ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 6, 2009

# CAUCHY-RASSIAS STABILITY OF SESQUILINEAR $n$-QUADRATIC MAPPINGS IN BANACH MODULES

CHUN-GIL PARK AND SUN-YOUNG JANG

ABSTRACT. We prove the Cauchy-Rassias stability of a sesquilinear $n$-quadratic mapping in a left Banach module over a unital $C^*$-algebra.

**1. Introduction.** Let $X$ and $Y$ be Banach spaces. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbf{R}$ for each fixed $x \in X$. Rassias [**11**] introduced the following inequality that we call *Cauchy-Rassias inequality*: Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Rassias [**11**] showed that there exists a unique $\mathbf{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p} ||x||^p$$

for all $x \in X$. The above inequality has provided a lot of influence in mathematical analysis in the development of what we now call *Hyers-Ulam-Rassias stability* of functional equations.

The norm on an inner product space satisfies the classical parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

2000 AMS *Mathematics subject classification.* Primary 39B52, 47Jxx, 46L05.
*Keywords and phrases.* Banach module over $C^*$-algebra, sesquilinear $n$-quadratic mapping, stability of functional equation, $n$-inner product space.
The first author was supported by grant No. F01-206-000-101110-0 from the Korea Science & Engineering Foundation.
The second author is the corresponding author.
Received by the editors on March 9, 2006, and in revised form on July 3, 2007.
DOI:10.1216/RMJ-2009-39-6-2015 

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic function*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [**18**] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Cholewa [**2**] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group. In [**3**], Czerwik, following the Cauchy-Rassias sequential approach originated by Rassias in [**10**], proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Several functional equations have been investigated in [**7**–**9, 12**–**17**].

Throughout this paper, let $\mathcal{A}$ be a unital $C^*$-algebra with unitary group $\mathcal{U}(\mathcal{A})$, and $X$ a left Banach $\mathcal{A}$-module.

**Definition 1.1.** A mapping $S : X^{n+2} \to \mathcal{A}$ is a *sesquilinear n-quadratic mapping* if

$$S(ax_1 + bx_2, y; z_1, \dots, z_n) = aS(x_1, y; z_1, \dots, z_n) + bS(x_2, y; z_1, \dots, z_n),$$

$$S(x, ay_1 + by_2; z_1, \dots, z_n) = S(x, y_1; z_1, \dots, z_n)a^* + S(x, y_2; z_1, \dots, z_n)b^*,$$

$$S(x, x; z_1, z_2, \dots, z_n) = S(z_1, z_1; x, z_2, \dots, z_n),$$

$$S(x, y; z_1, \dots, z_n) = S(x, y; z_{i_1}, \dots, z_{i_n})$$

for every even permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$,

$$S(x, y; z_1, \dots, z_n) = S(y, x; z_{i_1}, \dots, z_{i_n})^*$$

for every odd permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$,

$$S(x, y; z_1, \dots, z_n) = S(y, x; z_1, \dots, z_n)^*$$

for all $x, x_1, x_2, y, y_1, y_2, z_1, \dots, z_n \in X$ and all $a, b \in \mathcal{A}$.

Misiak defined $n$-inner product spaces and investigated the properties of the spaces, see [**1, 5, 6**]. An $n$-inner product space is defined as follows: Let $n$ be a nonzero natural number, and let $X$ be a linear space of dimension $\geq n$. A real function $(\cdot, \cdot|\cdot, \cdots, \cdot)$ on $X^{n+1}$ is called an $n$-inner product if it satisfies the following conditions:

(1) $(a, a|\ a_2, \dots, a_n) \geq 0$,

(2) $(a, a|\ a_2, \dots, a_n) = 0$ if and only if $a, a_2, \dots, a_n$ are linearly dependent,

(3) $(a, b|\ a_2, \dots, a_n) = (b, a|\ a_2, \dots, a_n)$,

(4) $(a, b|\ a_2, \dots, a_n) = (a, b|\ a_{i_2}, \dots, a_{i_n})$ for every permutation $(i_2, \dots, i_n)$ of $(2, \dots, n)$,

(5) If $n \geq 2$, then $(a, a|\ a_2, a_3, \dots, a_n) = (a_2, a_2|\ a, a_3, \dots, a_n)$,

(6) $(\alpha a, b|\ a_2, \dots, a_n) = \alpha(a, b|\ a_2, \dots, a_n)$ for every real $\alpha$,

(7) $(a + a', b|\ a_2, \dots, a_n) = (a, b|\ a_2, \dots, a_n) + (a', b|\ a_2, \dots, a_n)$.

The concept of an $n$-inner product space is a generalization of the concepts of an inner product space ($n = 1$), and of a 2-inner product space ($n = 2$). It is obvious that the $n$-inner product on a vector space $V$, defined by Misiak, is a sesquilinear $n$-quadratic form. Thus, it is important to understand the properties of sesquilinear $n$-quadratic forms to investigate the properties of $n$-inner product spaces.

In this paper, we extend the concept of a sesquilinear $n$-quadratic form on a vector space $V$ to the concept of a sesquilinear $n$-quadratic mapping on a left Banach module over a unital $C^*$-algebra and investigate sesquilinear $n$-quadratic mappings in left Banach modules over unital $C^*$-algebras.

**2. Sesquilinear $n$-quadratic mappings on Banach modules over $C^*$-algebras.** In [**9**] the Jensen type equation $f(x - y/2) = (f(x) - f(y))/2$ has been studied. It is well known that a mapping $f : X \to Y$ satisfies the Jensen type equation if and only if the mapping $f : X \to Y$ is a Cauchy additive mapping, i.e., $f(x + y) = f(x) + f(y)$, and $f(0) = 0$.

We prove the Cauchy-Rassias stability of a sesquilinear $n$-quadratic mapping in a left Banach module over a unital $C^*$-algebra.

**Theorem 2.1.** *Let $f : X^{n+2} \to \mathcal{A}$ be a mapping satisfying*

$$\begin{aligned} f(0, y; z_1, \ldots, z_n) &= f(x, 0; z_1, \ldots, z_n) = f(x, y; 0, z_2, \ldots, z_n) \\ &= f(x, y; z_1, 0, z_3, \ldots, z_n) = \cdots \\ &= f(x, y; z_1, \ldots, z_{n-1}, 0) = 0 \end{aligned}$$

*for all $x, y, z_1, \ldots, z_n \in X$ for which there exists a function $\varphi : X^{2n+4} \to [0, \infty)$ such that*

(2.1)

$$\widetilde{\varphi}(x_1, x_2, y_1, y_2, z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}) := \sum_{j=0}^{\infty} \frac{1}{2^{(2n+2)j}} \varphi(2^{j+1}x_1, 2^{j+1}x_2, 2^{j+1}y_1, 2^{j+1}y_2, 2^{j+1}z_1, \ldots, 2^{j+1}z_n, 2^{j+1}z_{n+1}, \ldots, 2^{j+1}z_{2n}) < \infty,$$

(2.2)

$$\begin{aligned} &\Bigg\| \sum_{i_1=\pm1, \ldots, i_n=\pm1} f\left(\frac{ux_1 - ux_2}{2}, \frac{y_1 + y_2}{2}; \frac{z_1 + i_1 z_{n+1}}{2}, \ldots, \frac{z_n + i_n z_{2n}}{2}\right) \\ &\qquad + \sum_{j_{-1}, j_0 = 1}^{2} (-1)^{j_0} \sum_{\substack{j_l = l, l+n \\ l = 1, \ldots, n}} \frac{u}{2^{n+2}} f(x_{j_{-1}}, y_{j_0}; z_{j_1}, \ldots, z_{j_n}) \Bigg\| \\ &\quad \le \varphi(x_1, x_2, y_1, y_2, z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}), \end{aligned}$$

$$(2.3) \quad \begin{aligned} &\|f(x, y; z_1, \ldots, z_n) - f(y, x; z_1, \ldots, z_n)^*\| \\ &\qquad \le \varphi(x, x, y, y, z_1, \ldots, z_n, z_1, \ldots, z_n), \end{aligned}$$

$$(2.4) \quad \begin{aligned} &\|f(x, x; z_1, z_2, \ldots, z_n) - f(z_1, z_1; x, z_2, \ldots, z_n)\| \\ &\qquad \le \varphi(x, x, x, x, z_1, \ldots, z_n, z_1, \ldots, z_n), \end{aligned}$$

$$(2.5) \quad \begin{aligned} &\|f(x, y; z_1, \ldots, z_n) - f(x, y; z_{i_1}, \ldots, z_{i_n})\| \\ &\qquad \le \varphi(x, x, y, y, z_1, \ldots, z_n, z_1, \ldots, z_n) \end{aligned}$$

*for every even permutation* $(i_1, \ldots, i_n)$ *of* $(1, \ldots, n)$,

$$\begin{aligned}(2.6)\quad &\|f(x,y;z_1,\dots,z_n)-f(x,y;z_{i_1},\dots,z_{i_n})^*\|\\ &\qquad\le\varphi(x,x,y,y,z_1,\dots,z_n,z_1,\dots,z_n)\\ &\qquad\textit{for every odd permutation}\quad(i_1,\dots,i_n)\quad\textit{of}\quad(1,\dots,n)\end{aligned}$$

*for all* $u\in\mathcal{U}(\mathcal{A})$ *and all* $x,y,x_1,x_2,y_1,y_2,z_1,\cdots,z_n,z_{n+1},\dots,z_{2n}\in X$*. Then there exists a unique* $\mathcal{A}$*-sesquilinear* $n$*-quadratic mapping* $S:X^{n+2}\to\mathcal{A}$ *such that*

$$\|f(x_1,y_1;z_1,\dots,z_n)-S(x_1,y_1;z_1,\dots,z_n)\|\le\frac{1}{2^n}\widetilde{\varphi}(x_1,0,y_1,0,z_1,\dots,z_n,\underbrace{0,\dots,0}_{n\text{ times}})$$

*for all* $x_1,y_1,z_1,\dots,z_n\in X$*.*

*Proof.* Set $u=1\in\mathcal{U}(\mathcal{A})$ and $x_2=y_2=z_{n+1}=\cdots=z_{2n}=0$ in (2.2). Then

$$\left\|2^nf\left(\frac{x_1}{2},\frac{y_1}{2};\frac{z_1}{2},\cdots,\frac{z_n}{2}\right)-\frac{1}{2^{n+2}}f(x_1,y_1;z_1,\dots,z_n)\right\|\le\varphi(x_1,0,y_1,0,z_1,\dots,z_n,\underbrace{0,\dots,0}_{n\text{ times}}$$

for all $x_1,y_1,z_1,\dots,z_n\in X$. Thus,

$$(2.7)\quad\left\|f(x_1,y_1;z_1,\dots,z_n)-\frac{1}{2^{2n+2}}f(2x_1,2y_1;2z_1,\dots,2z_n)\right\|\le\frac{1}{2^n}\varphi(2x_1,0,2y_1,0,2z_1,\dots,2z_n,\underbrace{0,\dots,0}_{n\text{ times}})$$

for all $x_1,y_1,z_1,\dots,z_n\in X$. Therefore,

$$\begin{aligned}&\Bigg\|\frac{1}{2^{(2n+2)j}}f(2^jx_1,2^jy_1;2^jz_1,\dots,2^jz_n)\\ &\qquad-\frac{1}{2^{(2n+2)(j+1)}}f(2^{j+1}x_1,2^{j+1}y_1;2^{j+1}z_1,\dots,2^{j+1}z_n)\Bigg\|\\ &\le\frac{1}{2^{(2n+2)j+n}}\varphi(2^{j+1}x_1,0,2^{j+1}y_1,0,2^{j+1}z_1,\dots,2^{j+1}z_n,\underbrace{0,\dots,0}_{n\text{ times}})\end{aligned}$$

for all $x_1, y_1, z_1, \dots, z_n \in X$. Hence,

$$\tag{2.8} \begin{aligned} &\Big\| \frac{1}{2^{(2n+2)l}} f(2^l x_1, 2^l y_1; 2^l z_1, \dots, 2^l z_n) \\ &\qquad - \frac{1}{2^{(2n+2)m}} f(2^m x_1, 2^m y_1; 2^m z_1, \dots, 2^m z_n) \Big\| \\ &\le \sum_{j=l}^{m-1} \frac{1}{2^{(2n+2)j+n}} \varphi(2^{j+1} x_1, 0, 2^{j+1} y_1, 0, 2^{j+1} z_1, \dots, 2^{j+1} z_n, \underbrace{0, \dots, 0}_{n \text{ times}}) \end{aligned}$$

for all nonnegative integers $l$ and $m$ with $m > l$ and all $x_1, y_1, z_1, \dots, z_n \in X$. It follows from (2.1) and (2.8) that the sequence $\{(1/2^{(2n+2)d}) f(2^d x_1, 2^d y_1; 2^d z_1, \dots, 2^d z_n)\}$ is Cauchy for all $x_1, y_1, z_1, \dots, z_n \in X$. Since $\mathcal{A}$ is complete, one can define the mapping $S : X^{n+2} \to \mathcal{A}$ by

$$\tag{2.9} S(x, y; z_1, \dots, z_n) := \lim_{d \to \infty} \frac{1}{2^{(2n+2)d}} f(2^d x, 2^d y; 2^d z_1, \dots, 2^d z_n)$$

for all $x, y, z_1, \dots, z_n \in X$.

Moreover, letting $l = 0$ and $m \to \infty$ in (2.8) gives

$$\begin{aligned} &\| f(x_1, y_1; z_1, \dots, z_n) - S(x_1, y_1; z_1, \dots, z_n) \| \\ &\qquad \le \frac{1}{2^n} \widetilde{\varphi}(x_1, 0, y_1, 0, z_1, \dots, z_n, \underbrace{0, \dots, 0}_{n \text{ times}}) \end{aligned}$$

for all $x_1, y_1, z_1, \dots, z_n \in X$.

Set $u = 1 \in \mathcal{U}(\mathcal{A})$ and $y_2 = y_1, z_1 = z_{n+1}, \dots, z_n = z_{2n}$ in (2.2). Then

$$\begin{aligned} &\Big\| f(\frac{x_1 - x_2}{2}, y_1; z_1, \dots, z_n) \\ &\qquad - \frac{1}{2} f(x_1, y_1; z_1, \dots, z_n) + \frac{1}{2} f(x_2, y_1; z_1, \dots, z_n) \Big\| \\ &\qquad\qquad \le \varphi(x_1, x_2, y_1, y_1, z_1, \dots, z_n, z_1, \dots, z_n) \end{aligned}$$

for all $x_1, x_2, y_1, z_1, \dots, z_n \in X$. Thus,

$$\begin{aligned} &\frac{1}{2^{(2n+2)j}} \Big\| f\left( \frac{2^j x_1 - 2^j x_2}{2}, 2^j y_1; 2^j z_1, \dots, 2^j z_n \right) \\ &\quad - \frac{1}{2} f(2^j x_1, 2^j y_1; 2^j z_1, \dots, 2^j z_n) + \frac{1}{2} f(2^j x_2, 2^j y_1; 2^j z_1, \dots, 2^j z_n) \Big\| \\ &\le \frac{1}{2^{(2n+2)j}} \varphi(2^j x_1, 2^j x_2, 2^j y_1, 2^j y_1, 2^j z_1, \dots, 2^j z_n, 2^j z_1, \dots, 2^j z_n), \end{aligned}$$

which converges to zero as $j \to \infty$ by (2.1). By (2.9),

$$S\left(\frac{x_1 - x_2}{2}, y_1; z_1, \dots, z_n\right) = \frac{1}{2}S(x_1, y_1; z_1, \dots, z_n) - \frac{1}{2}S(x_2, y_1; z_1, \dots, z_n)$$

for all $x_1, x_2, y_1, z_1, \dots, z_n \in X$. Therefore, the mapping $S : X^{n+2} \to \mathcal{A}$ is additive in the first variable.

Set $x_2 = 0$, and $y_1 = y_2, z_1 = z_{n+1}, \dots, z_n = z_{2n}$ in (2.2). Then

$$\left\| f\left(\frac{ux_1}{2}, y_1; z_1, \dots, z_n\right) - \frac{u}{2}f(x_1, y_1; z_1, \dots, z_n)\right\| \le \varphi(x_1, 0, y_1, y_1, z_1, \dots, z_n, z_1, \dots, z_n)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, y_1, z_1, \dots, z_n \in X$. Hence,

$$\frac{1}{2^{(2n+2)j}}\left\| f(2^{j-1}ux_1, 2^j y_1; 2^j z_1, \dots, 2^j z_n) - \frac{u}{2}f(2^j x_1, 2^j y_1; 2^j z_1, \dots, 2^j z_n)\right\| \le \frac{1}{2^{(2n+2)j}}\varphi(2^j x_1, 0, 2^j y_1, 2^j y_1, 2^j z_1, \dots, 2^j z_n, 2^j z_1, \dots, 2^j z_n),$$

which converges to zero as $j \to \infty$ by (2.1). By (2.9),

$$S\left(\frac{ux_1}{2}, y_1; z_1, \dots, z_n\right) = \frac{u}{2}S(x_1, y_1; z_1, \dots, z_n)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, y_1, z_1, \dots, z_n \in X$. Since $S$ is additive in the first variable,

$$\begin{aligned} S(ux_1, y_1; z_1, \dots, z_n) &= 2S\left(\frac{ux_1}{2}, y_1; z_1, \dots, z_n\right) \\ &= uS(x_1, y_1; z_1, \dots, z_n) \end{aligned}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, y_1, z_1, \dots, z_n \in X$.

Kadison and Pedersen's theorem [**4**] says that if $a \in \mathcal{A}$ and $|a| < 1-(2/m)$ for some integer $m$ greater than 2, then there are $u_1, \dots, u_m \in$

$\mathcal{U}(\mathcal{A})$ such that $ma = u_1 + \cdots + u_m$. Thus, let $a \in \mathcal{A}$, $a \neq 0$, and let $M$ be an integer greater than $3|a|$. Then $|a/M| < 1/3 = 1 - 2/3$. So there exist $u_1, u_2, u_3 \in \mathcal{U}(\mathcal{A})$ such that $3(a/M) = u_1 + u_2 + u_3$. Hence,

$$\begin{aligned} S(ax, y; z_1, \dots, z_n) &= S\left(\frac{M}{3} \cdot 3\frac{a}{M}x, y; z_1, \dots, z_n\right) \\ &= \frac{M}{3} \cdot S\left(3\frac{a}{M}x, y; z_1, \dots, z_n\right) \\ &= \frac{M}{3} S(u_1x + u_2x + u_3x, y; z_1, \dots, z_n) \\ &= \frac{M}{3}(S(u_1x, y; z_1, \dots, z_n) + S(u_2x, y; z_1, \dots, z_n) \\ &\qquad + S(u_3x, y; z_1, \dots, z_n)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)S(x, y; z_1, \dots, z_n) \\ &= \frac{M}{3} \cdot 3\frac{a}{M} S(x, y; z_1, \dots, z_n) \\ &= aS(x, y; z_1, \dots, z_n) \end{aligned}$$

for all $x, y, z_1, \dots, z_n \in X$. Hence, the mapping $S : X^{n+2} \to \mathcal{A}$ is $\mathcal{A}$-linear in the first variable.

It follows from (2.3) that

$$\begin{aligned} &\frac{1}{2^{(2n+2)j}} \|f(2^j x, 2^j y; 2^j z_1, \dots, 2^j z_n) - f(2^j y, 2^j x; 2^j z_1, \dots, 2^j z_n)^*\| \\ &\quad \leq \frac{1}{2^{(2n+2)j}} \varphi(2^j x, 2^j x, 2^j y, 2^j y, 2^j z_1, \dots, 2^j z_n, 2^j z_1, \dots, 2^j z_n), \end{aligned}$$

which converges to zero as $j \to \infty$ by (2.1). By (2.9),

$$S(x, y; z_1, \dots, z_n) = S(y, x; z_1, \dots, z_n)^*$$

for all $x, y, z_1, \dots, z_n \in X$. Thus,

$$\begin{aligned} &S(x, ay_1 + by_2; z_1, \dots, z_n) \\ &\qquad = S(ay_1 + by_2, x; z_1, \dots, z_n)^* \\ &\qquad = S(y_1, x; z_1, \dots, z_n)^* a^* + S(y_2, x; z_1, \dots, z_n)^* b^* \\ &\qquad = S(x, y_1; z_1, \dots, z_n) a^* + S(x, y_2; z_1, \dots, z_n) b^* \end{aligned}$$

for all $a, b \in \mathcal{A}$ and all $x, y_1, y_2, z_1, \dots, z_n \in X$. Therefore, the mapping $S : X^{n+2} \to \mathcal{A}$ is $\mathcal{A}$-conjugate-linear in the second variable.

Similarly, one can show that the mapping $S : X^{n+2} \to \mathcal{A}$ satisfies the rest of the properties of Definition 1.1. Hence, the mapping $S : X^{n+2} \to \mathcal{A}$ is an $\mathcal{A}$-sesquilinear $n$-quadratic mapping.

Now suppose $T : X^{n+2} \to \mathcal{A}$ is another $\mathcal{A}$-sesquilinear $n$-quadratic mapping satisfying

$$\begin{aligned}&\|f(x_1, y_1; z_1, \dots, z_n) - T(x_1, y_1; z_1, \dots, z_n)\| \\ &\qquad \le \frac{1}{2^n}\widetilde{\varphi}(x_1, 0, y_1, 0, z_1, \dots, z_n, \underbrace{0, \dots, 0}_{n \text{ times}})\end{aligned}$$

for all $x_1, y_1, z_1, \dots, z_n \in X$. Then we have

$$\begin{aligned}&\|S(x_1, y_1; z_1, \dots, z_n) - T(x_1, y_1; z_1, \dots, z_n)\| \\ &\quad = \frac{1}{2^{(2n+2)d}}\|S(2^d x_1, 2^d y_1; 2^d z_1, \dots, 2^d z_n) \\ &\qquad\qquad - T(2^d x_1, 2^d y_1; 2^d z_1, \dots, 2^d z_n)\| \\ &\quad \le \frac{1}{2^{(2n+2)d}}\|S(2^d x_1, 2^d y_1; 2^d z_1, \dots, 2^d z_n) \\ &\qquad\qquad - f(2^d x_1, 2^d y_1; 2^d z_1, \dots, 2^d z_n)\| \\ &\qquad + \frac{1}{2^{(2n+2)d}}\|f(2^d x_1, 2^d y_1; 2^d z_1, \dots, 2^d z_n) \\ &\qquad\qquad - T(2^d x_1, 2^d y_1; 2^d z_1, \dots, 2^d z_n)\| \\ &\le \sum_{j=d}^{\infty} \frac{2}{2^{(2n+2)j+n}}\varphi(2^{j+1}x_1, 0, 2^{j+1}y_1, 0, 2^{j+1}z_1, \dots, 2^{j+1}z_n, \underbrace{0, \dots, 0}_{n \text{ times}}),\end{aligned}$$

which converges to zero as $d \to \infty$ by (2.1). Hence,

$$S(x_1, y_1; z_1, \dots, z_n) = T(x_1, y_1; z_1, \dots, z_n)$$

for all $x_1, y_1, z_1, \dots, z_n \in X$. This proves the uniqueness of $S$. □

**Corollary 2.2.** *Let $p$ and $\theta$ be positive real numbers where $p < 2+2n$. Let $f : X^{n+2} \to \mathcal{A}$ be a mapping satisfying*

$$\begin{aligned}f(0, y; z_1, \dots, z_n) &= f(x, 0; z_1, \dots, z_n) = f(x, y; 0, z_2, \dots, z_n) \\ &= f(x, y; z_1, 0, z_3, \dots, z_n) = \cdots \\ &= f(x, y; z_1, \dots, z_{n-1}, 0) = 0\end{aligned}$$

*for all* $x, y, z_1, \dots, z_n \in X$*, and such that*

$$\Big\| \sum_{i_1=\pm1,\dots,i_n=\pm1} f\Big(\frac{ux_1 - ux_2}{2}, \frac{y_1+y_2}{2}; \frac{z_1 + i_1 z_{n+1}}{2}, \dots, \frac{z_n + i_n z_{2n}}{2}\Big) + \sum_{j_{-1},j_0=1}^{2} (-1)^{j_{-1}} \sum_{\substack{j_l=l,l+n \\ l=1,\dots,n}} \frac{u}{2^{n+2}} f(x_{j_{-1}}, y_{j_0}; z_{j_1}, \dots, z_{j_n}) \Big\| \le \theta\Big(||x_1||^p + ||x_2||^p + ||y_1||^p + ||y_2||^p + \sum_{i=1}^{2n} ||z_i||^p\Big),$$

$$\|f(x, y; z_1, \dots, z_n) - f(y, x; z_1, \dots, z_n)^*\| \le 2\theta\Big(||x||^p + ||y||^p + \sum_{i=1}^{n} ||z_i||^p\Big),$$

$$\|f(x, x; z_1, z_2, \dots, z_n) - f(z_1, z_1; x, z_2, \dots, z_n)\| \le 4\theta||x||^p + 2\theta \sum_{i=1}^{n} ||z_i||^p,$$

$$\|f(x,y; z_1, \dots, z_n) - f(x, y; z_{i_1}, \dots, z_{i_n})\| \le 2\theta\Big(||x||^p + ||y||^p + \sum_{i=1}^{n} ||z_i||^p\Big)$$

*for every even permutation* $(i_1, \dots, i_n)$ *of* $(1, \dots, n)$,

$$\|f(x,y; z_1, \dots, z_n) - f(x, y; z_{i_1}, \dots, z_{i_n})^*\| \le 2\theta\Big(||x||^p + ||y||^p + \sum_{i=1}^{n} ||z_i||^p\Big)$$

*for every odd permutation* $(i_1, \dots, i_n)$ *of* $(1, \dots, n)$

*for all* $u \in \mathcal{U}(\mathcal{A})$ *and all* $x, y, x_1, x_2, z_1, \dots, z_n, z_{n+1}, \dots, z_{2n} \in X$. *Then there exists a unique* $\mathcal{A}$*-sesquilinear* $n$*-quadratic mapping* $S : X^{n+2} \to \mathcal{A}$ *such that*

$$\|f(x, y; z_1, \dots, z_n) - S(x, y; z_1, \dots, z_n)\| \\ \le \frac{2^{n+2+p}\theta}{2^{2n+2} - 2^p}\Big(||x||^p + ||y||^p + \sum_{i=1}^{n} ||z_i||^p\Big)$$

*for all* $x, y, z_1, \dots, z_n \in X$.

*Proof.* Define

$$\varphi(x_1, x_2, y_1, y_2, z_1, \dots, z_n, z_{n+1}, \dots, z_{2n}) \\ = \theta\Big(||x_1||^p + ||x_2||^p + ||y_1||^p + ||y_2||^p + \sum_{i=1}^{2n} ||z_i||^p\Big),$$

and apply Theorem 2.1. □

**Theorem 2.3.** *Let* $f : X^{n+2} \to \mathcal{A}$ *be a mapping satisfying*

$$f(0, y; z_1, \dots, z_n) = f(x, 0; z_1, \dots, z_n) = f(x, y; 0, z_2, \dots, z_n) \\ = f(x, y; z_1, 0, z_3, \dots, z_n) = \cdots \\ = f(x, y; z_1, \dots, z_{n-1}, 0) = 0$$

*for all* $x, y, z_1, \dots, z_n \in X$, *and for which there exists a function* $\varphi : X^4 \times X^{2n} \to [0, \infty)$ *satisfying* (2.2)–(2.6) *such that*

$$(2.10) \quad \widetilde{\varphi}(x_1, x_2, y_1, y_2, z_1, \dots, z_n, z_{n+1}, \dots, z_{2n}) \\ := \sum_{j=0}^{\infty} 2^{(2n+2)j}\varphi\Big(\frac{x_1}{2^j}, \frac{x_2}{2^j}, \frac{y_1}{2^j}, \frac{y_2}{2^j}, \frac{z_1}{2^j}, \dots, \frac{z_n}{2^j}, \frac{z_{n+1}}{2^j}, \dots, \frac{z_{2n}}{2^j}\Big) < \infty$$

*for all* $x_1, x_2, y_1, y_2, z_1, \dots, z_n, z_{n+1}, \dots, z_{2n} \in X$. *Then there exists a unique* $\mathcal{A}$*-sesquilinear* $n$*-quadratic mapping* $S : X^{n+2} \to \mathcal{A}$ *such that*

$$\|f(x, y; z_1, \dots, z_n) - S(x, y; z_1, \dots, z_n)\| \\ \le 2^{n+2}\widetilde{\varphi}(x, 0, y, 0, z_1, \dots, z_n, \underbrace{0, \dots, 0}_{n \text{ times}})$$

*for all* $x_1, y_1, z_1, \dots, z_n \in X$.

*Proof.* It follows from (2.7) that

$$\left\| f(x_1, y_1; z_1, \dots, z_n) - 2^{2n+2} f\left(\frac{x_1}{2}, \frac{y_1}{2}; \frac{z_1}{2}, \dots, \frac{z_n}{2}\right) \right\| \\ \leq 2^{n+2} \varphi(x_1, 0, y_1, 0, z_1, \dots, z_n, \underbrace{0, \dots, 0}_{n \text{ times}})$$

for all $x_1, y_1, z_1, \dots, z_n \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 2.4.** *Let $p$ and $\theta$ be positive real numbers where $p > 2+2n$. Let $f : X \times X \times X^n \to \mathcal{A}$ be a mapping satisfying the conditions given in Corollary 2.2. Then there exists a unique $\mathcal{A}$-sesquilinear $n$-quadratic mapping $S : X^{n+2} \to \mathcal{A}$ such that*

$$\|f(x, y; z_1, \dots, z_n) - S(x, y; z_1, \dots, z_n)\| \\ \leq \frac{2^{n+2+p}\theta}{2^p - 2^{2n+2}} \left( ||x||^p + ||y||^p + \sum_{i=1}^{n} ||z_i||^p \right)$$

*for all $x, y, z_1, \dots, z_n \in X$.*

*Proof.* Define

$$\varphi(x_1, x_2, y_1, y_2, z_1, \dots, z_n, z_{n+1}, \dots, z_{2n}) \\ = \theta \left( ||x_1||^p + ||x_2||^p + ||y_1||^p + ||y_2||^p + \sum_{i=1}^{2n} ||z_i||^p \right),$$

and apply Theorem 2.3. □

## REFERENCES

**1.** Y.J. Cho, P.C.S. Lin, S.S. Kim and A. Misiak, *Theory of 2-inner product spaces*, Nova Science Publishers, New York, 2001.

**2.** P.W. Cholewa, *Remarks on the stability of functional equations*, Aequat. Math. **27** (1984), 76–86.

**3.** S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.

**4.** R.V. Kadison and G. Pedersen, *Means and convex combinations of unitary operators*, Math. Scand. **57** (1985), 249–266.

**5.** A. Misiak, *$n$-inner product spaces*, Math. Nachr. **140** (1989), 299–319.

**6.** ———, *Orthogonality and orthogonormality in $n$-inner product spaces*, Math. Nachr. **143** (1989), 249–261.

**7.** C. Park, *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl. **275** (2002), 711–720.

**8.** ———, *Generalized Hyers-Ulam-Rassias stability of $n$-sesquilinear-quadratic mappings on Banach modules over $C^*$-algebras*, J. Comput. Appl. Math. **180** (2005), 279–291.

**9.** ———, *Multilinear Trif d-mappings in Banach modules over a $C^*$-algebra*, Rocky Mountain J. Math. **35** (2005), 641–654.

**10.** J.M. Rassias and M.J. Rassia, *On the Ulam stability of Jensen and Jensen type mappings on restricted domains*, J. Math. Anal. Appl. **281** (2003), 516–524.

**11.** Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

**12.** ———, *On the stability of the quadratic functional equation and its applications*, Studia Univ. Babes-Bolyai **43** (1998), 89–124.

**13.** ———, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.

**14.** ———, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.

**15.** ———, *On the stability of the quadratic functional equation*, Mathematica, to appear.

**16.** Th.M. Rassias and P. Šemrl, *On the Hyers-Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.

**17.** Th.M. Rassias and K. Shibata, *Variational problem of some quadratic functionals in complex analysis*, J. Math. Anal. Appl. **228** (1998), 234–253.

**18.** F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.

Department of Mathematics, Hanyang University, Seoul 133-791, Republic of Korea
**Email address: baak@hanyang.ac.kr**

Department of Mathematics, University of Ulsan, Ulsan 680 749, Republic of Korea
**Email address: jsym@uou.ulsan.ac.kr**