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COMPOSITION OPERATORS BETWEEN BLOCH-TYPE SPACES AND MÖBIUS INVARIANT $\mathcal{Q}_K$ SPACES

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ABSTRACT. A characterization of the boundedness and compactness of a composition operator $C_\varphi f = f \circ \varphi$ acting from the Bloch type spaces $\mathcal{B}^\alpha$ to the Möbius invariant spaces $\mathcal{Q}_K$ is given. In particular, estimates for the essential norm of such an operator are obtained.

1. Introduction and main results. Let $\phi : \mathbf{D} \to \mathbf{D}$ be an analytic map of the unit disc $\mathbf{D} = \{z : |z| < 1\}$ into itself. The map ϕ induces a linear composition operator $C_\phi f = f \circ \phi$ on space $\mathcal{H}(\mathbf{D})$ of all analytic functions on the unit disc. A fundamental problem in the study of composition operators is to characterize in terms of the function theoretic properties of ϕ, the boundedness and compactness of restrictions of C_ϕ to various Banach spaces of analytic functions.

Recall that a bounded linear map T from a Banach space X into a Banach space Y is called *compact* (weakly compact) if it maps the closed unit ball of X onto a relatively compact (a relatively weakly compact) set in Y. The *essential norm* of T is defined to be the distance to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - S\| : S \text{ is compact}\}.$$

Since $\|T\|_e = 0$ if and only if T is compact, estimates for $\|T\|_e$ give conditions for T to be compact.

For $s > -1$, consider the weighted Dirichlet space D_s of all analytic functions on the unit disc $\mathbf{D}$ for which

$$\|f\|_{D_s}^2 = \int_{\mathbf{D}} |f'(z)|^2 \,(1 - |z|^2)^s \, dA(z) < \infty.$$

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For $s \geq 0$, let Q_s be the space of all analytic functions on the unit disc with

$$\|f\|_{Q_s}^2 = \sup_{w\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 \, (1 - |\varphi_w(z)|^2) \, dA(z) < \infty,$$

where $\varphi_w(z) = (w - z)/(1 - \overline{w}z)$ is a Möbius map. We note that Q_s is the Möbius invariant space generated by D_s, that is,

$$\|f\|_{Q_s}^2 = \sup_{w\in\mathbf{D}} \|f \circ \varphi_w\|_{D_s}^2.$$

Note that $Q_0 = \mathcal{D}$ is the classical Dirichlet space, $Q_1 = BMOA$, the space of all analytic functions of bounded mean oscillation, and for $s > 1$, Q_s coincides with the Bloch space $\mathcal{B}$ of all analytic functions on $\mathbf{D}$ with

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbf{D}} (1 - |z|^2) \, |f'(z)| < \infty.$$

Let $\alpha > 0$. The Bloch-type space $\mathcal{B}^\alpha$ consists of all analytic functions on $\mathbf{D}$ such that

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbf{D}} (1 - |z|^2)^\alpha \, |f'(z)| < \infty.$$

We also consider the little Bloch type space $\mathcal{B}_0^\alpha$ of those functions in $\mathcal{B}^\alpha$ with

$$\lim_{|z|\to 1^-} (1 - |z|^2)^\alpha |f'(z)| = 0.$$

For composition operators on weighted Dirichlet spaces D_s with $-1 < s < 1$, a characterization of the boundedness and compactness can be found in [**12**]. Compactness of C_ϕ in $BMOA$ is studied in [**1, 7**]. For composition operators between Bloch-type spaces, we refer to [**3, 4, 8**].

For $1 < p < \infty$ and a nondecreasing function $K : (0, +\infty) \to (0, +\infty)$, consider the Besov type space B_K^p of all analytic functions f on the unit disc for which

$$\|f\|_{B_K^p}^p = \int_{\mathbf{D}} |f'(z)|^p \, (1 - |z|^2)^{p-2} \, K(1 - |z|^2) \, dA(z) < \infty.$$

Let $\mathcal{Q}_K(p)$ be the bounded Möbius invariant space generated by B_K^p, that is, an analytic function f is in $\mathcal{Q}_K(p)$ if

$$\|f\|_{K,p}^p := \sup_{w\in\mathbf{D}} \|f \circ \varphi_w\|_{B_K^p}^p < \infty.$$

Note that for $p = 2$ and $K(t) = t^s$ with $s \ge 0$ we obtain the Q_s spaces defined before. We assume throughout the paper that

$$(1) \qquad \int_0^1 (1-r^2)^{p-2}\, K\left(\log\frac{1}{r}\right) r\,dr < \infty.$$

Otherwise, the space $\mathcal{Q}_K(p)$ only contains constant functions (see [**9**]). Also, from [**9**], we have that $\mathcal{Q}_K(p) \subset \mathcal{B}$.

In this note we study composition operators from $\mathcal{B}^\alpha$ into $\mathcal{Q}_K(p)$, and a description of the boundedness and compactness of C_ϕ is given in terms of the function ϕ. We also prove that the essential norm of $C_\phi : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$ is equivalent to the quantity

$$\lim_{r\to 1^-} \sup_{w\in\mathbf{D}} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}\,(1-|z|^2)^{p-2}\,K(1-|\varphi_w(z)|^2)\,dA(z),$$

generalizing the result given in [**2**], where the result is obtained for the case $\alpha = 1$, $p = 2$ and $K(t) = t^s$, that is, for composition operators from $\mathcal{B}$ to Q_s.

We use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ such that $a \le C\,b$, and the notation $a \approx b$ (a is comparable with b) means that $a \lesssim b \lesssim a$. The paper is organized as follows. In Section 2 we study composition operators from $\mathcal{B}^\alpha$ to $\mathcal{Q}_K(p)$ and prove the main results of the paper, and Section 3 is devoted to the study of composition operators from $\mathcal{Q}_K(p)$ to $\mathcal{B}^\alpha$.

2. Composition operators from $\mathcal{B}^\alpha$ to $\mathcal{Q}_K(p)$. We begin this section with a description of when a composition operator from $\mathcal{B}^\alpha$ to $\mathcal{Q}_K(p)$ exists as a bounded operator.

Theorem 1. *Let $\alpha \in (0, +\infty)$, $1 < p < \infty$, and $\phi : \mathbf{D} \to \mathbf{D}$ be analytic, and let $K : (0, +\infty) \to (0, +\infty)$ be a nondecreasing function. A composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$ is bounded if and only if*

$$\sup_{w\in\mathbf{D}} \int_{\mathbf{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\,\alpha}}\,(1-|z|^2)^{p-2}\,K(1-|\varphi_w(z)|^2)\,dA(z) < \infty.$$

Proof. Let

$$M = \sup_{w\in\mathbf{D}} \int_{\mathbf{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\,\alpha}}\,(1-|z|^2)^{p-2}\,K(1-|\varphi_w(z)|^2)\,dA(z).$$

Suppose first that $M < \infty$. Then, given $f \in \mathcal{B}^\alpha$ we have

$$\begin{aligned}\|C_\phi f\|_{K,p}^p \\ &= \sup_{w\in\mathbf{D}} \int_{\mathbf{D}} |f'(\phi(z))|^p \, |\phi'(z)|^p \, (1-|z|^2)^{p-2} \, K(1-|\varphi_w(z)|^2) \, dA(z) \\ &\le M \, \|f\|_{\mathcal{B}^\alpha}^p,\end{aligned}$$

that is, $C_\phi : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$ is bounded.

Suppose now that $\|C_\phi f\|_{K,p} \le C\|f\|_{\mathcal{B}^\alpha}$ whenever f is in $\mathcal{B}^\alpha$. From Theorem 2.1.1 of [**10**] there exist two functions f_1 and f_2 in $\mathcal{B}^\alpha$ such that

$$(|f_1'(z)| + |f_2'(z)|) \approx (1-|z|^2)^{-\alpha}, \quad z \in \mathbf{D}.$$

Then

$$M \le \|C_\phi f_1\|_{K,p}^p + \|C_\phi f_2\|_{K,p}^p \le C(\|f_1\|_{\mathcal{B}^\alpha}^p + \|f_2\|_{\mathcal{B}^\alpha}^p) < \infty,$$

and the proof is complete. □

It is a well-known result that, under the usual integral pairing, the dual of the little Bloch space $\mathcal{B}_0$ is isomorphic to the Bergman space A^1 of all analytic functions on the unit disc with

$$\int_{\mathbf{D}} |f(z)| \, dA(z) < \infty.$$

We will need a similar result for $\mathcal{B}_0^\alpha$ with another natural pairing.

Lemma 2. *The map $h \mapsto \langle \cdot, h \rangle_{\mathcal{B}^\alpha}$ defines an isomorphism from $A^1 \oplus \mathbf{C}$ onto the dual of $\mathcal{B}_0^\alpha$. Here*

$$\langle f, g \rangle_{\mathcal{B}^\alpha} = \int_{\mathbf{D}} f'(z) g(\bar{z}) \, (1-|z|^2)^\alpha \, dA(z) + c f(0)$$

for $f \in \mathcal{B}^\alpha$ and $h = (g, c) \in A^1 \oplus \mathbf{C}$.

Proof. Consider the space $A^{-\alpha}$ of all analytic functions on $\mathbf{D}$ with

$$\|f\|_{A^{-\alpha}} := \sup_{z\in\mathbf{D}} (1-|z|^2)^\alpha |f(z)| < \infty,$$

and its closed subspace $A_0^{-\alpha}$ consisting of all functions $f \in A^{-\alpha}$ with

$$\lim_{|z|\to 1^-}(1-|z|^2)^\alpha |f(z)| = 0.$$

These spaces are denoted by $A_\infty(\varphi)$ and $A_0(\varphi)$ in [**6**] with $\varphi(z) = (1-|z|^2)^\alpha$. Choosing $\psi(z) := 1$, the pair $\{\varphi, \psi\}$ is a *normal pair* of weight functions in the sense of [**6**, page 291]. Therefore, by Theorem 2 of [**6**], the map $g \mapsto \langle \cdot, g\rangle_{A^{-\alpha}}$ defines an isomorphism from A^1 onto the dual of $A_0^{-\alpha}$, where

$$\langle f, g\rangle_{A^{-\alpha}} = \int_{\mathbf{D}} f(z) g(\bar{z})\,(1-|z|^2)^\alpha\, dA(z).$$

Therefore, duality

$$(A_0^{-\alpha} \oplus \mathbf{C})^* = (A_0^{-\alpha})^* \oplus \mathbf{C}^* = A^1 \oplus \mathbf{C}$$

holds with the pairing $\langle y, h\rangle = \langle f, g\rangle_{A^{-\alpha}} + bc$, where $y = (f, b) \in A_0^{-\alpha} \oplus \mathbf{C}$ and $h = (g, c) \in A^1 \oplus \mathbf{C}$.

We also note that the map $I : f \to (f', f(0))$ is a linear isometric bijection from $\mathcal{B}_0^\alpha$ to $A_0^{-\alpha} \oplus \mathbf{C}$, with the direct sums endowed with the sum-norm. Therefore, the result follows from the above remarks and the fact that

$$\langle f, h\rangle_{\mathcal{B}^\alpha} = \langle If, h\rangle$$

holds for $f \in \mathcal{B}^\alpha$ and $h \in A^1 \oplus \mathbf{C}$. □

We also need the following result due to Yamashita, see [**10**, page 13] or [**11**].

Lemma 3. *Let* $0 < \alpha < \infty$, *and let* $f(z) = \sum_{k=0}^\infty a_k z^{n_k}$ *be a Hadamard gap series, that is,* $n_{k+1} \geq C n_k$ *for some constant* $C > 1$ *and all* $k \in \mathbf{N}$. *Then* $f \in \mathcal{B}^\alpha$ *if and only if* $\sup_k |a_k|\, n_k^{1-\alpha} < \infty$. *Also,* $f \in \mathcal{B}_0^\alpha$ *if and only if* $\lim_k |a_k|\, n_k^{1-\alpha} = 0$.

Now we are going to study the compactness of a composition operator from $\mathcal{B}^\alpha$ to $\mathcal{Q}_K(p)$ when K is a nondecreasing weight. With the same proof given in [**2**] (where the case $\alpha = 1$, $p = 2$, and $K(t) = t^s$ is proved) we have that a composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$ is compact if and only if it is weakly compact. The following result, where the essential norm of a composition operator from $\mathcal{B}^\alpha$ to $\mathcal{Q}_K(p)$ is estimated, is the main result of the paper.

Theorem 4. *Let $\alpha \in (0,+\infty)$, $1 < p < \infty$, and $\phi : \mathbf{D} \to \mathbf{D}$ be analytic, and let $K : (0,+\infty) \to (0,+\infty)$ be a nondecreasing function. Let C_ϕ be a bounded operator from $\mathcal{B}^\alpha$ into $\mathcal{Q}_K(p)$. Then*

$$\|C_\phi\|_e^p \approx \lim_{r\to 1^-} \sup_{w\in\mathbf{D}} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}} (1-|z|^2)^{p-2} \cdot K(1-|\varphi_w(z)|^2)\, dA(z).$$

Proof. We adapt the argument given in [**2**] to our case. We first show the lower estimate of the essential norm. Choose a sequence $\{\lambda_n\}$ in $\mathbf{D}$ converging to 1 as $n\to\infty$, and let

$$f_{n,m,\theta}(z) = \sum_{k=1}^\infty \frac{2^k}{2^k+2^m} 2^{-k(1-\alpha)} (\lambda_n e^{i\theta})^{2^k} z^{2^k+2^m}, \quad 0\le\theta\le 2\pi, \quad n,m\in\mathbf{N}$$

and

$$f(z) = \sum_{k=1}^\infty 2^{-k(1-\alpha)} z^{2^k}.$$

We note that

$$f'_{n,m,\theta}(z) = z^{2^m}\lambda_n e^{i\theta} f'(\lambda_n e^{i\theta} z) = z^{2^m-1}\sum_{k=1}^\infty 2^{\alpha k}(\lambda_n e^{i\theta} z)^{2^k}.$$

By Lemma 3 we have that $f\in\mathcal{B}^\alpha$ and $f_{n,m,\theta}\in\mathcal{B}_0^\alpha$, and normalizing we can assume that $\|f\|_{\mathcal{B}^\alpha}\le 1$ and $\|f_{n,m,\theta}\|_{\mathcal{B}^\alpha}\le 1$. Given $F\in(\mathcal{B}_0^\alpha)^*$, let $h=(g,c)\in A^1\oplus\mathbf{C}$ be such that $F(f_{n,m,\theta}) = \langle f_{n,m,\theta}, h\rangle_{\mathcal{B}^\alpha}$. Then

$$\begin{aligned}\sup_{n,\theta}|F(f_{n,m,\theta})| &\le \sup_{n,\theta}\int_{\mathbf{D}} |f'_{n,m,\theta}(z)||g(z)|\,(1-|z|^2)^\alpha\, dA(z)\\ &\le \int_{\mathbf{D}} |z|^{2^m-1}|g(z)|\, dA(z),\end{aligned}$$

which converges to 0 as $m \to \infty$ by the Lebesgue dominated convergence theorem, since $g \in A^1$. Therefore,

$$\lim_{m\to\infty} \sup_{n,\theta} |F(f_{n,m,\theta})| = 0. \tag{2}$$

Now we claim that, for any compact operator $T : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$, we have that

$$\lim_{m\to\infty} \sup_{n,\theta} \|Tf_{n,m,\theta}\|_{K,p} = 0.$$

Indeed, if not, then there is a subsequence $(m_k)_{k=1}^\infty$ such that for each k we can find n_k and θ_k such that for all k we have

$$\|Tf_{n_k,m_k,\theta_k}\|_{K,p} \ge c > 0, \tag{3}$$

for some constant c. By (2) we have that $f_{n_k,m_k,\theta_k} \to 0$ weakly in $\mathcal{B}_0^\alpha$ when $k \to \infty$. But, since T is compact, this is a contradiction with (3). Therefore, if T is an arbitrary compact operator, we have

$$\begin{aligned}\|C_\phi - T\| &\ge \lim_{m\to\infty} \sup \sup_{n,\theta} \|(C_\phi - T)f_{n,m,\theta}\|_{K,p}\\ &\ge \lim_{m\to\infty} \sup \sup_{n,\theta} (\|(C_\phi f_{n,m,\theta}\|_{K,p} - \|Tf_{n,m,\theta}\|_{K,p})\\ &= \lim_{m\to\infty} \sup \sup_{n,\theta} \|C_\phi f_{n,m,\theta}\|_{K,p}.\end{aligned}$$

Hence, we obtain

$$\|C_\phi\|_e^p \ge \lim_{m\to\infty} \sup \sup_{n,\theta} \Big(\sup_{w\in\mathbf{D}} \int_{\mathbf{D}} |f'_{n,m,\theta}(\phi(z))|^p \, |\phi'(z)|^p \, dA_{K,w}(z) \Big),$$

where $dA_{K,w}(z) = (1-|z|^2)^{p-2} \, K(1 - |\varphi_w(z)|^2) \, dA(z)$.

Now, given $\varepsilon > 0$, choose $m_0 \in \mathbf{N}$ such that for $m \ge m_0$ we have

$$\begin{aligned}&\|C_\phi\|_e^p + \varepsilon\\ &\ge \sup_{w\in\mathbf{D}} \int_{\mathbf{D}} |\phi(z)|^{2^{m+1}} \Big| \sum_{k=0}^\infty 2^{\alpha k} (\lambda_n \phi(z))^{2^k-1} (e^{i\theta})^{2^k} \Big|^p |\phi'(z)|^p \, dA_{K,w}(z)\end{aligned}$$

for all θ and all n. Let $w \in \mathbf{D}$ be fixed. Integrating with respect to θ and using Fubini's theorem, we get

$$\begin{aligned}\|C_\phi\|_e^p &+ \varepsilon \\ &\geq \int_{\mathbf{D}} |\phi(z)|^{2^{m+1}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{\infty} 2^{\alpha k} (\lambda_n \phi(z))^{2^k - 1} (e^{i\theta})^{2^k} \right|^p d\theta \right) \\ &\qquad \times |\phi'(z)|^p \, dA_{K,w}(z).\end{aligned}$$

By [**13**, Theorem 8.20], if g is a function given by a Hadamard gap series, we have

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta \right)^{1/p} \gtrsim \left(\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^2 \, d\theta \right)^{1/2}. \tag{4}$$

Therefore, by (4) and the Parseval identity, we get

$$\begin{aligned}\|C_\phi\|_e^2 &+ \varepsilon \\ &\gtrsim \int_{\mathbf{D}} |\phi(z)|^{2^{m+1}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{\infty} 2^{\alpha k} (\lambda_n \phi(z))^{2^k - 1} (e^{i\theta})^{2^k} \right|^2 d\theta \right)^{p/2} \\ &\qquad \times |\phi'(z)|^p \, dA_{K,w}(z) \\ &= \int_{\mathbf{D}} |\phi(z)|^{2^{m+1}} \left(\sum_{k=0}^{\infty} 2^{2\alpha k} |\lambda_n \phi(z)|^{2(2^k-1)} \right)^{p/2} \\ &\qquad \times |\phi'(z)|^p \, dA_{K,w}(z) \\ &\gtrsim \int_{\{|\phi| > 1 - 2^{-(m+1)}\}} \left(\sum_{k=0}^{\infty} 2^{2\alpha k} |\lambda_n \phi(z)|^{2(2^k-1)} \right)^{p/2} \\ &\qquad \times |\phi'(z)|^p \, dA_{K,w}(z).\end{aligned}$$

We claim that, for $3/4 \leq r < 1$, we have

$$\sum_{k=0}^{\infty} 2^{2\alpha k} r^{2^{k+1}-2} \geq C_\alpha (1-r)^{-2\alpha} \tag{5}$$

for some positive constant C_α depending only on α. Indeed, it is straightforward to see that

$$r^{2^{k+1}-2} \geq \exp\{-2^{k+2}(1-r)\}, \quad 1/2 \leq r < 1. \tag{6}$$

Now, for $3/4 \le r < 1$, there is an integer k such that $\alpha/2 \le 2^k(1-r) < (\alpha+1)/2$. Therefore,

$$2^{2\alpha k}\exp\{-2^{k+2}(1-r)\} \ge \left(\frac{\alpha+1}{2}\right)^{2\alpha} e^{-2(\alpha+1)}(1-r)^{-2\alpha}, \tag{7}$$

since for $\alpha > 0$, the function $t^{2\alpha}e^{-4t}$ is decreasing in $[\alpha/2, (\alpha+1)/2]$. Now, estimate (5) follows from (6) and (7).

Hence, by (5) we have

$$\sum_{k=0}^{\infty} 2^{2\alpha k}|\lambda_n\phi(z)|^{2(2^k-1)} \ge C_\alpha(1-|\lambda_n\phi(z)|^2)^{-2\alpha}$$

for all $z \in \mathbf{D}$ with $|\phi(z)| > 1 - 2^{-(m+1)}$ if n and m are big enough. Therefore, by Fatou's lemma we have

$$\begin{aligned}\|C_\phi\|_e^p + \varepsilon &\gtrsim \liminf_{n\to\infty} \int_{\{|\phi(z)|>1-2^{-(m+1)}\}} \frac{|\phi'(z)|^p}{(1-|\lambda_n\phi(z)|^2)^{p\alpha}}\, dA_{K,w}(z)\\ &\ge \int_{\{|\phi(z)|>1-2^{-(m+1)}\}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}\, dA_{K,w}(z).\end{aligned}$$

Thus, since $w \in \mathbf{D}$ is arbitrary, we obtain

$$\|C_\phi\|_e^p + \varepsilon \gtrsim \lim_{r\to 1^-} \sup_{w\in\mathbf{D}} \int_{\{|\phi(z)|>r\}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}\, dA_{K,w}(z),$$

that proves the lower estimate.

Now we are going to compute the upper estimate. For each $k \in \mathbf{N}$, define a sequence of compact linear operators $C_k : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ by

$$C_k f(z) = f\left(\frac{k}{k+1}\, z\right), \quad z \in \mathbf{D}.$$

Let $\psi_k(z) = (kz)/(k+1)$ so that $C_k f = f \circ \psi_k$. Then we have

$$\begin{aligned}\|C_\phi\|_e^p &\le \|C_\phi - C_\phi C_k\|^p = \|C_\phi(Id - C_k)\|^p\\ &= \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} \sup_{w\in\mathbf{D}} \int_{\mathbf{D}} |(f - f\circ\psi_k)'(\phi(z))|^p\, |\phi'(z)|^p\, dA_{K,w}(z),\end{aligned}$$

which is less than

$$\begin{aligned}
&\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{w\in\mathbf{D}}\int_{\{|\phi|>r\}}|(f-f\circ\psi_k)'(\phi(z))|^p\,|\phi'(z)|^p\,dA_{K,w}(z)\\
&\quad+\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{w\in\mathbf{D}}\int_{\{|\phi|\le r\}}|(f-f\circ\psi_k)'(\phi(z))|^p\,|\phi'(z)|^p\,dA_{K,w}(z)\\
&:=I_k+J_k,
\end{aligned}$$

where $0<r<1$ is fixed. To estimate the first term I_k, note that for $\|f\|_{\mathcal{B}^\alpha}\le 1$ and $z\in\mathbf{D}$, we have

$$|f'(z)|\le(1-|z|^2)^{-\alpha}.$$

Since $\|f\circ\psi_k\|_{\mathcal{B}^\alpha}\le\|f\|_{\mathcal{B}^\alpha}$, we obtain

$$I_k\le 2^p\sup_{w\in\mathbf{D}}\int_{\{|\phi|>r\}}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}\,dA_{K,w}(z).$$

Therefore, it is enough to show that $\lim_{k\to\infty}J_k=0$. Since $C_\phi z=\phi\in\mathcal{Q}_K(p)$, we get

$$M:=\sup_{w\in\mathbf{D}}\int_{\mathbf{D}}|\phi'(z)|^p\,(1-|z|^2)^{p-2}\,K(1-|\varphi_w(z)|^2)\,dA(z)<\infty.$$

Therefore,

$$J_k\le M\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{\{|\phi|\le r\}}|g_k'(\phi(z))|^p,$$

where $g_k=f-f\circ\psi_k$. Since g_k converges to 0 uniformly on compact subsets of $\mathbf{D}$, then g_k' also converges to 0 uniformly on compact subsets of $\mathbf{D}$. Hence, we obtain that

$$\lim_{k\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{\{|\phi|\le r\}}|g_k'(\phi(z))|^p=0,$$

and the proof is complete. □

As an immediate consequence of Theorem 4, we get the following characterization of compact composition operators from $\mathcal{B}^\alpha$ to $\mathcal{Q}_K(p)$.

Corollary 5. *Let $0 < \alpha < \infty$ and $1 < p < \infty$. A composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$ is compact if and only if $\phi \in \mathcal{Q}_K(p)$ and*

$$\lim_{r \to 1^-} \sup_{w \in \mathbf{D}} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}} (1-|z|^2)^{p-2} \times K(1-|\varphi_w(z)|^2)\, dA(z) = 0.$$

If $\phi : \mathbf{D} \to \mathbf{D}$ is univalent, we can provide some geometric characterizations of the boundedness and compactness of C_ϕ. This requires some background on the hyperbolic metric. Recall that the hyperbolic distance $\beta_{\mathbf{D}}(z,w)$ between two points $z, w \in \mathbf{D}$ is defined by $\beta_{\mathbf{D}}(z,w) = \log_2(1+\rho(z,w))/(1-\rho(z,w))$, where $\rho(z,w) = |(z-w)/(1-\overline{w}z)|$. We note that

$$(1-|z|)^{-1} \le 2^{\beta_{\mathbf{D}}(0,z)} \le 2(1-|z|)^{-1}. \tag{8}$$

This distance is invariant under Möbius transformations and therefore transfers to a conformally invariant metric on any simply connected proper subset Ω of $\mathbf{C}$. If $f : \mathbf{D} \to \Omega$ is any conformal map, the hyperbolic distance on Ω is given by $\beta_\Omega(w_1, w_2) = \beta_{\mathbf{D}}(z_1, z_2)$ where $w_j = f(z_j)$ for $j = 1, 2$. We denote by $d(z, \partial\Omega)$ the Euclidian distance from z to the boundary of Ω.

Theorem 6. *Let $0 < \alpha < \infty$, $1 < p < \infty$, and let K be a nondecreasing function. Let $\phi : \mathbf{D} \to \mathbf{D}$ be univalent, and let $\Omega = \phi(\mathbf{D})$. Then*

(i) *$C_\phi : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$ exists as a bounded operator if and only if*

$$\sup_{w \in \Omega} \int_\Omega \frac{K(2^{-\beta_\Omega(w,z)})}{(1-|z|^2)^{p\alpha}} d(z, \partial\Omega)^{p-2}\, dA(z) < \infty.$$

(ii) *$C_\phi : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$ exists as a compact operator if and only if $\phi \in \mathcal{Q}_K(p)$ and*

$$\lim_{r \to 1} \sup_{w \in \Omega} \int_{\Omega \cap \{|z|>r\}} \frac{K(2^{-\beta_\Omega(w,z)})}{(1-|z|^2)^{p\alpha}} d(z, \partial\Omega)^{p-2}\, dA(z) = 0.$$

Proof. Let $\Omega = \phi(\mathbf{D})$. Since $\phi : \mathbf{D} \to \mathbf{D}$ is conformal, the fact that $a \in \mathbf{D}$ is equivalent to the fact that $\phi(a) \in \Omega$. If ψ denotes the inverse map of ϕ, then by (8) we have

$$1 - |\varphi_a(\psi(z))|^2 \approx 2^{-\beta_{\mathbf{D}}(0,\varphi_a(\psi(z)))} = 2^{-\beta_\Omega(\phi(a),z)}.$$

Also, by Koebe's distortion theorem we have $(1 - |z|^2)|\phi'(z)| \approx d(\phi(z), \partial\Omega)$. Hence,

$$\begin{aligned}\int_{\mathbf{D}} \frac{|\phi'(w)|^p\, K(1 - |\varphi_a(w)|^2)}{(1 - |\phi(w)|^2)^{p\alpha}}\, dA(w) & \\ &\approx \int_{\mathbf{D}} \frac{|\phi'(w)|^2\, K(1 - |\varphi_a(w)|^2)}{(1 - |\phi(w)|^2)^{p\alpha}}\, d(\phi(w), \partial\Omega)^{p-2}\, dA(w) \\ &= \int_{\Omega} \frac{K(1 - |\varphi_a(\psi(z))|^2)}{(1 - |z|^2)^{p\alpha}}\, d(z, \partial\Omega)^{p-2}\, dA(z) \\ &\approx \int_{\Omega} \frac{K(2^{-\beta_\Omega(w,z)})}{(1 - |z|^2)^{p\alpha}}\, d(z, \partial\Omega)^{p-2}\, dA(z).\end{aligned}$$

This, together with Theorem 1 and Corollary 5 leads to (i) and (ii). □

3. Composition operators from $\mathcal{Q}_K(p)$ to $\mathcal{B}^\alpha$. We begin this section with two lemmas. The first one is standard and can be found for example in [**5**].

Lemma 7. *Let $\sigma > -1$ and $a, b > 0$ be such that $a + b - \sigma > 2$ and $a - \sigma, b - \sigma < 2$. Then*

$$\int_{\mathbf{D}} \frac{(1 - |\zeta|^2)^\sigma}{|1 - \bar{\zeta} z|^a\, |1 - \bar{\zeta} w|^b}\, dA(\zeta) \lesssim |1 - \overline{w} z|^{2+\sigma-a-b}.$$

Lemma 8. *Let $1 < p < \infty$, and let K be a nondecreasing function such that for some $s > 0$ we have that $t^{-s}K(t)$ is increasing for $0 < t \le 1$. Then, for each $w \in \mathbf{D}$, the function*

$$f_w(z) = -\log(1 - \overline{w} z)$$

belongs to $Q_K(p)$ *with* $\|f_w\|_{K,p} \le C$, *where* C *is a constant independent of* w.

Proof. Let $w \in \mathbf{D}$. By assumption, there is an $s > 0$ such that $t^{-s}K(t)$ is increasing for $0 < t \le 1$. Therefore,

$$\begin{aligned}\|f_w\|_{K,p}^p &= \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f_w'(z)|^p\,(1-|z|^2)^{p-2} K(1-|\varphi_a(z)|^2)\,dA(z)\\ &\le K(1)\,\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f_w'(z)|^p\,(1-|z|^2)^{p-2}\,(1-|\varphi_a(z)|^2)^s\,dA(z)\\ &\le K(1)\,\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \frac{(1-|z|^2)^{s+p-2}(1-|a|^2)^s}{|1-\overline{a}z|^{2s}\,|1-\overline{w}z|^p}\,dA(z)\end{aligned}$$

which, by Lemma 7, is bounded by

$$C\sup_{a\in\mathbf{D}} \frac{(1-|a|^2)^s}{|1-\overline{a}w|^{2s+p-s-p+2-2}} \le C. \qquad \square$$

We note that, for $s > 0$, the function $K(t) = t^s$ satisfies the condition for K given in Lemma 8. Also this condition implies that $\int_0^1 K(t)/t\,dt < \infty$.

For the case of composition operators from $\mathcal{Q}_K(p)$ to $\mathcal{B}^\alpha$ we have the following description of boundedness and compactness.

Theorem 9. *Let* $\alpha \in (0,+\infty)$, $1 < p < \infty$, *and let* $\phi : \mathbf{D} \to \mathbf{D}$ *be analytic. Let* $K : (0,+\infty) \to (0,+\infty)$ *be a nondecreasing function such that for some* $s > 0$ *the function* $t^{-s}K(t)$ *is increasing for* $0 < t \le 1$. *Then*

(i) $C_\phi : \mathcal{Q}_K(p) \to \mathcal{B}^\alpha$ *is bounded if and only if*

$$\sup_{z\in\mathbf{D}} \frac{|\phi'(z)|}{1-|\phi(z)|^2}\,(1-|z|^2)^\alpha < \infty.$$

(ii) $C_\phi : \mathcal{Q}_K(p) \to \mathcal{B}^\alpha$ *is compact if and only if* $\phi \in \mathcal{B}^\alpha$ *and*

$$\lim_{r\to 1} \sup_{z:|\phi(z)|>r} \frac{|\phi'(z)|}{1-|\phi(z)|^2}\,(1-|z|^2)^\alpha = 0.$$

Proof. By Lemma 8, if $w \in \mathbf{D}$, then the function

$$f_w(z) = -\log(1 - \overline{w}z)$$

belongs to $\mathcal{Q}_K(p)$. Then one can repeat the proof for the Q_s case given in [**10**]. We omit the details. □

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Added in proof. After the acceptance of that paper, the author noticed the article by M. Kotilainen, "*On composition operators in Q_K type spaces*," J. Function Spaces Appl. **5** (2007), 103–122, which has already appeared and contains some overlaps with the present work.

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