rho\}\}$, for each ρ in κ. Since κ is a regular uncountable cardinal and C is closed, unbounded in κ, we have for each ρ, γ in κ with $\rho<\gamma$, $g(\rho)<g(\gamma)$, and for each $\lambda\in\kappa\cap Lim$, $g(\lambda) = \sup\{g(i) : i\in\lambda\}$. Let $g(\rho)$ be denoted by γ_ρ for each ρ in κ and $D = \{z_{\gamma_\rho} : \rho\in\kappa\}$. It easy to see that D is a closed and discrete subspace of X. Therefore, $\cap\{(z_{\gamma_i}, z_{\gamma_{\lambda+1}}) : i<\lambda\}$ is open in X, for each $\lambda\in\kappa\cap Lim$. We define the following families;

$$\begin{aligned}
\mathcal{V}_1 &= \left\{\left(z_{\gamma_\rho}, z_{\gamma_{\rho+2}}\right) : \rho\in\kappa \quad\text{and}\quad \rho \quad \text{odd}\right\},\\
\mathcal{V}_2 &= \left\{\left(z_{\gamma_\rho}, z_{\gamma_{\rho+2}}\right) : \rho\in\kappa \quad\text{and}\quad \rho \quad \text{even}\right\},\\
\mathcal{V}_3 &= \left\{\bigcap_{i<\lambda}\left(z_{\gamma_i}, z_{\gamma_{\lambda+1}}\right) : \lambda\in\kappa\cap Lim\right\}.
\end{aligned}$$

Since $\{z_{\gamma_\rho} : \rho<\kappa\}\subseteq st(x_m,\mathcal{O})$ and each element of $\mathcal{O}$ is convex, the above families are disjoint open, partial refinements of $\mathcal{O}$. In addition, there is an element O of $\mathcal{O}$ such that $[x_m, z_{\gamma_0}]\subseteq O$ and so the set $st^{\infty}(x,\mathcal{O})\cap[x,\rightarrow)$ is covered by $\{O\}\cup\mathcal{O}_0\cup\mathcal{V}_1\cup\mathcal{V}_2\cup\mathcal{V}_3$ which is a σ-disjoint open, partial refinement of $\mathcal{O}$. $\square$

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