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CYCLIC COVERS OF RATIONAL ELLIPTIC SURFACES

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ABSTRACT. We compute the maximal rank of a cyclic cover for a class of rational elliptic surfaces.

1. Introduction. Let $\pi_1 : E_1 \to \mathbf{P}^1$ be a smooth complex relatively minimal nonisotrivial elliptic surface with section, and consider the map $\mathbf{P}^1 \to \mathbf{P}^1$ defined by $t \to t^r$. Define $\pi_r : E_r \to \mathbf{P}^1$ to be the minimal compactification of the Neron model of the generic fiber of $E \times_{\mathbf{P}^1} \mathbf{P}^1$.

For $t \in \mathbf{P}^1$, let E_1^t be the fiber of E_1 over t with conductor f_t and Euler characteristic e_t. If the fiber is of type I_n or I_n^*, let $n_t = n$ and set $n_t = 0$ otherwise. In [**2**] we give a bound for the rank of E_r if

$$\gamma = \sum_{t \neq 0, \infty} (f_t - e_t/6) - \frac{n_0 + n_\infty}{6} < 1.$$

However, this bound is far from sharp.

Persson lists all 287 possible configurations of singular fibers on a rational elliptic surface [**3**]. Thirty-eight of these have $\gamma < 1$. When $\gamma = 0$, either E_1 is semi-stable or all fibers are of type I_n or I_n^*. We have already shown [**2**] that in the nine cases where $\gamma = 0$, E_r is extremal for all r and thus E_r has rank 0 for all r.

In this paper, we consider the remaining 29 cases where E_1 is a rational elliptic surface with $0 < \gamma < 1$ and compute the rank of E_r in most of these and significantly improve the bound given in [**2**] in the rest.

We will see that our bounds depend only on the fibers at $t = 0$ and $t = \infty$. Because of this, our bounds hold for all (not necessarily rational) elliptic surfaces with $\gamma < 1$ and the given fiber types at 0 and ∞.

2. Preliminary results. Unless otherwise noted, proofs of the results in this section can be found in [**2**, Section 1]. For any elliptic surface $\pi : E \to \mathbf{P}^1$,

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$$\operatorname{Rank} E = \dim H^1(\mathbf{P}^1, R^1\pi_*\mathbf{Q}) \cap H^{1,1}(E, \mathbf{C}) \quad (\text{see } [\mathbf{1}]).$$

Define $H_r^{1,1} = H^1(\mathbf{P}^1, R^1\pi_{r*}\mathbf{C}) \cap H^{1,1}(E_r, \mathbf{C})$.

Let σ_r be the automorphism of E_r lifted from the automorphism of $\mathbf{P}^1$ defined by $t \to \zeta_r t$ where $\zeta_r = e^{2\pi i/r}$. Then σ_r acts on $H^1(\mathbf{P}^1, R^1\pi_{r*}\mathbf{C})$, all eigenvalues of σ_r on $H_r^{1,1}$ are of the form ζ_r^n and when $\gamma < 1$, no value ζ_r^n appears more than twice. Consider the set S_r of eigenvalues on $H_r^{1,1}$, ζ, such that if ζ is a primitive dth root of unity, all primitive dth roots of unity are also eigenvalues on $H_r^{1,1}$, counting multiplicity. We have shown that the dimension of $H^1(\mathbf{P}^1, R^1\pi_{r*}\mathbf{Q}) \cap H^{1,1}(E, \mathbf{C})$ and hence the rank of E_r is bounded by $\#S_r$ and if $\gamma < 1$, $\#S_r$ depends only on E_1 and is independent of r. Further, we get equality if there is only one singular fiber on E_1 away from $t = 0, \infty$ and that fiber is of multiplicative type.

For $t = 0, \infty$ we have $\operatorname{tr}(\sigma_r, N_{r,t}) = \zeta_r^{a_t}$ for some integer a_t where N_r is the pushforward to the normal bundle to a section of π_r and $N_{r,t}$ is its stalk at t. We show that if we remove the eigenvalues of σ_r on $H^1(E_r^0, \mathbf{C})$ and $H^1(E_r^\infty, \mathbf{C})$ from the set

$$\zeta_r^{a_0}, \ldots, \zeta_r^{-a_0}, \zeta_r^{-a_\infty}, \ldots, \zeta_r^{a_\infty}$$

we are left with S_r. Note that some values may appear more than once in the list above.

Now $\operatorname{tr}(\sigma_r, N_{r,t})$ is just the conjugate of the action of σ_r on E_r^t for $t = 0, \infty$. So to compute a_0 and a_∞ we need only understand how σ_r acts on the fibers of a smooth point at $t = 0$ and $t = \infty$.

3. Local computations. In this section we compute a_t and the eigenvalues of σ_r on $H^1(E_r^t, \mathbf{C})$ for $t = 0, \infty$. These depend only on the local minimal Weierstrass equation for E_1 and, in particular, on the fiber types of E_1^0, E_1^∞. Because none of our examples have smooth fibers or fibers of type I_0^* at $t = 0, \infty$, in what follows we assume that $n > 0$ for fibers of type I_n, I_n^*. We begin by computing a_0 for each of the eight possible fiber types.

Assume that a local minimal Weierstrass equation for E_1 at $t = 0$ is

$$y^2 = x^3 + A(t)t^l x + B(t)t^m$$

where $A(t), B(t)$ are polynomials with $A(0), B(0) \neq 0$ and the values of l and m are given in Table 1. The discriminant $\Delta(t) = 4A(t)^3 + 27B(t)^2$ at $t = 0$ vanishes to order n for I_n fibers and $n + 6$ for fibers of type I_n^*. We rewrite r as in Table 1. An equation for E_r is then

$$y^2 = x^3 + A(t^r)t^{rl}x + B(t^r)t^{rm},$$

and a local minimal equation at $t = 0$ is

$$Y^2 = X^3 + A(t^r)t^{l_r}X + B(t^r)t^{m_r},$$

where $(X, Y) = (x/t^{p_r}, y/t^{q_r})$ and the values of p_r, q_r are in Table 1. We will not need to know the values of l_r, m_r.

We now determine how σ_r acts at a smooth point of E_r^0, the fiber of E_r at $t = 0$. The origin $(0, 1, 0)$ is smooth for all r with local affine coordinates $(X/Y, Z/Y)$ and local parameter X/Y. Then,

$$\begin{aligned} \sigma_r(X, Y, Z) &= \sigma_r(x/t^{p_r}, y/t^{q_r}, z) = (\zeta_r^{-p_r}X, \zeta_r^{-q_r}Y, Z) \\ &= (\zeta_r^{q_r - p_r}X, Y, \zeta_r^{q_r}Z) \end{aligned}$$

so that σ_r acts on E_r^0 as multiplication by $\zeta_r^{q_r - p_r}$ and $a_0 = r - (q_r - p_r)$. These results are summarized in Table 1. Note that $a_\infty = r - a_0$.

Finally, we need to compute the eigenvalues of σ_r on $H^1(E_r^t, \mathbf{C})$ for $t = 0, \infty$. Since $H^1(E_r^t, \mathbf{C})$ is zero-dimensional when E_r^t is an additive fiber, we need only consider the case where the fiber is smooth or of multiplicative type.

For fibers of type II, III, IV, IV^*, III^* and II^* the fiber is of additive type when $s' \neq 0$ and smooth when $s' = 0$, so we assume that $s' = 0$. As in [**2**], the Lefschetz fixed point formula implies that

$$\operatorname{tr}(\sigma_r^i, H^1(E_r^t, \mathbf{C})) = 2 - \#\{\text{points fixed by } \sigma_r^i\}.$$

TABLE 1. The values of a_0.

Fiber Type	r	l, m	p_r, q_r	a_0
II	$r = 6s + s'$ $0 \le s' \le 5$	$l \ge 1$ $m = 1$	$p_r = 2s,\ q_r = 3s$	$5s + s'$
III	$r = 4s + s'$ $0 \le s' \le 3$	$l = 1$ $m \ge 2$	$p_r = 2s,\ q_r = 3s$	$3s + s'$
IV	$r = 3s + s'$ $0 \le s' \le 2$	$l \ge 2$ $m = 2$	$p_r = 2s,\ q_r = 3s$	$2s + s'$
IV^*	$r = 3s + s'$ $0 \le s' \le 2$	$l \ge 3$ $m = 4$	$s' = 0$: $p_r = 4s,\ q_r = 6s$ $s' > 0$: $p_r = 4s + 2(s'-1)$ $q_r = 6s + 3(s'-1)$	s $s+1$
III^*	$r = 4s + s'$ $0 \le s' \le 3$	$l = 3$ $m \ge 5$	$s' = 0$: $p_r = 6s,\ q_r = 9s$ $s' > 0$: $p_r = 6s + 2(s'-1)$ $q_r = 9s + 3(s'-1)$	s $s+1$
II^*	$r = 6s + s'$ $0 \le s' \le 5$	$l \ge 4$ $m = 5$	$s' = 0$: $p_r = 10s,\ q_r = 15s$ $s' > 0$: $p_r = 10s + 2(s'-1)$ $q_r = 15s + 3(s'-1)$	s $s+1$
I_n	$r = s$	$l = 0$ $m = 0$	$p_r = 0,\ q_r = 0$	0
I_n^*	$r = 2s + s'$ $0 \le s' \le 1$	$l = 2$ $m = 3$	$p_r = 2s,\ q_r = 3s$	$s + s'$

When E_1^t is of type II or II^* and $r = 6s$, σ_r has order 6, and σ_r^i has 1, 3, 4, 3, 1 fixed points respectively for $i = 1, \dots, 5$. It follows that the eigenvalues of σ_r on $H^1(E_r^t, \mathbf{C})$ are ζ^s, ζ^{-s}. Similarly, when E_1^t is of type III or III^* and $r = 4s$, σ_r has order 4 and the eigenvalues of σ_r on $H^1(E_r^t, \mathbf{C})$ are ζ^s, ζ^{-s}. For fibers of type IV or IV^* and $r = 3s$, σ_r has order 3 and the eigenvalues of σ_r are again ζ^s, ζ^{-s}.

For fibers of type I_n, $n > 0$, $H^1(E_r^t, \mathbf{C})$ is one-dimensional and σ_r fixes the curve for all r so that 1 is the only eigenvalue of σ_r on $H^1(E_r^t, \mathbf{C})$. Finally, for fibers of type I_n^*, the fiber is of type I_{rn} when $r = 2s$ and σ_r acts as multiplication by -1, so that the eigenvalue of σ_r on $H^1(E_r^t, \mathbf{C})$ is -1.

TABLE 2. Eigenvalues of σ_r on $H^1(E_r^t, \mathbf{C})$.

Fiber Type	r	Order of σ_r	Eigenvalues of σ on $H^1(E_r^t, \mathbf{C})$
II, II^*	$6s$	6	ζ^s, ζ^{-s}
III, III^*	$4s$	4	ζ^s, ζ^{-s}
III, III^*	$3s$	3	ζ^s, ζ^{-s}
I_n^*, $n > 0$	$2s$	2	-1
I_n, $n > 0$	s	1	1

4. The set S_r. From Persson's list, there are 14 possible combinations of fiber types for E_1^0 and E_1^∞. We will compute the set $S = \cup S_r$ for each choice. Write $r = 12s + s'$ with $0 \le s' \le 11$ and define U to be the smallest open interval or union of two open intervals such that, for all r, if $e^{i\theta}$ is an eigenvalue of σ_r on $H_r^{1,1}$, $\theta \in U$, counting multiplicities.

Once we have found U, we can find S as follows: For each positive integer d, the set of primitive dth roots of unity is in S if and only if the argument of each primitive dth root of unity is in U. The maximum rank of E_r is now easily computed:

$$\operatorname{Rank} E_r \le \sum_{\substack{d|r \\ \zeta_d \in S}} \phi(d). \tag{1}$$

If two sets of primitive dth roots of unity are in S, each contributes to the bound on the rank and is counted separately. Let

$$k = \# \left\{ \begin{matrix} \text{Multiplicative fibers} \\ \text{over } \mathbf{P}^1 \setminus \{0, \infty\} \end{matrix} \right\} + 2 \cdot \# \left\{ \begin{matrix} \text{Additive fibers} \\ \text{over } \mathbf{P}^1 \setminus \{0, \infty\} \end{matrix} \right\}.$$

It follows from the proof given in [**2**] that when $k = 1$, (1) is an equality. For rational elliptic surfaces, γ can only be less than 1 when $k \le 2$. We will see in the next section that we get equality in many of those cases where $k = 2$ and will also give an example of a rational elliptic surface where the rank of E_r is less than the bound given by (1).

Using the local computations in Section 3, for each possible choice of E_1^0 and E_1^∞, we find U, the values of d for which $\zeta_d \in S$ and the maximum rank of E_r. The results are given in Table 3. We also list the value for k. Note that for configuration N, each eigenvalue appears twice, so the maximum rank is 2, not 1. For configuration L, the two intervals overlap, so that the eigenvalue $\zeta^0 = 1$ may appear twice and the maximum rank of E_r is 4.

TABLE 3. The rank of E_r.

Configuration	E_1^0	E_1^∞	k	U	d	Maximum Rank of E_r
A	II^*	I_n	1	$(\pi/3, 5\pi/3)$	2,3,4,5	9
B	III^*	I_n	1	$(\pi/2, 3\pi/2)$	2,3	3
C	III^*	II	1	$(\pi/2, 3\pi/2) \cup (5\pi/3, \pi/3)$	1,2,3,7,8, 10,12,15, 18,20,42	56
D	IV^*	I_n	1	$(2\pi/3, 4\pi/3)$	2	1
E	IV^*	III	1	$(2\pi/3, 4\pi/3) \cup (3\pi/2, \pi/2)$	1,2,5,6,8, 9,12,14, 20,21,30	56
F	IV^*	II	1	$(2\pi/3, 4\pi/3) \cup (5\pi/3, \pi/3)$	1,2,8,12,20	18
G	I_n^*	IV	1	$(4\pi/3, 2\pi/3)$	1,4,6,10	9
H	I_n^*	III	1	$(3\pi/2, \pi/2)$	1,6	3
I	I_n^*	II	1	$(5\pi/3, \pi/3)$	1	1
J	IV	I_n	2	$(4\pi/3, 2\pi/3)$	1,4,6,10	9
K	III	I_n	2	$(3\pi/2, \pi/2)$	1,6	3
L	III	II	2	$(3\pi/2, \pi/2) \cup (5\pi/3, \pi/3)$	1,6	4
M	II	I_n	2	$(5\pi/3, \pi/3)$	1	1
N	II	II	2	$(5\pi/3, \pi/3)$	1	2

5. The rational elliptic surfaces. We now consider the 29 rational elliptic surfaces for which $0 < \gamma < 1$, or equivalently, for which [**2**, Theorem 1] applies. Each falls into one of the 14 categories A–N listed in Section 4. In Table 4, we list the singular fibers, with E_1^0 first, E_1^∞ second and the remaining singular fibers afterwards. A number of the surfaces have more than one configuration for which our theorem applies. We will only include those which have different values of a_0, a_∞. In particular, we may always interchange the placement of the I_n fibers for different values of n without changing the rank calculations. The surfaces are ordered as in Persson's paper. Finally, recall that for configurations A–I, the maximum rank of E_r is achieved. We will look more closely at the ranks for configurations J–N in the next section.

6. The surfaces with $k = 2$. Recall that $k = m + 2a$ where m is the number of multiplicative fibers and a is the number of additive fibers, not including the fibers at 0 and ∞. While the bounds given in Table 3 are not sharp when $k > 1$, we can often get more information about the rank of E_r.

Configuration J. Consider the surface E_1 with configuration J and fibers I_6, IV, I_1, I_1 (number 18 in Table 4). E_1 has rank 1 and is the double cover ramified over 0 and ∞ of a surface E_1' with singular fibers I_3, IV^*, I_1 which has configuration D. Now $\text{Rank}\,(E_r) = \text{Rank}\,(E_{2r}') = 1$ for all r, which is smaller than the bound of 9 given in (1). The two other rational elliptic surfaces with configuration J also have rank 1, so for these $1 \le \text{Rank}\,(E_r) \le 9$.

Configurations K and L. The six surfaces E_1 with configuration K all have rank 1 so that $\text{Rank}\,(E_r) = 1$ if $6 \nmid r$ and (E_{6s}) has rank 1 or 3. Similarly, the four rational elliptic surfaces with configuration L have rank 2 so that $\text{Rank}\,(E_r) = 2$ if $6 \nmid r$ and $\text{Rank}\,(E_{6s}) = \text{Rank}\,(E_6) = 2$ or 4.

Configurations M and N. The eight surfaces, E_1, with configuration M all have rank 1, the maximum rank of E_r. Similarly, the four surfaces with configuration N have maximal rank 2 so (1) is an equality in these 12 cases.

The values of k given for configurations A–N in Table 3 apply only when E_1 is rational. For nonrational E_1 with $\gamma < 1$, $k \ge 2$, regardless of the configuration so that, without any additional information, (1) is always an inequality.

TABLE 4. The rational elliptic surfaces with $0 < \gamma < 1$.

No.	No. in Persson's List	E_1^0	E_1^∞	Other Singular Fibers	γ	Configuration	Rank E_1	Max Rank of E_r
1	2	II^*	I_1	I_1	2/3	A	0	9
2	6	III^*	I_2	I_1	1/2	B	0	3
3	7	III^*	II	I_1	5/6	C	1	56
		II	I_1	III^*	1/3	M	1	1
4	9	I_3^*	II	I_1	1/3	I	1	1
		II	I_1	I_3^*	1/3	M	1	1
5	12	I_8	II	I_1, I_1	1/3	M	1	1
6	16	IV^*	I_3	I_1	1/3	D	0	1
7	17	IV^*	III	I_1	5/6	E	1	56
		III	I_1	IV^*	1/2	K	1	3
8	18	IV^*	II	I_2	2/3	F	1	18
		II	I_2	IV^*	1/3	M	1	1
9	21	I_2^*	III	I_1	1/2	H	1	3
		III	I_1	I_2^*	1/2	K	1	3
10	24	II	II	I_2^*	2/3	N	2	2
11	27	I_7	III	I_1, I_1	1/2	K	1	3
12	28	I_7	II	I_2, I_1	1/3	M	1	1
13	30	II	II	I_7, I_1	2/3	N	2	2
14	34	I_1^*	IV	I_1	2/3	G	1	9
		IV	I_1	I_1^*	2/3	J	1	9
15	35	I_1^*	II	I_3	1/3	I	1	1
		I_3	II	I_1^*	1/3	M	1	1
16	37	I_1^*	III	I_2	1/2	H	1	3
		III	I_2	I_1^*	1/2	K	1	3
17	38	III	II	I_1^*	5/6	L	2	4
18	46	I_6	IV	I_1, I_1	2/3	J	1	9
19	49	I_6	III	I_2, I_1	1/2	K	1	3
20	50	III	II	I_6, I_1	5/6	L	2	4
21	54	II	II	I_6, I_2	2/3	N	2	2
22	82	I_5	II	I_4, I_1	1/3	M	1	1
23	84	I_5	IV	I_2, I_1	2/3	J	1	9
24	87	I_5	III	I_3, I_1	1/2	K	1	3
25	88	I_5	II	I_3, I_2	1/3	M	1	1
26	93	III	II	I_5, I_2	5/6	L	2	4
27	109	II	II	I_4, I_4	2/3	N	2	2
28	119	I_4	III	I_3, I_2	1/2	K	1	3
29	120	III	II	I_4, I_3	5/6	L	2	4

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