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THE LITTLEWOOD-ORLICZ OPERATOR IDEAL

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ABSTRACT. In this paper, we show that each continuous linear operator from an $\mathcal{L}_2$-space to an $\mathcal{L}_\infty$-space is Littlewood-Orlicz, and each Littlewood-Orlicz operator from a Banach space to an $\mathcal{L}_2$-space is 2-summing. As a consequence, Littlewood-Orlicz operators from a Banach space with cotype 2 to an $\mathcal{L}_2$-space coincide with 2-summing operators.

1. Introduction.

The classes of p-summing operators ($1 \leq p < \infty$) were introduced by Pietsch [**16**] in 1967. Actually, the class of 1-summing operators was studied before in Grothendieck's Résumé [**9**] in 1953. Later, Mityagin and Pelczynski [**13**] and Kwapien [**11**] studied the classes of (q,p)-summing operators, i.e., operators that take weakly p-summable sequences to absolutely q-summable sequences. Cohen [**4**] and Apiola [**1**] studied the classes of 'Cohen's (q,p)-summing operators,' i.e., operators that take weakly p-summable sequences to strongly q-summable sequences. In 1980, Piestch in his monograph [**17**] introduced the classes of (r,q,p)-summing operators. In particular, $(1,q,p)$-summing operators coincide with 'Cohen's (q,p)-summing operators.' Now, with the help of sequential representations of $\ell_p \widehat{\otimes} X$ given by Bu and Diestel [**3**], $(1,q,p)$-summing operators from a Banach space X to a Banach space Y are nothing else but operators that take members in $\ell_p \check{\otimes} X$, the injective tensor product of ℓ_p and X, to members in $\ell_q \widehat{\otimes} Y$, the projective tensor product of ℓ_q and Y.

Bu in [**2**] used $(1,1,2)$-summing operators (called Littlewood-Orlicz operators in [**2**]) to characterize G.T. spaces with cotype 2. In this paper, we will give some properties of Littlewood-Orlicz operators between $\mathcal{L}_p$-spaces. The reasons why we are interested in such operators and why we called them Littlewood-Orlicz operators are as follows.

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Littlewood's inequality [**12**]. *There is a constant K such that, for any finite $n \times n$ scalar matrix (a_{ij}),*

$$\sum_{i=1}^n \bigg(\sum_{j=1}^n |a_{ij}|^2\bigg)^{1/2} \le K \cdot \max\bigg\{\bigg|\sum_{i,j=1}^n a_{ij}s_it_j\bigg| : |s_i| \le 1, |t_j| \le 1\bigg\}.$$

Orlicz theorem [**14**]. *If $\sum_n f_n$ is an unconditionally convergent series in $L_1[0,1]$, then $\sum_{n=1}^\infty \|f_n\|^2 < \infty$.*

With the help of sequential representations of projective and injective tensor products of ℓ_p and X (see [**3**] and [**5**, page 90]), we can formulate Littlewood's inequality such that if $X = \ell_1$, then $\ell_1 \check{\otimes} X \subseteq \ell_2 \widehat{\otimes} X$ and formulate Orlicz's theorem such that if $X = L_1[0,1]$, then $\ell_1 \check{\otimes} X \subseteq \ell_2^{\text{strong}}(X)$. Moreover, Grothendieck in his Résumé [**9**] showed that if X is an $\mathcal{L}_1$-space, then $\ell_1 \check{\otimes} X \subseteq \ell_2 \widehat{\otimes} X$ (for an exposition of this result, see [**7**]). Therefore, the identity operator on an $\mathcal{L}_1$-space X takes members in $\ell_1 \check{\otimes} X$ into members in $\ell_2 \widehat{\otimes} X$. Diestel called such operators between Banach spaces Littlewood-Orlicz operators.

2. Preliminaries. For any Banach space X, let X^* denote its dual and B_X denote its closed unit ball. Let $(e_n)_n$ denote the unit vector basis of ℓ_2. For $1 < p < \infty$, let p' denote its conjugate, i.e., $1/p + 1/p' = 1$. If $p = 1$, then let $\ell_{p'} = c_0$; and if $p = \infty$, then let $(\sum_{n=1}^\infty |t_n|^p)^{1/p} = \sup_{n \ge 1} |t_n|$.

Given a Banach space X and $1 \le p < \infty$, we denote (see [**8**, pages 32–36]) by $\ell_p^{\text{strong}}(X)$ the Banach space of all sequences $(x_n)_n$ in X such that $\sum_{n=1}^\infty \|x_n\|^p < \infty$ with the norm

$$\|(x_n)_n\|_{\ell_p^{\text{strong}}(X)} = \bigg(\sum_{n=1}^\infty \|x_n\|^p\bigg)^{1/p},$$

and by $\ell_p^{\text{weak}}(X)$ the Banach space of all sequences $(x_n)_n$ in X such that $\sum_{n=1}^\infty |x^*(x_n)|^p < \infty$ for each $x^* \in X^*$ with the norm

$$\|(x_n)_n\|_{\ell_p^{\text{weak}}(X)} = \sup\bigg\{\bigg(\sum_{n=1}^\infty |x^*(x_n)|^p\bigg)^{1/p} : x^* \in B_{X^*}\bigg\};$$

and by $\ell_p^{\text{weak},0}(X)$ the closed subspace of $\ell_p^{\text{weak}}(X)$ consisting of such sequences whose tails converge to zero, i.e.,

$$\begin{aligned}&\ell_p^{\text{weak},0}(X)\\&=\Big\{(x_n)_n\in\ell_p^{\text{weak}}(X):\lim_n\|(0,\dots,0,x_n,x_{n+1},\dots)\|_{\ell_p^{\text{weak}}(X)}=0\Big\}.\end{aligned}$$

For $1<p<\infty$, let $\ell_p\langle X\rangle$ denote the space of all sequences $(x_n)_n$ in X such that $\sum_{n=1}^\infty|x_n^*(x_n)|<\infty$ for each $(x_n^*)_n\in\ell_{p'}^{\text{weak}}(X^*)$, normed by

$$\|(x_n)_n\|_{\ell_p\langle X\rangle}=\sup\bigg\{\bigg|\sum_{n=1}^\infty x_n^*(x_n)\bigg|:\|(x_n^*)_n\|_{\ell_{p'}^{\text{weak}}(X^*)}\le 1\bigg\},$$

where p' is the conjugate of p. With this norm $\ell_p\langle X\rangle$ is a Banach space (see [**3, 4**]). For convenience, let $\ell_1\langle X\rangle:=\ell_1^{\text{strong}}(X)$. From the definitions, we have for $1\le p<\infty$,

$$\ell_p\langle X\rangle\subseteq\ \ell_p^{\text{strong}}(X)\ \subseteq\ell_p^{\text{weak},0}(X)\subseteq\ell_p^{\text{weak}}(X),$$

and

$$\|\cdot\|_{\ell_p^{\text{weak}}(X)}\le\|\cdot\|_{\ell_p^{\text{strong}}(X)}\le\|\cdot\|_{\ell_p\langle X\rangle}.$$

Moreover, by the Dvoretzky-Rogers theorem (see [**6**, page 61]), if X is an infinite-dimensional Banach space then $\ell_p\langle X\rangle\ne\ell_p^{\text{strong}}(X)$ for $1<p<\infty$ and $\ell_p^{\text{strong}}(X)\ne\ell_p^{\text{weak},0}(X)$ for $1\le p<\infty$.

For Banach spaces X and Y, let $X\widehat{\otimes}Y$ and $X\check{\otimes}Y$ denote the projective and the injective tensor product of X and Y, respectively. For $1\le p\le\infty$, define

$$\psi:\ell_p\otimes X\longrightarrow X^{\mathbf{N}}$$

by

$$\sum_{k=1}^n s^{(k)}\otimes y_k\longmapsto\bigg(\sum_{k=1}^n s_i^{(k)}y_k\bigg)_i.$$

Then ψ is a well-defined linear map. Combining results in [**5**, page 92], [**3, 10**], we have the following.

Proposition 1. *Let $1\le p<\infty$. Then, under the mapping ψ, $\ell_p\check{\otimes}X=\ell_p^{\text{weak},0}(X)$ and $\ell_p\widehat{\otimes}X=\ell_p\langle X\rangle$ isometrically.*

We should mention here Khinchin's inequality (see [**8**, page 10]) which will play a critical role in this paper. Let $r_n(t)$ denote the Rademacher functions, namely, $r_n : [0,1] \longrightarrow \mathbf{R}$, $n \in \mathbf{N}$, defined by $r_n(t) := \operatorname{sign}(\sin 2^n \pi t)$.

Khinchin's inequality. *For any $0 < p < \infty$, there are positive constants A_p and B_p such that for any scalar $a_1, \dots, a_n$, we have*

$$A_p \cdot \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \leq \left(\int_0^1 \left| \sum_{k=1}^{n} a_k r_k(t) \right|^p dt \right)^{1/p} \leq B_p \cdot \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2}.$$

3. Littlewood-Orlicz operators. Recall that a continuous linear operator u from a Banach space X to a Banach space Y is called *Littlewood-Orlicz* (see [**2**]) if there exists a positive constant c such that for any finite sequence $x_1, \dots, x_n$ in X and any finite sequence $y_1^*, \dots, y_n^*$ in Y^*,

$$\sum_{k=1}^{n} |\langle u x_k, y_k^* \rangle| \leq c \cdot \sup_{x^* \in B_{X^*}} \sum_{k=1}^{n} |x^*(x_k)| \cdot \sup_{y \in B_Y} \left(\sum_{k=1}^{n} |y_k^*(y)|^2 \right)^{1/2}.$$

The infimum of all possible c above is called the *Littlewood-Orlicz norm* of u, denoted by $\|u\|_{LO}$. That is, u is a Littlewood-Orlicz operator if and only if u takes sequences in $\ell_1^{\text{weak}}(X)$ into sequences in $\ell_2\langle X \rangle$, equivalently, u takes sequences in $\ell_1 \check{\otimes} X$ into sequences in $\ell_2 \widehat{\otimes} Y$.

Note that each 1-summing operator from X to Y takes sequences in $\ell_1 \check{\otimes} X$ into sequences in $\ell_1 \widehat{\otimes} Y$ and each 2-dominated operator from X to Y takes sequences in $\ell_2 \check{\otimes} X$ into sequences in $\ell_2 \widehat{\otimes} Y$. Thus, all 1-summing operators and all 2-dominated operators are Littlewood-Orlicz. It was shown in [**2**, page 745] that $\ell_1^{\text{weak}}(X) \subseteq \operatorname{Rad}(X)$ for any Banach space X and [**2**, Theorem 12] stated that each 1-factorable operator takes sequences in $\operatorname{Rad}(X)$ into sequences in $\ell_2 \widehat{\otimes} Y$. Thus, all 1-factorable operators are Littlewood-Orlicz. An example of non Littlewood-Orlicz operators is given as follows.

Example. The inclusion map $i_p : L_\infty[0,1] \to L_p[0,1]$ is *not* Littlewood-Orlicz for $1 < p < \infty$.

Proof. *Case* 1. $1 < p < 2$. Let $E_n = (1/((n+1)^{p-1}), 1/(n^{p-1})]$, $n = 1, 2, \ldots$. Then $\{E_n\}_1^\infty$ is a partition of $(0,1]$ with

$$m(E_n) = \frac{(1+(1/n))^{p-1} - 1}{(n+1)^{p-1}}, \quad n = 1, 2, \ldots .$$

Define $f_n = \chi_{E_n}$, $n = 1, 2, \ldots$. Then $f_n \in L_\infty[0,1]$, and it is easy to see that $(f_n)_n \in \ell_1^{\text{weak}}(L_\infty[0,1])$. Define $g_n = [1/(m(E_n))]^{1/p'} \chi_{E_n}$, $n = 1, 2, \ldots$. Then $g_n \in L_{p'}[0,1]$ and, for each $h \in L_p[0,1]$,

$$\begin{aligned}
\left(\sum_{n=1}^\infty |\langle g_n, h\rangle|^2 \right)^{1/2} &= \left(\sum_{n=1}^\infty \left| \int_{E_n} \left[\frac{1}{m(E_n)} \right]^{1/p'} h(t)\, dt \right|^2 \right)^{1/2} \\
&\le \left(\sum_{n=1}^\infty \left[\frac{1}{m(E_n)} \right]^{2/p'} \left[\int_{E_n} |h(t)|\, dt \right]^2 \right)^{1/2} \\
&\le \left(\sum_{n=1}^\infty \left[\frac{1}{m(E_n)} \right]^{2/p'} \left[\int_{E_n} 1^{p'}\, dt \right]^{2/p'} \right. \\
&\qquad \left. \cdot \left[\int_{E_n} |h(t)|^p\, dt \right]^{2/p} \right)^{1/2} \\
&= \left(\sum_{n=1}^\infty \left[\int_{E_n} |h(t)|^p\, dt \right]^{2/p} \right)^{1/2} \\
&= \left[\left\| \left(\int_{E_n} |h(t)|^p\, dt \right)_n \right\|_{\ell_{2/p}} \right]^{1/p} \\
&\le \left[\left\| \left(\int_{E_n} |h(t)|^p\, dt \right)_n \right\|_{\ell_1} \right]^{1/p} \\
&= \left[\sum_{n=1}^\infty \int_{E_n} |h(t)|^p\, dt \right]^{1/p} \\
&= \|h\|_{L_p[0,1]} < \infty.
\end{aligned}$$

Thus, $(g_n)_n \in \ell_2^{\text{weak}}(L_{p'}[0,1])$. But

$$\begin{aligned}\sum_{n=1}^{\infty} |\langle i_p f_n, g_n\rangle| &= \sum_{n=1}^{\infty} \Big| \int_0^1 f_n(t) g_n(t)\, dt \Big| \\ &= \sum_{n=1}^{\infty} \left[\frac{(1+(1/n))^{p-1}-1}{(n+1)^{p-1}}\right]^{1/p} \\ &= \infty.\end{aligned}$$

Therefore, $(i_p f_n)_n \notin \ell_2\langle L_p[0,1]\rangle$. It follows that i_p is not a Littlewood-Orlicz operator.

Case 2. $2 \le p < \infty$. Let $E_n = (1/(n+1), (1/n)]$, $n = 1, 2, \dots$. Then $\{E_n\}_1^\infty$ is a petition of $(0,1]$ with $m(E_n) = 1/(n(n+1))$, $n = 1, 2, \dots$. Define $f_n = \chi_{E_n}$, $n = 1, 2, \dots$. Then, $f_n \in L_\infty[0,1]$ and $(f_n)_n \in \ell_1^{\text{weak}}(L_\infty[0,1])$. Define $g_n = \sqrt{n(n+1)}\chi_{E_n}$, $n = 1, 2, \dots$. Then, $g_n \in L_{p'}[0,1]$ and for each $h \in L_p[0,1]$,

$$\begin{aligned}\left(\sum_{n=1}^{\infty} |\langle g_n, h\rangle|^2\right)^{1/2} &= \left(\sum_{n=1}^{\infty} \Big| \int_0^1 \sqrt{n(n+1)}\chi_{E_n} h(t)\, dt\Big|^2\right)^{1/2} \\ &\le \left(\sum_{n=1}^{\infty} n(n+1)\cdot \left[\int_{E_n} |h(t)|\, dt\right]^2\right)^{1/2} \\ &\le \left(\sum_{n=1}^{\infty} n(n+1)\cdot \left[\int_{E_n} 1^2\, dt\right]^{2/2} \right. \\ &\qquad \left. \cdot \left[\int_{E_n} |h(t)|^2\, dt\right]^{2/2}\right)^{1/2} \\ &= \|h\|_{L_2[0,1]} \le \|h\|_{L_p[0,1]} < \infty.\end{aligned}$$

Thus, $(g_n)_n \in \ell_2^{\text{weak}}(L_{p'}[0,1])$. But

$$\sum_{n=1}^{\infty} |\langle i_p f_n, g_n\rangle| = \sum_{n=1}^{\infty} \Big| \int_0^1 f_n(t) g_n(t)\, dt \Big| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} = \infty.$$

Therefore, $(i_p f_n)_n \notin \ell_2\langle L_p[0,1]\rangle$. It follows that i_p is not a Littlewood-Orlicz operator. $\square$

Recall that (see [**8**, page 60]) a Banach space X is called an $\mathcal{L}_p$-*space* for $1 \leq p \leq \infty$ if every finite-dimensional subspace E of X is contained in a finite-dimensional subspace F of X for which there is an isomorphism $v : F \to \ell_p^{\dim F}$ with $\|v\| \cdot \|v^{-1}\| < \lambda$ for some $\lambda > 1$.

Lemma 2. *Let $n, m \in \mathbf{N}$ and $u : \ell_2^n \to \ell_\infty^m$. Then $\|u\|_{\mathrm{LO}} \leq K_G \cdot B_1 \cdot \|u\|$, where K_G is the Grothendieck's constant and B_1 is the constant in Khinchin's inequality.*

Proof. Let u^* denote the adjoint operator of u. By the Grothendieck theorem (see [**9**] or [**8**, page 15]), $u^* : \ell_1^m \to \ell_2^n$ is 1-summing with

$$\pi_1(u^*) \leq K_G \cdot \|u^*\|.$$

By the Pietsch domination theorem (see [**16**] or [**8**, page 44]), there is a regular probability measure μ on $B_{\ell_\infty^m}$ such that, for each $y \in \ell_1^m$,

$$\|u^* y\| \leq \pi_1(u^*) \cdot \int_{B_{\ell_\infty^m}} |\langle y, z\rangle| \, d\mu(z).$$

Let $x_1, \dots, x_k \in \ell_2^n$ and $y_1, \dots, y_k \in \ell_1^m = (\ell_\infty^m)^*$. By Khinchin's inequality,

$$\begin{aligned}
\Big|\sum_{i=1}^k \langle u x_i, y_i\rangle\Big| &= \Big|\int_0^1 \Big\langle \sum_{i=1}^k r_i(t) u x_i, \sum_{i=1}^k r_i(t) y_i \Big\rangle \, dt\Big| \\
&\leq \int_0^1 \Big\|\sum_{i=1}^k r_i(t) x_i\Big\| \cdot \Big\|\sum_{i=1}^k r_i(t) u^* y_i\Big\| \, dt \\
&\leq \sup_{t\in[0,1]} \Big\|\sum_{i=1}^k r_i(t) x_i\Big| \cdot \int_0^1 \Big|u^*\Big(\sum_{i=1}^k r_i(t) y_i\Big)\Big\| \, dt \\
&\leq \|(x_i)_1^k\|_{\ell_1^w(\cdot)} \cdot \pi_1(u^*) \\
&\quad \cdot \int_0^1 \Big(\int_{B_{\ell_\infty^m}} \Big|\Big\langle \sum_{i=1}^k r_i(t) y_i, z\Big\rangle\Big| \, d\mu(z)\Big) \, dt \\
&= \pi_1(u^*) \cdot \|(x_i)_1^k\|_{\ell_1^w(\cdot)} \\
&\quad \cdot \int_{B_{\ell_\infty^m}} \Big(\int_0^1 \Big|\sum_{i=1}^k r_i(t)\langle y_i, z\rangle\Big| \, dt\Big) \, d\mu(z)
\end{aligned}$$

$$\begin{aligned}&\leq K_G \cdot \|u^*\| \cdot \|(x_i)_1^k\|_{\ell_1^w(\cdot)} \\ &\quad \cdot \int_{B_{\ell_\infty^m}} \left(B_1 \cdot \left(\sum_{i=1}^k |\langle y_i, z\rangle|^2 \right)^{1/2} \right) d\mu(z) \\ &\leq K_G \cdot B_1 \cdot \|u\| \cdot \|(x_i)_1^k\|_{\ell_1^w(\cdot)} \cdot \|(y_i)_1^k\|_{\ell_2^w(\cdot)}.\end{aligned}$$

It follows that $\|u\|_{\mathrm{LO}} \leq K_G \cdot B_1 \cdot \|u\|$. □

By localization, we have

Theorem 3. *Each continuous linear operator from an $\mathcal{L}_2$-space to an $\mathcal{L}_\infty$-space is Littlewood-Orlicz.*

Lemma 4. *Let $n \in \mathbf{N}$ and $u : X \to \ell_2^n$. Then $\pi_2(u) \leq \|u\|_{\mathrm{LO}}$.*

Proof. Let $x_1, \dots, x_m \in X$. Define a linear operator $v : \ell_2^m \to X$ by $ve_k = x_k$ for $k = 1, 2, \dots, m$. Then

$$\|v\| = \sup_{x^* \in B_{X^*}} \left(\sum_{k=1}^m |x^*(x_k)|^2 \right)^{1/2}.$$

Note that $uv : \ell_2^m \to \ell_2^n$ is an operator between Hilbert spaces. By the proof of [**2**, Theorem 14],

$$\pi_2(uv) = \|uv\|_{HS} \leq \|uv\|_{\mathrm{LO}} \leq \|u\|_{\mathrm{LO}} \cdot \|v\|.$$

Thus,

$$\begin{aligned}\left(\sum_{k=1}^m \|ux_k\|^2 \right)^{1/2} &= \left(\sum_{k=1}^m \|uve_k\|^2 \right)^{1/2} \leq \pi_2(uv) \\ &\leq \|u\|_{\mathrm{LO}} \cdot \|v\| \\ &= \|u\|_{\mathrm{LO}} \cdot \sup_{x^* \in B_{X^*}} \left(\sum_{k=1}^m |x^*(x_k)|^2 \right)^{1/2}.\end{aligned}$$

Therefore, $\pi_2(u) \leq \|u\|_{\mathrm{LO}}$. □

By localization, we have

Theorem 5. *Each Littlewood-Orlicz operator from a Banach space to an $\mathcal{L}_2$-space is 2-summing.*

Remark. It follows from Corollary 11.16 in [**8**, page 224] that each 2-summing operator from a Banach space with cotype 2 to a Banach space is 1-summing and, hence, Littlewood-Orlicz. Thus, Littlewood-Orlicz operators from a Banach space with cotype 2 to an $\mathcal{L}_2$-space coincide with 2-summing operators. Consequently, Littlewood-Orlicz operators between Hilbert spaces coincide with Hilbert-Schmidt operators.

Recall that a Banach space X is said to have the *property* V if for every Banach space Y, a continuous linear operator u from X to Y is weakly compact if and only if u is unconditionally converging, i.e., u takes a weakly unconditionally Cauchy (wuC) series in X into an unconditionally convergent (uc) series in Y. For any Hausdorff compact metric space K, $C(K)$ has property V (see [**15**]). The identity operator on ℓ_1 is 1-factorable, and hence Littlewood-Orlicz, but it is not weakly compact. However, if the domain space X has property V, then we have the following theorem.

Theorem 6. *Let X and Y be Banach spaces such that X has property V. Then each Littlewood-Orlicz operator from X to Y is weakly compact.*

Proof. Let $u : X \to Y$ be a Littlewood-Orlicz operator. To show that u is weakly compact, we need only to show that u is unconditionally converging. If $\sum_n x_n$ is a wuC series in X for which $\sum_n ux_n$ is not a uc series in Y, then there exists a subsequence $\{x'_n\}_1^\infty$ of $\{x_n\}_1^\infty$ such that $\sum_n ux'_n$ does not converge. Hence, $\{\sum_{k=1}^n ux'_k\}_1^\infty$ is not a Cauchy sequence. It follows that there exist $\varepsilon_0 > 0$ and an integer sequence $1 \le m_1 < n_1 < m_2 < n_2 < \cdots$ such that

$$(*) \qquad \Big\| \sum_{i=m_k+1}^{n_k} ux'_i \Big\| \ge \varepsilon_0, \quad k = 1, 2, \dots .$$

Let $y_k = \sum_{i=m_k+1}^{n_k} x'_i$. Then, for each $x^* \in X^*$,

$$\sum_{k=1}^{\infty} |x^*(y_k)| \le \sum_{k=1}^{\infty} \sum_{i=m_k+1}^{n_k} |x^*(x_i')| \le \sum_{i=1}^{\infty} |x^*(x_i')| \le \sum_{i=1}^{\infty} |x^*(x_i)| < \infty.$$

Thus, $(y_k)_k \in \ell_1^{\mathrm{weak}}(X)$. Therefore, $(uy_k)_k \in \ell_2\langle Y\rangle \subseteq \ell_2^{\mathrm{strong}}(Y)$. It follows that $\lim_k \|uy_k\| = 0$ which contradicts $(*)$. This contradiction shows that u is unconditionally converging. □

Corollary 7. *Let K be a Hausdorff compact metric space. Then each Littlewood-Orlicz operator from $C(K)$ to a Banach space is weakly compact.*

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