ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 6, 2009

A NOTE ON ARONSSON'S EQUATION

DANIEL DRUCKER AND STEPHEN A. WILLIAMS

ABSTRACT. This note gives reasons why the equation in the title is interesting, shows that constant multiples of solutions of the eikonal equation are solutions, and proves that, for other local solutions, curves on which the length of the gradient remains constant propagate with a normal velocity depending on curvature and "time." Under reasonable assumptions, this conforms to the setting used by Souganidis in [**10**] to study propagating fronts.

1. Introduction. This note considers C^2 solutions f, in some open set $\Omega \subseteq \mathbf{R}^2$, of the equation

$$(f_x)^2 f_{xx} + 2 f_x f_y f_{xy} + (f_y)^2 f_{yy} = 0. \tag{1}$$

As shown in [**8**], equation (1) is the condition under which each curve $\sigma(t)$ of steepest descent on the graph of f is asymptotic, i.e., its acceleration vector $\sigma''(t)$ remains tangent to the graph of f at each point on the curve $\sigma(t)$. Indeed, if $\sigma(t) = (x(t), y(t), f(x(t), y(t)))$, then to say that σ is a steepest descent curve means that its horizontal velocity (x', y') is a negative multiple of (f_x, f_y), the gradient of f. Thus, the condition that σ be asymptotic, namely,

$$(x')^2 f_{xx} + 2x'y' f_{xy} + (y')^2 f_{yy} = 0,$$

is equivalent to (1). Alternatively, (1) can be viewed as the requirement that at each point $P \in \Omega$ the second derivative of the values of f along the line through P in the direction of $\nabla f(P)$ is 0 at P, since the condition

2000 AMS *Mathematics subject classification.* Primary 35K55, Secondary 35A25.
Keywords and phrases. Eikonal equation, propagating front, curvature-dependent normal velocity, viscosity solutions, Aronsson, infinity Laplacian, steepest descent curve, asymptotic curve.
Received by the editors on February 27, 2006, and in revised form on April 23, 2007.
DOI:10.1216/RMJ-2009-39-6-1859

$$
\begin{aligned}
0 &= \frac{d^2}{dt^2}[f(P + t\,\nabla f(P))]\Big|_{t=0} \\
&= \begin{pmatrix} f_x(P) & f_y(P) \end{pmatrix} \begin{pmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{pmatrix} \begin{pmatrix} f_x(P) \\ f_y(P) \end{pmatrix},
\end{aligned}
$$

says that (1) holds at P.

Aronsson studied (1) in [**2, 3**] after discovering in [**1**] that, for a C^2 function f on an open set $\Omega \subseteq \mathbf{R}^2$, (1) is necessary and sufficient for f to have the property, for any bounded connected open set D with $\overline{D} \subseteq \Omega$, that its minimal Lipschitz constant on $\overline{D}$ is the same as its minimal Lipschitz constant on ∂D. Researchers who have continued in this direction have defined the "infinity Laplacian" operator Δ_∞ by

$$
\Delta_\infty f = \sum_{i,j=1}^{n} f_i f_j f_{ij},
$$

where each subscript indicates a partial derivative with respect to the appropriate variable (in $\mathbf{R}^n$), and then have studied the equation $\Delta_\infty f = 0$. For $n = 2$, this is (1). Reference [**6**] is a recent paper in this area that gives good new results and references to the literature. Reference [**4**], a noteworthy survey paper on the topic, is even more recent. Both [**6**] and [**4**] single out *viscosity solutions* as being the ones of interest. (It is assumed here that the reader knows the basic facts about viscosity solutions. For readers for whom this is not true, [**7**] is an excellent reference, suitable for beginning, intermediate or advanced users of viscosity solutions.) Research in this direction characteristically deals with boundary value problems. The present note deals instead with an initial-value problem.

It is clear that any linear function f is a solution of (1). One of the many remarkable results in [**2**] is that linear functions are the only C^2 solutions of (1) on all of $\mathbf{R}^2$. Since global solutions of (1) are well understood, we shift our attention to local solutions.

It is a rather complicated matter to find nontrivial solutions of $\Delta_\infty u = 0$. (See [**3**], for example.) This note shows that there are two large families of local solutions of (1), both of them surprising:

(i) Any constant multiple of a solution u of the eikonal equation $(u_x)^2 + (u_y)^2 = 1$ of geometric optics is a solution of (1).

(ii) In a region in which (i) does not hold and in which f is not constant, if f satisfies (1) and if we define $h = \sqrt{(f_x)^2 + (f_y)^2}$, then the level curves $h = h_0$ (for constants $h_0 \neq 0$) propagate with a curvature-dependent (and "time"-dependent) normal velocity, and from this family of level curves we can recover the associated solutions f of (1).

It is natural to ask whether these results extend to the three-dimensional case. Unfortunately, they do not.

2. Local solutions of equation (1).

Let's begin with some elementary observations. First, with $a = (f_x)^2$, $b = f_x f_y$ and $c = (f_y)^2$, the calculation $ac - b^2 = (f_x)^2(f_y)^2 - (f_x f_y)^2 \equiv 0$ shows that (1) is parabolic. (See [**5**, pages 163–164] for the definition of "parabolic" used here.) Second, a simple calculation shows that radial solutions of (1) are of the form $f(x,y) = A\sqrt{x^2+y^2} + B$ for constants A and B; the solutions are smooth for $(x,y) \neq (0,0)$. Lastly, separation of variables gives the solutions $f(x,y) = a[(x+b)^{4/3} - (y+c)^{4/3}] + d$ for any constants a, b, c and d; the solutions are smooth for $x \neq -b$ and $y \neq -c$. (See Example 3 in [**8**] for the details.)

Now let u be a solution of the eikonal equation $(u_x)^2 + (u_y)^2 = 1$. Differentiating with respect to x, respectively y, gives $2u_x u_{xx} + 2u_y u_{yx} = 0$, respectively $2u_x u_{xy} + 2u_y u_{yy} = 0$. Thus,

$$0 = u_x(u_x u_{xx} + u_y u_{yx}) + u_y(u_x u_{xy} + u_y u_{yy}),$$

so that u (hence any constant multiple of u) satisfies (1). Note that the linear functions and the radial solutions discussed above are constant multiples of eikonal solutions (or are constant).

If $f = Cu$, where u is an eikonal solution and C is constant, then by defining $h = \sqrt{(f_x)^2 + (f_y)^2}$, as we do in what follows, we obtain $h \equiv |C|$, so that $\nabla h \equiv \mathbf{0}$. Since solutions of the eikonal equation are well understood (for example, see [**9**, pages 40–43]), we focus our attention on local solutions satisfying $\nabla h \neq \mathbf{0}$. The following theorem also assumes that h is never 0. This was proved in Theorem 6 of [**2**] to be true on any connected open set on which f is not constant.

Theorem 2.1. *Let $\Omega \subseteq \mathbf{R}^2$ be a connected open set. Let f be a C^2 solution of (1) on Ω. Let $h = \sqrt{(f_x)^2 + (f_y)^2}$. Assume that h and $|\nabla h|$*

are never 0 *on* Ω. *Then, for any level curve* $h = h_0$ *in* Ω, *where* h_0 *is a constant, we have*

$$\kappa = -s\frac{\sqrt{(h_x)^2 + (h_y)^2}}{h} \tag{2}$$

at each point of that curve, where κ *is the signed curvature of the curve at that point and where* $s = 1$ *or* $s = -1$, *depending on the direction in which the curve is traced.*

Remark. Thus, if $\kappa > 0$ for the curve as traced, then $s = -1$. If the curve is traced in the reverse direction, then $\kappa < 0$ and $s = 1$.

Proof. Using (1), it is easy to check that $f_x h_x + f_y h_y = 0$ on Ω. Thus,

$$\frac{(f_x,\, f_y)}{h} = s\frac{(-h_y,\, h_x)}{\sqrt{(h_x)^2 + (h_y)^2}} \tag{3}$$

holds at all points of Ω, with either $s = 1$ or $s = -1$. Let (x, y) be a point of the level curve $h = h_0$. Then there is a steepest ascent curve on the graph of f whose projection $\alpha(t) = (x(t), y(t))$ onto the xy-plane passes through (x, y). Since α is the projection of a steepest ascent curve, we can assume without loss of generality that $dx/dt = f_x(x(t),\, y(t))$ and $dy/dt = f_y(x(t),\, y(t))$. Then all the points of α lie on the level curve $h = h_0$ since $h(x, y) = h_0$ and

$$\frac{d}{dt}\,h(\alpha(t)) = h_x\frac{dx}{dt} + h_y\frac{dy}{dt} = f_x h_x + f_y h_y = 0.$$

Thus, by using (3), we see that along $h = h_0$ we have

$$\begin{aligned}
\kappa(t) &= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[(\dot{x})^2 + (\dot{y})^2]^{3/2}} = \frac{f_x[f_{yx}f_x + f_{yy}f_y] - f_y[f_{xx}f_x + f_{xy}f_y]}{[(f_x)^2 + (f_y)^2]^{3/2}} \\
&= \frac{f_x[hh_y] - f_y[hh_x]}{h^3} \\
&= \frac{f_x[-sf_x\sqrt{(h_x)^2 + (h_y)^2}] - f_y[sf_y\sqrt{(h_x)^2 + (h_y)^2}]}{h^3} \\
&= -s\sqrt{(h_x)^2 + (h_y)^2}\,\frac{[(f_x)^2 + (f_y)^2]}{h^3} = -s\frac{\sqrt{(h_x)^2 + (h_y)^2}}{h}. \qquad \square
\end{aligned}$$

Suppose that the hypotheses of the preceding theorem are satisfied. For some particular $h_0 \neq 0$, let $h = h_0$ be our starting curve at time $t = 0$. For any $t > 0$ with $t < h_0$, let $h = h_0 - t$ be the new curve at that time. Then the normal velocity V of the propagating front (in the unit normal direction $\mathbf{n}$ opposite to the direction of $(h_x,\, h_y)$) is $V = 1/\sqrt{(h_x)^2 + (h_y)^2}$. (To see this, pick any point P on the level curve of h at time t, and let $(x(s),\, y(s))$ parametrize the normal line of the level curve at P so that s is arclength, $(x(0),\, y(0)) = P$, and $(x'(s),\, y'(s))$ points opposite to the direction of $(h_x(P),\, h_y(P))$. Let $\Delta s = s - 0$ and $\Delta h = h(x(s), y(s)) - h(P)$. Then, for small Δs, $-\Delta t = \Delta h \approx -s\sqrt{(h_x(P))^2 + (h_y(P))^2}$, so $\Delta s/\Delta t \approx 1/\sqrt{(h_x(P))^2 + (h_y(P))^2}$, with the approximation becoming better and better as $\Delta t \to 0$.) From (2) above, we see that $V = -s/(h\kappa) = -s/[(h_0 - t)\kappa]$. (A rigorous derivation of the form of V can be given, but the informal argument given here is likely easier to understand.) Thus, V is a curvature- (and time-)dependent normal velocity.

We are now ready to use the apparatus of [**10**] to investigate the time evolution of these curves. From (3) we see that $\mathbf{n} = -\nabla h/|\nabla h| = (-sf_y/h,\, sf_x/h)$ along the propagating curve. Let $D\mathbf{n}$ be the 2 by 2 matrix whose first row is the gradient of $-sf_y/h$ and whose second row is the gradient of sf_x/h. A simple calculation shows that the trace of $D\mathbf{n}$ is $-s\kappa$ along the propagating curve. Thus, (1.1) of [**10**] holds with

$$v(D\mathbf{n}, \mathbf{n}, \mathbf{x}, t) = \frac{(-s)^2}{(h_0 - t)\operatorname{tr} D\mathbf{n}} = \frac{1}{(h_0 - t)\operatorname{tr} D\mathbf{n}}.$$

Following the formulas of [**10**], we obtain, with

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \neq \mathbf{0}$$

and

$$\mathbf{p} \otimes \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} (\, p_1 \quad p_2 \,) = \begin{pmatrix} p_1p_1 & p_1p_2 \\ p_2p_1 & p_2p_2 \end{pmatrix},$$

that

$$F(X, \mathbf{p}, \mathbf{x}, t) = |\mathbf{p}| \frac{-1}{(h_0 - t)\left\{(1/|\mathbf{p}|)(x_{11} + x_{22}) - (1/|\mathbf{p}|^3)[(p_1)^2 x_{11} + 2p_1p_2x_{12} + (p_2)^2 x_{22}]\right\}}.$$

This function F is not acceptable. (For example, it is not defined when X is the zero matrix.) To see how to modify F, let's decide what to take for the initial condition function u_0 of [**10**]. To meet the requirements there, u_0 must be uniformly continuous on $\mathbf{R}^2$ and the starting curve $h = h_0$ must be the 0-level set of u_0, with $u_0 > 0$ on one side (the side with $\mathbf{n}$ as its exterior normal) and with $u_0 < 0$ on the other side. By Lemmas 1 and 2 in [**2**], the starting curve is locally of the form $y = \psi(x)$ with $\psi''(x) > 0$ and with ψ a C^∞ function (after some rotation of the axes). Make that rotation of the axes. (It is not hard to see that (1) still holds–with a new f, x and y–if the axes are rotated.) Thus, we can reasonably assume that our starting curve is of the form $y = \psi(x)$, where $\psi \in C^\infty(\mathbf{R})$, with ψ uniformly continuous and $\psi'' > 0$. (By first restricting its domain to a slightly smaller interval if necessary, the locally defined ψ above can be extended so as to satisfy these conditions.) Then the function u_0 defined by $u_0(x, y) = y - \psi(x)$ can be shown to satisfy all of the requirements above. A simple calculation shows that

$$\begin{aligned}
&\frac{1}{|\nabla u_0|}[(u_0)_{xx} + (u_0)_{yy}] \\
&\quad - \frac{1}{|\nabla u_0|^3}\Big\{[(u_0)_x]^2(u_0)_{xx} + 2(u_0)_x(u_0)_y(u_0)_{xy} + [(u_0)_y]^2(u_0)_{yy}\Big\} \\
&= \frac{-\psi''(x)}{\{1 + [\psi'(x)]^2\}^{3/2}} < 0.
\end{aligned}$$

Let $K \subset \mathbf{R}^2$ be a compact set containing a set in which we want our local solution to be defined. Let $\varepsilon > 0$ be chosen so that $\varepsilon < \psi''(x)/\{1 + [\psi'(x)]^2\}^{3/2}$ for all (x, y) in K. Define g_ε by $g_\varepsilon(r) = \min\{-\varepsilon, r\}$ for every $r \in \mathbf{R}$. Define G_δ for any $0 < \delta < h_0$ by $G_\delta(r) = \max\{\delta, r\}$ for every $r \in \mathbf{R}$. Then define

$$\begin{aligned}
&F_{\delta,\varepsilon}(X, \mathbf{p}, \mathbf{x}, t) \\
&\quad = |\mathbf{p}| \frac{-1}{G_\delta(h_0 - t)\, g_\varepsilon((1/|\mathbf{p}|)(x_{11} + x_{22}) - (1/|\mathbf{p}|^3)[(p_1)^2 x_{11} + 2p_1p_2x_{12} + (p_2)^2 x_{22}])}.
\end{aligned}$$

Note that $F_{\delta,\varepsilon}$ agrees with F whenever the argument of g_ε is $\leq -\varepsilon$ and $0 \leq t < h_0 - \delta$. Note also that $F_{\delta,\varepsilon}$ depends on *all* of the above: u_0, ψ, K, ε and δ.

Now let $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{pmatrix}$ with $X \geq Y$, i.e., with $X - Y$ nonnegative definite. Then we clearly have

$$\begin{pmatrix} \frac{p_2}{|\mathbf{p}|} & \frac{-p_1}{|\mathbf{p}|} \end{pmatrix} \begin{pmatrix} x_{11} - y_{11} & x_{12} - y_{12} \\ x_{12} - y_{12} & x_{22} - y_{22} \end{pmatrix} \begin{pmatrix} \frac{p_2}{|\mathbf{p}|} \\ \frac{-p_1}{|\mathbf{p}|} \end{pmatrix} \geq 0,$$

so

$$\begin{aligned} (4) \quad & y_{11} + y_{22} - \frac{1}{|\mathbf{p}|^2}[(p_1)^2 y_{11} + 2p_1p_2y_{12} + (p_2)^2 y_{22}] \\ & \qquad \leq x_{11} + x_{22} - \frac{1}{|\mathbf{p}|^2}[(p_1)^2 x_{11} + 2p_1p_2x_{12} + (p_2)^2 x_{22}]. \end{aligned}$$

Dividing both sides of (4) by $|\mathbf{p}|$, applying g_ε to both sides of the resulting inequality (which preserves the direction of the inequality since g_ε is nondecreasing), and using the fact that all the values of g_ε are negative, it is easy to verify (1.8) in [**10**] for $F_{\delta,\varepsilon}$; that is,

$$F_{\delta,\varepsilon}(X, \mathbf{p}, \mathbf{x}, t) \geq F_{\delta,\varepsilon}(Y, \mathbf{p}, \mathbf{x}, t) \quad \text{if} \quad X \geq Y.$$

$F_{\delta,\varepsilon}$ also satisfies (1.7) of [**10**], once one corrects a misprint by inserting $\lambda\mathbf{p}$ as the second of four arguments on the left-hand side. Explicitly, (1.7) of [**10**] for $F_{\delta,\varepsilon}$ should read

$$\begin{gathered} F_{\delta,\varepsilon}(\lambda X + \mu(\mathbf{p} \otimes \mathbf{p}), \lambda\mathbf{p}, \mathbf{x}, t) = \lambda F_{\delta,\varepsilon}(X, \mathbf{p}, \mathbf{x}, t) \\ \text{for all} \quad \lambda > 0 \quad \text{and} \quad \mu \in \mathbf{R}. \end{gathered}$$

When $\mathbf{p} = \mathbf{0}$, define $F_{\delta,\varepsilon}(X, \mathbf{p}, \mathbf{x}, t) = 0$. Then $F_{\delta,\varepsilon}$ is continuous everywhere (the proof is left to the reader), so (1.10) of [**10**] is easily seen to hold, i.e.,

$$F^*_{\delta,\varepsilon}(0, 0, \mathbf{x}, t) = (F_{\delta,\varepsilon})_*(0, 0, \mathbf{x}, t) \quad \text{for} \quad (x, t) \in \mathbf{R}^2 \times (0, \infty),$$

where (see, for example, (4.1) in [**7**]), for any function u, the expressions u^* and u_* denote its upper and lower semi-continuous envelopes, respectively. ($u = u^* = u_*$ when u is continuous, as it is here.) Using Proposition 1.2 of [**10**], we obtain the unique viscosity solution of the equation $u_t = F_{\delta,\varepsilon}(D^2u, Du, \mathbf{x}, t)$ subject to the initial condition $u(\mathbf{x}, 0) = u_0(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^2$. Then we throw away any portions of the viscosity solution leading to arguments r of g_ε where the derivatives involved are not classical derivatives or where $g_\varepsilon(r) = -\varepsilon$. (The reader should be warned that no results are known to the authors which guarantee that the viscosity solution has continuous second-order derivatives, even locally.) The propagating front at time $t > 0$ with $t < h_0 - \delta$ is then $\{\mathbf{x} : u(\mathbf{x}, t) = 0\}$, i.e., the 0-level set at that time.

Let us now consider the question of how to get our solutions f, once we know what the $h =$ constant curves are in some open set Ω. Since $f_x h_x + f_y h_y = 0$, ∇f is tangent at any point of a level curve $h = h_0$ of h. Since $h = |\nabla f| = h_0$ on this level curve, f is a linear function of arclength along the curve. Consider a family of disjoint curves orthogonal to the $h =$ constant curves. Since ∇f is everywhere tangent to the level curves of h, clearly f is constant along curves of the orthogonal family. With this in mind, we can now understand how to get our solutions f to (1) from the family $h =$ constant. Pick a particular curve $h = h_0$. Define a function g along this curve so that g is a nonconstant linear function of arclength s along this curve. Then extend these values of g off the curve $h = h_0$ by keeping its values constant along curves of the orthogonal family. (If g is now defined at more than one point of a new curve $h = h_1$, we can extend it to be defined on all of $h = h_1$ by making it a linear function of arclength along $h = h_1$. These new values of g can then be extended off $h = h_1$ as before. This process can obviously be iterated.) Finally, $f = Ag + B$ for some constants $A \neq 0$ and B. (To see this, note that since f and g are both linear functions of arclength s on the level curve $h = h_0$ and g is nonconstant, there are constants $A \neq 0$ and B such that $f = Ag + B$ on this curve. Since the values of both f and g remain constant along curves of the orthogonal family, $f = Ag + B$ remains true for the first extension off the curve $h = h_0$. Simple considerations show that $f = Ag + B$ remains true for later extensions as well.) In fact, for any constants $C \neq 0$ and D, the function $\tilde{f} = Cg + D$ will also satisfy (1), and the function $\tilde{h}$ derived from $\tilde{f}$ will have the same level curves that h does.

REFERENCES

1. G. Aronsson, *Extension of functions satisfying Lipschitz conditions*, Ark. Mat. **6** (1967), 551–561.

2. ———, *On the partial differential equation* $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, Ark. Mat. **7** (1968), 395–425.

3. G. Aronsson, *On certain singular solutions of the partial differential equation* $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, Manuscripta Math. **47** (1984), 133–151.

4. G. Aronsson, M.G. Crandall and P. Juutinen, *A tour of the theory of absolutely minimizing functions*, Bull. Amer. Math. Soc. **41** (2004), 439–505 (electronic).

5. R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. II, Interscience, New York, 1962.

6. M.G. Crandall, L.C. Evans and R.F. Gariepy, *Optimal Lipschitz extensions and the infinity Laplacian*, Calculus Variations **13** (2001), 123–139.

7. M.G. Crandall, H. Ishii and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1–67.

8. D. Drucker and S.A. Williams, *When does water find the shortest path downhill? The geometry of steepest descent curves*, Amer. Math. Monthly **110** (2003), 869–885.

9. P.R. Garabedian, *Partial differential equations*, Wiley, New York, 1964.

10. P.E. Souganidis, *Front propagation: theory and applications*, in *Viscosity solutions and applications*, Lecture Notes Math. **1660**, Springer, Berlin, (1997), 186–242.

Department of Mathematics, Wayne State University, Detroit, MI 48202
Email address: drucker@math.wayne.edu

Department of Mathematics, Wayne State University, Detroit, MI 48202
Email address: willistep@gmail.com