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GENERALIZED BASKAKOV-BETA OPERATORS

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ABSTRACT. Very recently Wang [9] introduced the modified form of Baskakov-beta operators and obtained a Voronovskaja type asymptotic formula for these operators. We extend the study and here we estimate a direct result in terms of higher order modulus of continuity and an inverse theorem in simultaneous approximation for these new modified Baskakov-beta operators.

1. Introduction. For $f \in C_\gamma[0,\infty) \equiv \{f \in C[0,\infty) : f(t) = O(t^\gamma)$ as $t \to \infty$ for some $\gamma > 0\}$ and $\alpha > 0$, Wang [9] introduced modified Baskakov-beta operators as

(1)
$$B_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} p_{n,k,\alpha}(x) \int_0^\infty b_{n,k,\alpha}(t) f(t)\,dt = \int_0^\infty W_{n,\alpha}(x,t) f(t)\,dt$$

where

$$p_{n,k,\alpha}(x) = \frac{\Gamma(n/\alpha+k)}{\Gamma(k+1)\Gamma(n/\alpha)}.\frac{(\alpha x)^k}{(1+\alpha x)^{(n/\alpha)+k}},$$

$$b_{n,k,\alpha}(t) = \frac{1}{B(n/\alpha,k+1)}\frac{\alpha(\alpha t)^k}{(1+\alpha t)^{n/\alpha+k+1}}$$

and

$$W_{n,\alpha}(x,t) = \sum_{k=0}^{\infty} p_{n,k,\alpha}(x) b_{n,k,\alpha}(t).$$

The norm- $||.||_\gamma$ on the class $C_\gamma[0,\infty)$ is defined as $||f||_\gamma = \sup_{0<t<\infty} |f(t)|t^{-\gamma}$.

As a special case $\alpha = 1$, the operators defined by (1) reduce to the well known Baskakov-beta operators [5]. Wang [9] recently obtained an asymptotic formula for the operators (1). In the present paper we

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establish an error estimation and inverse theorem in the simultaneous approximation for the unbounded functions of growth of the order t^γ. Also very recently Gupta [**6**] established some direct results for the generalized operators of Finta [**2**].

Throughout the present paper we denote by M the positive constant which has different meaning at each occurrence.

2. Basic results. In this section we mention certain lemmas which will be used in the sequel.

Lemma 1 [**3**]. *For $m \in N \cup \{0\}$, if the mth order moment is defined as*

$$U_{n,m,\alpha}(x) = \sum_{k=0}^{\infty} p_{n,k,\alpha}(x)\left(\frac{k}{n} - x\right)^m,$$

then $U_{n,0,\alpha}(x) = 1, U_{n,1,\alpha}(x) = 0$ and

$$nU_{n,m+1,\alpha}(x) = x(1+\alpha x)(U^{(1)}_{n,m,\alpha}(x) + mU_{n,m-1,\alpha}(x)).$$

Consequently, we have $U_{n,m,\alpha}(x) = O(n^{-[(m+1)/2]})$.

Lemma 2. *Let the function $T_{n,m,\alpha}(x)$, $m \in N \cup \{0\}$, be defined as*

$$T_{n,m,\alpha}(x) = B_{n,\alpha}\big((t-x)^m, x\big) = \sum_{k=0}^{\infty} p_{n,k,\alpha}(x)\int_0^{\infty} b_{n,k,\alpha}(t)(t-x)^m\,dt.$$

Then

$$\begin{aligned}T_{n,0,\alpha}(x) &= 1,\\ T_{n,1,\alpha} &= (1+\alpha x)/(n-\alpha),\\ T_{n,2,\alpha}(x) &= (2\alpha(n+\alpha)x^2 + 2(n+2\alpha)x + 2)/((n-\alpha)(n-2\alpha)),\end{aligned}$$

and for $n > (m+1)\alpha$, the following recurrence relation holds

$$\begin{aligned}&[n-(m+1)\alpha]T_{n,m+1,\alpha}(x)\\ &\qquad = x(1+\alpha x)\lfloor T^{(1)}_{n,m,\alpha}(x) + 2mT_{n,m-1,\alpha}(x)\rfloor\\ &\qquad\quad + [(m+1)(1+2\alpha x) - \alpha x]T_{n,m,\alpha}(x).\end{aligned}$$

Corollary 3. *Let δ be a positive number and $s = 1, 2, 3, \ldots$. Then, for every $\gamma > 0$ and $x \in (0, \infty)$, there exists a constant $M(s, x)$ independent of n and dependent on s and x such that*

$$\left\| \int_{|t-x|>\delta} W_{n,\alpha}(x,t) t^{\gamma}\, dt \right\|_{C[a,b]} \leq M(s,x) n^{-s}.$$

Lemma 4. *There exist the polynomials $Q_{i,j,r,\alpha}(x)$ of degree at most r in x and independent of n and k such that*

$$\{x(1+\alpha x)\}^r D^r \left[p_{n,k,\alpha}(x)\right] = \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i (k-nx)^j Q_{i,j,r,\alpha}(x) p_{n,k,\alpha}(x),$$

where $D \equiv d/(dx)$.

Proof. By simple computation, it is easily verified that $x(1 + \alpha x)p^{(1)}_{n,k,\alpha}(x) = (k - nx)p_{n,k,\alpha}(x)$. In order to prove the result, we assume that the result is true for $r = m$; we can easily prove that it is also true for $r = m + 1$. Thus, by the principle of mathematical induction, the lemma follows. □

By C_0, we denote the class of continuous functions on the interval $(0, \infty)$ having a compact support, and C_0^r is the class of r-times continuously differentiable functions with $C_0^r \subset C_0$. The function f is said to belong to the generalized Zygmund class $\text{Liz}\,(\beta, k, a, b)$, if there exists a constant M such that $\omega_{2k}(f, \delta) \leq M\delta^{\beta k}$, $\delta > 0$, where $\omega_{2k}(f, \delta)$ denotes the modulus of continuity of $2k$th order on the interval $[a, b]$. The class $\text{Liz}\,(\beta, 1, a, b)$ is more commonly denoted by $\text{Lip}^*(\beta, a, b)$. Suppose $G^{(r)} = \{g : g \in C_0^{r+2}, \text{supp}\, g \subset [a', b'] \text{ where } [a', b'] \subset (a, b)\}$. For r times continuously differentiable functions f with $\text{supp}\, f \subset [a', b']$, the Peetre's K-functional is defined as

$$K_r(\xi, f) = \inf_{g\in G^{(r)}} [||f^{(r)} - g^{(r)}||_{C[a',b']} + \xi\{||g^{(r)}||_{C[a',b']} + ||g^{(r+2)}||_{C[a',b']}\}],$$
$$0 < \xi \leq 1.$$

For $0 < \beta < 2$, $C_0^r(\beta, a, b)$ denotes the set of functions for which

$$\sup_{0<\xi\leq 1} \xi^{-\beta/2} K_r(\xi, f) < M.$$

Theorem 5 [**9**]. *Let* $f \in C_\gamma[0,\infty)$. *If* $f^{(r+2)}$ *exists at a point* $x \in (0,\infty)$, *then*

$$\lim_{n\to\infty} n[B_{n,\alpha}^{(r)}(f,x) - f^{(r)}(x)]$$
$$= \alpha r^2 f^{(r)}(x) + [(1+r) + \alpha x(1+2r)] f^{(r+1)}(x) + x(1+\alpha x) f^{(r+2)}(x).$$

Further if $f^{(r+2)}$ *exists and is continuous on* $(a-\eta, b+\eta) \subseteq [0,\infty), \eta > 0$, *then the above limit holds uniformly on* $[a,b]$.

Lemma 6. *Let* $0 < \beta < 2$, $0 < a < a' < a'' < b'' < b' < b < \infty$. *If* $f \in C_0^r$ *with* $\operatorname{supp} f \subset [a'', b'']$, $|f(t)| \le Mt^\gamma$ *for some* $M > 0$, $\gamma > 0$ *and* $||B_{n,\alpha}^{(r)}(f,.) - f^{(r)}||_{C[a,b]} = O(n^{-\beta/2})$, *then*

$$K_r(\xi, f) \le M\{n^{-\beta/2} + n\xi K_r(n^{-1}, f)\}.$$

Consequently, $K_r(\xi, f) \le M\xi^{\beta/2}$, $M > 0$.

Proof. It is sufficient to prove

$$K_r(\xi, f) \le M\{n^{-\beta/2} + n\xi K_r(n^{-1}, f)\},$$

for sufficiently large n. Since $\operatorname{supp} f \subset [a'', b'']$, there is an $h \in G^{(r)}$ (see also [**8**]), such that

$$||B_{n,\alpha}^{(i)}(f,\bullet) - h^{(i)}||_{C[a',b']} \le Mn^{-1}, \quad i = r, \quad r+2.$$

Therefore,

$$K_r(\xi, f) \le 3Mn^{-1} + ||B_{n,\alpha}^{(r)}(f,\bullet) - f^{(r)}||_{C[a',b']}$$
$$+ \xi \left\{ ||B_{n,\alpha}^{(r)}(f,\bullet)||_{C[a',b']} + ||B_{n,\alpha}^{(r+2)}(f,\bullet)||_{C[a',b']} \right\}.$$

Next, it is sufficient to show that there exists an absolute constant M such that, for each $g \in G^{(r)}$,

(2)
$$||B_{n,\alpha}^{(r)}(f,\bullet)||_{C[a',b']} \le M.n\{||f^{(r)} - g^{(r)}||_{C[a',b']} + n^{-1}||g^{(r+2)}||_{C[a',b']}\}.$$

By the linearity property, we have

(3)
$$||B_{n,\alpha}^{(r+2)}(f,\bullet)||_{C[a',b']} \le ||B_{n,\alpha}^{(r+2)}(f-g,\bullet)||_{C[a',b']} + ||B_{n,\alpha}^{(r+2)}(g,\bullet)||_{C[a',b']}.$$

Applying Lemma 4, we get

$$\int_0^\infty \left|\frac{\partial^{r+2}}{\partial x^{r+2}} W_{n,\alpha}(x,t)\right| dt$$
$$\leq \sum_{\substack{2i+j\leq r+2 \\ i,j\geq 0}} \sum_{k=0}^{\infty} n^i |k-nx|^j \frac{|Q_{i,j,r,\alpha}(x)|}{\{x(1+\alpha x)\}^{r+2}} p_{n,k,\alpha}(x) \int_0^\infty b_{n,k,\alpha}(t)\, dt.$$

Therefore, by the Cauchy-Schwarz inequality and Lemma 1, we obtain

$$(4) \qquad ||B_{n,\alpha}^{(r)}(f-g,\bullet)||_{C[a',b']} \leq M.n||f^{(r)}-g^{(r)}||_{C[a',b']},$$

where the above constant M is independent of f and g. By Taylor's expansion, we have

$$g(t) = \sum_{i=0}^{r+1} \frac{g^{(i)}(x)}{i!}(t-x)^i + \frac{g^{(r+2)}(\xi)}{(r+2)!}(t-x)^{r+2},$$

where ξ lies between t and x. Using the above expansion we get

$$(5) \quad ||B_{n,\alpha}^{(r+2)}(g,\bullet)||_{C[a',b']} \leq M||g^{(r+2)}||_{C[a',b']}\cdot \left\| \frac{\partial^{r+2}}{\partial x^{r+2}} W_{n,\alpha}(x,t)(t-x)^{r+2}\, dt \right|_{C[a',b']}.$$

Also by Lemma 4 and the Cauchy-Schwarz inequality, we have

$$(6) \qquad \|B_{n,\alpha}^{(r+2)}(g,\bullet)\|_{C[a',b']} \leq M\|g^{(r+2)}\|_{C[a',b']}.$$

Combining the estimates of (3)–(6), we get (2). The other consequence follows from [**1**]. This completes the proof of the lemma. □

Lemma 7. *Let* $0 < \beta < 2$, $0 < a' < a'' < b'' < b' < b < \infty$ *and* $f^{(r)} \in C_0$ *with* $\operatorname{supp} f \subset [a'', b'']$ *and, if* $f \in C_0^r(\beta, a', b')$, *then we have* $f^{(r)} \in \mathrm{Lip}^*(\beta, a', b')$.

Proof. Let $g \in G^{(r)}$ and $|h| < \delta$. Then, for $f \in C_0^r(\beta, 1, a', b')$, we have

$$\begin{aligned} |\triangle_h^2 f^{(r)}(x)| &\leq |\triangle_h^2 (f^{(r)} - g^{(r)})(x)| + |\triangle_h^2 g^{(r)}(x)| \\ &\leq 2^2 ||f^{(r)} - g^{(r)}||_{C[a',b']} + h^2 ||g^{(r+2)}||_{C[a',b']} \\ &\leq MK_r(h^2, f) \leq Mh^\beta, \end{aligned}$$

which implies that

$$\omega_2(f^{(r)}, \delta) = \sup_{|h|<\delta} |\triangle_h^2 f^{(r)}(x)| \le M\delta^\beta.$$

Thus, $f^{(r)} \in \mathrm{Lip}^*(\beta, a', b')$, which completes the proof of the lemma (see [**7**] also). □

Lemma 8. *If f is r times differentiable on $[0,\infty)$, such that $f^{(r-1)} = O(t^\gamma)$, $\gamma > 0$ as $t \to \infty$, then for $r = 1, 2, 3, \dots$ and $n > \gamma + r$ we have*

$$B_{n,\alpha}^{(r)}(f, x) = \frac{\Gamma(n/\alpha + r)\Gamma(n/\alpha - r)}{(\Gamma(n/\alpha))^2} \sum_{k=0}^{\infty} p_{n+\alpha r,k,\alpha}(x) \int_0^\infty b_{n-\alpha r,k+r,\alpha}(t) f^{(r)}(t)\, dt.$$

3. Rate of approximation. In this section we present the following results.

Theorem 9. *Let $f \in C_\gamma[0,\infty)$, and suppose $0 < a < a_1 < b_1 < b < \infty$. Then, for all n sufficiently large, we have*

$$\|B_{n,\alpha}^{(r)}(f, \bullet) - f^{(r)}(.)\|_{C[a_1,b_1]} \le M.\big\{\omega_2(f^{(r)}, n^{-1/2}, a, b) + n^{-1}\|f\|_\gamma\big\}.$$

Proof. For sufficiently small $\delta > 0$, we define a function $f_{2,\delta}(t)$ corresponding to $f \in C_\gamma[0,\infty)$ by

$$f_{2,\delta}(t) = \delta^{-2} \int_{-\delta/2}^{\delta/2} \int_{-\delta/2}^{\delta/2} \big(f(t) - \Delta_\eta^2 f(t)\big)\, dt_1\, dt_2,$$

where $\eta = (t_1 + t_2)/2$, $t \in [a, b]$, and $\Delta_\eta^2 f(t)$ is the second forward difference of f with step length η. Following [**3**], see also [**8**, page 325], it is easily checked that:

(i) $f_{2,\delta}$ has continuous derivatives up to order 2 on $[a, b]$,

(ii) $\|f_{2,\delta}^{(r)}\|_{C[a_1,b_1]} \le M\delta^{-r}\omega_2(f,\delta,a,b)$,

(iii) $\|f - f_{2,\delta}\|_{C[a_1,b_1]} \le M\omega_2(f,\delta,a,b)$,

(iv) $\|f_{2,\delta}\|_{C[a_1,b_1]} \le M||f||_{C[a_1,b_1]} \le M\|f\|_\gamma$.

We can write

$$\begin{aligned}
\||B_{n,\alpha}^{(r)}(f,\bullet) - f^{(r)}||_{C[a_1,b_1]} \\
&\le ||B_{n,\alpha}^{(r)}(f - f_{2,\delta},\bullet)||_{C[a_1,b_1]} + ||B_{n,\alpha}^{(r)}(f_{2,\delta},\bullet) - f_{2,\delta}^{(r)}||_{C[a_1,b_1]} \\
&\quad + ||f^{(r)} - f_{2,\delta}^{(r)}||_{C[a_1,b_1]} \\
&=: H_1 + H_2 + H_3.
\end{aligned}$$

Since $f_{2,\delta}^{(r)} = (f^{(r)})_{2,\delta}(t)$, by property (iii) of the function $f_{2,\delta}$, we get

$$H_3 \le M\omega_2(f^{(r)},\delta,a,b).$$

Next, on an application of Theorem 5, it follows that

$$H_2 \le Mn^{-1}\sum_{j=r}^{r+2}\|f_{2,\delta}^{(j)}\|_{C[a_1,b_1]}.$$

Using the interpolation property due to Goldberg and Meir [**4**], for each $j = r,\ r+1,\ r+2$, it follows that

$$||f_{2,\delta}^{(j)}||_{C[a_1,b_1]} \le M\left\{||f_{2,\delta}||_{C[a_1,b_1]} + ||f_{2,\delta}^{(r+2)}||_{C[a_1,b_1]}\right\}.$$

Therefore, by applying properties (ii) and (iv) of the function $f_{2,\delta}$, we obtain

$$H_2 \le M.n^{-1}\big\{||f||_\gamma + \delta^{-2}\omega_2(f^{(r)},\delta,a,b)\big\}.$$

Finally we shall estimate H_1, choosing a^*, b^* satisfying the conditions $0 < a < a^* < a_1 < b_1 < b^* < b < \infty$. Suppose $\psi(t)$ denotes the characteristic function of the interval $[a^*, b^*]$, then

$$\begin{aligned}
H_1 &\le ||B_{n,\alpha}^{(r)}(\psi(t)(f(t) - f_{2,\delta}(t)),\bullet)||_{C[a_1,b_1]} \\
&\quad + \|B_{n,\alpha}^{(r)}((1-\psi(t))(f(t) - f_{2,\delta}(t)),\bullet)\|_{C[a_1,b_1]} \\
&=: H_4 + H_5.
\end{aligned}$$

Using Lemma 8, it is clear that

$$B_{n,\alpha}^{(r)}\big(\psi(t)(f(t)-f_{2,\delta}(t)),x\big) = \frac{\Gamma(n/\alpha+r)\Gamma(n/\alpha-r)}{\Gamma(n/\alpha))^2}\sum_{k=0}^{\infty}p_{n+\alpha r,k,\alpha}(x) \times\int_0^\infty b_{n-\alpha r,k+r,\alpha}(t)\psi(t)(f^{(r)}(t)-f_{2,\delta}^{(r)}(t))\,dt.$$

Hence,

$$||B_{n,\alpha}^{(r)}(\psi(t)(f(t)-f_{2,\delta}(t)),\bullet)||_{C[a_1,b_1]}\le M||f^{(r)}-f_{2,\delta}^{(r)}||_{C[a^*,b^*]}.$$

Next, for $x\in[a_1,b_1]$ and $t\in[0,\infty)\setminus[a^*,b^*]$, we choose a $\delta_1>0$ satisfying $|t-x|\ge\delta_1$.

Therefore, by Lemma 4 and the Cauchy-Schwarz inequality, we have

$$I\equiv B_{n,\alpha}^{(r)}((1-\psi(t))(f(t)-f_{2,\delta}(t),x)$$

and

$$|I|\le\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\frac{|Q_{i,j,r,\alpha}(x)|}{\{x(1+\alpha x)\}^r}\sum_{k=0}^{\infty}p_{n,k,\alpha}(x)|k-nx|^j \times\int_{|t-x|>\delta_1}b_{n,k,\alpha}(t)(1-\psi(t))|f(t)-f_{2,\delta}(t)|\,dt.$$

Thus, by property (ii) of $f_{2,\delta}$, we have

$$\begin{aligned}|I|&\le M||f||_{C[a_1,b_1]}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\sum_{k=0}^{\infty}p_{n,k,\alpha}(x)|k-nx|^j\int_{|t-x|\ge\delta_1}b_{n,k,\alpha}(t)\,dt\\
&\le M||f||_\gamma\delta_1^{-2m}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\sum_{k=0}^{\infty}p_{n,k,\alpha}(x)|k-nx|^j\left(\int_0^\infty b_{n,k,\alpha}(t)dt\right)^{1/2}\\
&\qquad\times\left(\int_0^\infty b_{n,k,\alpha}(t)(t-x)^{4m}\,dt\right)^{1/2}\\
&\le M||f||_\gamma\delta_1^{-2m}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i\left\{\sum_{k=0}^{\infty}p_{n,k,\alpha}(x)(k-nx)^{2j}\right\}^{1/2}\\
&\qquad\times\left\{\sum_{k=0}^{\infty}p_{n,k,\alpha}(x)\int_0^\infty b_{n,k,\alpha}(t)(t-x)^{4m}\,dt\right\}^{1/2}.\end{aligned}$$

Hence, by using Lemmas 1 and 2, we have

$$|I| \leq M||f||_\gamma \delta_1^{-2m} O(n^{(i+(j/2)-m)}) \leq Mn^{-q}||f||_\gamma,$$

where $q = m-(r/2)$. Now, choosing $m > 0$ satisfying $q \geq 1$, we obtain $I \leq Mn^{-1}\|f\|_\gamma$. Therefore, by property (iii) of the function $f_{2,\delta}(t)$, we get

$$\begin{aligned} H_1 &\leq M||f^{(r)} - f_{2,\delta}^{(r)}||_{C[a^*,b^*]} + Mn^{-1}||f||_\gamma \\ &\leq M\omega_2(f^{(r)},\delta,a,b) + Mn^{-1}||f|||_\gamma. \end{aligned}$$

Finally, combining the estimates of $H_1 - H_3$, we get

$$\begin{aligned} \|B_{n,\alpha}^{(r)}(f,\bullet) - f^{(r)}(.)\|_{C[a_1,b_1]} \leq\; & M\omega_2(f^{(r)},\delta,a,b) \\ & + Mn^{-1}\{||f||_\gamma + \delta^{-2}\omega_2(f^{(r)},\delta,a,b)\} \\ & + M\omega_2(f^{(r)},\delta,a,b) + Mn^{-1}||f|||_\gamma, \end{aligned}$$

and choosing $\delta = n^{-1/2}$, we get the desired result. This completes the proof of the theorem. □

4. Inverse theorem. This section is devoted to the following inverse theorem in simultaneous approximation:

Theorem 10. *Let* $0 < \beta < 2$, $0 < a_1 < a_2 < b_2 < b_1 < \infty$, $f \in C_0^r$ *and* $f(t) = O(t^\gamma)$. *Then in the following statements* (i) ⇒ (ii)

(i) $||B_{n,\alpha}^{(r)}(f,\bullet) - f^{(r)}(.)||_{C[a_1,b_1]} = O(n^{-\beta/2})$

(ii) $f^{(r)} \in \mathrm{Lip}^*(\beta,a_2,b_2)$.

Proof. Let us choose a', a'', b', b'' in such a way that $a_1 < a' < a'' < a_2 < b_2 < b'' < b' < b_1$. Also, suppose $g \in C_0^\infty$ with $\operatorname{supp} g \in [a'',b'']$ and $g(x) = 1$ on the interval $[a_2,b_2]$. For $x \in [a',b']$ with $D \equiv d/dx$, we have

$$\begin{aligned} B_{n,\alpha}^{(r)}(fg,x) - (fg)^{(r)}(x) &= D^r(B_{n,\alpha}((fg)(t) - (fg)(x)),x) \\ &= D^r(B_{n,\alpha}(f(t)(g(t) - g(x)),x)) \\ &\quad + D^r(B_{n,\alpha}(g(x)(f(t) - f(x)),x)) \\ &=: J_1 + J_2. \end{aligned}$$

Using Leibniz's formula, we have

$$\begin{aligned}
J_1 &= \frac{d^r}{dx^r}\int_0^\infty W_{n,\alpha}(x,t)f(t)[g(t)-g(x)]\,dt\\
&= \sum_{i=0}^{r}\binom{r}{i}\int_0^\infty W_{n,\alpha}^{(i)}(x,t)\frac{\partial^{r-i}}{\partial x^{r-i}}[f(t)(g(t)-g(x))]\,dt\\
&= -\sum_{i=0}^{r-1}\binom{r}{i}g^{(r-i)}(x)B_{n,\alpha}^{(i)}(f,x)\\
&\quad+\int_0^\infty W_{n,\alpha}^{(r)}(x,t)f(t)(g(t)-g(x))\,dt\\
&=: J_3+J_4.
\end{aligned}$$

Applying Theorem 9, we have

$$J_3 = -\sum_{i=0}^{r-1}\binom{r}{i}g^{(r-i)}(x)f^{(i)}(x)+O(n^{-\beta/2}),$$

uniformly in $x \in [a', b']$. Applying Theorem 5, the Cauchy-Schwarz inequality, Taylor's expansions of f and g and Lemma 2, we are led to

$$\begin{aligned}
J_4 &= \sum_{i=0}^{r}\frac{g^{(i)}(x)f^{(r-i)}(x)}{i!(r-i)!}r!+o(n^{-1/2})\\
&= \sum_{i=0}^{r}\binom{r}{i}g^{(i)}(x)f^{(r-i)}(x)+o(n^{-\beta/2}),
\end{aligned}$$

uniformly in $x \in [a', b']$. Again using Leibniz's formula, we have

$$\begin{aligned}
J_2 &= \sum_{i=0}^{r}\binom{r}{i}\int_0^\infty W_{n,\alpha}^{(i)}(x,t)\frac{\partial^{r-i}}{\partial x^{r-i}}[g(t)(f(t)-f(x))]\,dt\\
&= \sum_{i=0}^{r}\binom{r}{i}g^{(r-i)}(x)B_{n\alpha}^{(i)}(f,x)-(fg)^{(r)}(x)\\
&= \sum_{i=0}^{r}\binom{r}{i}g^{(r-i)}(x)f^{(i)}(x)-(fg)^{(r)}(x)+o(n^{-\beta/2})\\
&= O(n^{-\beta/2}),
\end{aligned}$$

uniformly in $x \in [a', b']$. Combining the above estimates, we get

$$||B_{n,\alpha}^{(r)}(fg, \bullet) - (fg)^{(r)}||_{C[a',b']} = O(n^{-\beta/2}).$$

Thus, by Lemmas 4 and 6, we have $(fg)^{(r)} \in \mathrm{Lip}^*(\beta, a', b')$ and also $g(x) = 1$ on the interval $[a_2, b_2]$, which proves that $f^{(r)} \in \mathrm{Lip}^*(\beta, a_2, b_2)$. This completes the validity of the implication (i) $\Rightarrow$ (ii) for the case $0 < \beta \leq 1$. To prove the result for $1 < \beta < 2$ for any interval $[a^*, b^*] \subset (a_1, b_1)$, let a_2^*, b_2^* be such that $(a_2, b_2) \subset (a_2^*, b_2^*)$ and $(a_2^*, b_2^*) \subset (a_1^*, b_1^*)$. Let $\delta > 0$. We shall prove the assertion $\beta < 2$. From the previous case this implies that $f^{(r)}$ exists and belongs to $\mathrm{Lip}\,(1-\delta, a_1^*, b_1^*)$. Let $g \in C_0^\infty$ be such that $g(x) = 1$ on $[a_2, b_2]$ and $\mathrm{supp}\, g \subset (a_2^*, b_2^*)$. Then with $\chi_2(t)$ denoting the characteristic function of the interval $[a_1^*, b_1^*]$, we have

$$\begin{aligned} ||B_{n,\alpha}^{(r)}&(fg, \bullet) - (fg)^{(r)}||_{C[a_2^*,b_2^*]} \\ &\leq ||D^r[B_{n,\alpha}(g(.)(f(t) - f(.)), \bullet)]||_{C[a_2^*,b_2^*]} \\ &\quad + ||D^r[B_{n,\alpha}(f(t)(g(t) - g(.)), \bullet)]||_{C[a_2^*,b_2^*]} \\ &=: P_1 + P_2. \end{aligned}$$

To estimate P_1, by Theorem 9, we have

$$\begin{aligned} P_1 &= ||D^r[B_{n,\alpha}(g(.)(f(t), \bullet)] - (fg)^{(r)}||_{C[a_2^*,b_2^*]} \\ &= \left\| \sum_{i=0}^{r} \binom{r}{i} g^{(r-i)}(.) B_{n,\alpha}^{(i)}(f, \bullet) - (fg)^{(r)} \right\|_{C[a_2^*,b_2^*]} \\ &= \left\| \sum_{i=0}^{r} \binom{r}{i} g^{(r-i)}(.) f^{(i)} - (fg)^{(r)} \right\|_{C[a_2^*,b_2^*]} + O(n^{-\beta/2}) \\ &= O(n^{-\beta/2}). \end{aligned}$$

Also by Leibniz's formula and Theorem 5, have

$$\begin{aligned} P_2 &\leq \Bigg\| \sum_{i=0}^{r} \binom{r}{i} g^{(r-i)}(.) B_{n,\alpha}(f, \bullet) \\ &\qquad + B_{n,\alpha}^{(r)}(f(t)(g(t) - g(.))\chi_2(t), \bullet) \Bigg\|_{C[a_2^*,b_2^*]} + O(n^{-1}) \\ &=: ||P_3 + P_4||_{C[a_2^*,b_2^*]} + O(n^{-1}). \end{aligned}$$

Then by Theorem 9, we have

$$P_3 = \sum_{i=0}^{r-1} \binom{r}{i} g^{(r-i)}(x) f^{(i)}(x) + O(n^{-\beta/2}),$$

uniformly in $x \in [a_2^*, b_2^*]$. Applying Taylor's expansion of f, we have

$$\begin{aligned} P_4 &= \int_{i=0}^{\infty} W_{n,\alpha}^{(r)}(x,t)[f(t)(g(t)-g(x))\chi_2(t)\,dt \\ &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^{\infty} W_{n,\alpha}^{(r)}(x,t)(t-x)^i(g(t)-g(x))\,dt \\ &\quad + \int_0^{\infty} W_{n,\alpha}^{(r)}(x,t) \frac{(f^{(r)}(\xi)-f^{(r)}(x))}{r!}(t-x)^r(g(t) \\ &\qquad\qquad - g(x))\chi_2(t)\,dt, \end{aligned}$$

where ξ lies between t and x. Next by Theorem 9, the first term in the above expression is given by

$$\sum_{m=0}^{r} \binom{r}{m} g^{(m)} f^{(r-m)}(x) + O(n^{-\beta/2}),$$

uniformly in $x \in [a_2^*, b_2^*]$. Also by the mean value theorem and using Lemma 4, we can obtain the second term as follows:

$$\begin{aligned} &\left\| \int_0^{\infty} W_{n,\alpha}^{(r)}(x,t) \frac{(f^{(r)}(\xi)-f^{(r)}(x))}{r!}(t-x)^r(g(t)-g(x))\chi_2(t)\,dt \right\|_{C[a_2^*,b_2^*]} \\ &\leq \sum_{\substack{2m+s\leq r \\ m,s\geq 0}} n^{m+s} \left\| \frac{|Q_{m,s,r,\alpha}(x)|}{x(1+\alpha x)}^r \int_0^{\infty} W_{n,\alpha}(x,t)|t-x|^{\delta+r+1} \right. \\ &\qquad\qquad \left. \times \frac{|f^{(r)}(\xi)-f^{(r)}(x)|}{r!}|g'(\eta)|\chi_2(t)\,dt \right\|_{C[a_2^*,b_2^*]} \\ &= O(n^{-\delta/2}), \end{aligned}$$

choosing δ such that $0 \leq \delta \leq 2-\beta$. Combining the above estimates, we get

$$||B_{n,\alpha}^{(r)}(fg,\bullet) - (fg)^{(r)}||_{C[a_2^*,b_2^*]} = O(n^{-\beta/2}).$$

Since supp $fg \subset (a_2^*, b_2^*)$, it follows from Lemmas 4 and 6 that $(fg)^{(r)} \in \mathrm{Lip}^*(\beta, a_2^*, b_2^*)$. Since $g(x) = 1$ on $[a_2, b_2]$, we have $f^{(r)} \in \mathrm{Lip}^*(\beta, a_2, b_2)$. This completes the proof of the theorem. $\square$

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