ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 6, 2009

ON THE PRIME NUMBER THEOREM FOR A COMPACT RIEMANN SURFACE

M. AVDISPAHIĆ AND L. SMAJLOVIĆ

ABSTRACT. We improve the estimate of the error term in Selberg's and Huber's formula for distribution of the eigenvalues of the Laplace-Beltrami operator on a compact Riemann surface.

1. Introduction. When Riemann [**6**] introduced complex analysis into the field of number theory, his main goal apparently was to outline the eventual proof of the prime number theorem. His hypothesis about the zeros of his zeta function remains the most famous unsolved problem of mathematics today. In the case of the Selberg zeta function, the situation is different. The analogue of Riemann's hypothesis is known to be true in this setting. However, the number of nontrivial zeros is significantly higher than in the classical case. The purpose of this note is to improve the estimate of the error term in the analogue of the prime number theorem.

2. Preliminaries. Let $\mathcal{H} = \{z = x + iy : y > 0\}$ denote the upper half-plane equipped with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$. Möbius transformations $z \to (az + b)/(cz + d)$, where $a, b, c, d \in \mathbf{R}$ and $ad - bc = 1$ form the group $PSL(2, \mathbf{R})$ that acts on $\mathcal{H}$ by homeomorphisms which preserve the hyperbolic distance.

Discrete subgroups of $PSL(2, \mathbf{R})$ are called Fuchsian groups. We shall consider a strictly hyperbolic Fuchsian group Γ, in which case the quotient space $\Gamma \backslash \mathcal{H}$ can be identified with a compact Riemann surface F of a genus $g \geq 2$. An element $\gamma \in \Gamma$ has the trace $|a + d| > 2$ and possesses two fixed points $z_1 \neq z_2$ lying in $\mathbf{R}$. We shall denote by P the conjugacy class in Γ generated by γ. All transformations having the same fixed points as γ form an infinite cyclic subgroup. By [**2**, Proposition 6.1, page 24] there exists the generator γ_0 of this cyclic

2000 AMS *Mathematics subject classification.* Primary 11N45, 11F72.
Keywords and phrases. Prime number theorem, Selberg zeta function.
Received by the editors on January 22, 2004, and in revised form on June 4, 2007.

DOI:10.1216/RMJ-2009-39-6-1837

group such that $\gamma = \gamma_0^m$ for some $m \in \mathbf{N}$. By P_0 we shall denote the conjugacy class of γ_0 and call it a primitive hyperbolic conjugacy class. Obviously, $P = P_0^m$.

Now, $\gamma \in P$ has exactly two real eigenvalues. The square of the larger eigenvalue is denoted by $N(P)$ and called the norm of class P. If $P = P_0^m$, for P_0 and $m \in \mathbf{N}$ as above, then $N(P) = N(P_0)^m$. For this reason, $N(P_0)$ is also known as a pseudo-prime. The norm $N(P_0)$ is determined by the length of the geodesic joining the fixed points z_1 and z_2. For more details, see, e.g., [**5**, Chapters 15.7 and 15.9].

For any $x > 0$, we are interested in the number of classes $\{P_0\}$ such that $N(P_0) \le x$.

3. Distribution of eigenvalues of the Laplace-Beltrami operator. The main tool in the proof of the analogue of the prime number theorem is the Selberg zeta function on $\Gamma \setminus \mathcal{H}$, defined by the Euler product as

$$Z_\Gamma(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} \left(1 - N(P_0)^{-s-k}\right), \quad \mathrm{Re}\,(s) > 1,$$

where the product is taken over all primitive conjugacy classes $\{P_0\}$.

The Selberg zeta function can be continued to the whole complex plane as a meromorphic function of a finite order. Furthermore, it fulfills the functional equation

$$Z_\Gamma(s) = \Psi_\Gamma(s) Z_\Gamma(1-s),$$

where the factor of the functional equation is

$$\Psi_\Gamma(s) = \exp\left(-|F| \int_0^{s-(1/2)} t \cdot \tan(\pi t)\, dt \right),$$

$|F|$ denoting the area of the Riemann surface F.

Nontrivial zeros $s_n = 1/2 + ir_n$ and $\tilde{s}_n = 1/2 - ir_n$ of the function Z_Γ are closely related to the eigenvalues λ_n of the Laplace-Beltrami operator on F by the equation $r_n = \sqrt{\lambda_n - 1/4}$ for $\lambda_n \ge 1/4$ and $r_n = -i\sqrt{-\lambda_n + 1/4}$ for $\lambda_n < 1/4$. Numbers λ_n are nonnegative and tend to infinity; hence, there are finitely many of them less than $1/4$. Let us assume there are exactly M values of λ_n less than $1/4$. Nontrivial zeros of Z_Γ can be split into two parts: zeros $s_n = 1/2 + ir_n$ and $\tilde{s}_n = 1/2 - ir_n$, for $r_n \geq 0$ lying on the critical line $\operatorname{Re}(s) = 1/2$ and zeros $s_n = 1/2 + ir_n$ and $\tilde{s}_n = 1/2 - ir_n$ lying on the segment $[0, 1]$.

The logarithmic derivative of the Selberg zeta function is given by

$$\frac{Z'_\Gamma}{Z_\Gamma}(s) = \sum_{\{P\}} \frac{\log N(P_0)}{1 - N(P)^{-1}} N(P)^{-s},$$

where the sum is over all hyperbolic conjugacy classes P and P_0 is the primitive such class associated to P.

An important ingredient in the proof of our main result is the following estimate for the rate of growth of this logarithmic derivative, see [**2**, Proposition 10.9, page 155 and Theorem 8.1, page 119].

$$\frac{Z'_\Gamma}{Z_\Gamma}(s) = O\left(\frac{1}{\varepsilon}\left(\frac{T}{\log T}\right)^{2\max(0,1+\varepsilon-\sigma)}\right)$$

for $s = \sigma + iT$, $\sigma \geq 1/2 + \varepsilon$, $T \geq 1000$, $T \neq$ all r_n.

The main result of the paper is the following theorem.

Theorem 3.1. *Let $\pi_0(x)$ denote the number of distinct primitive classes $\{P_0\}$ such that $N(P_0) \leq x$. Then,*

$$\pi_0(x) = li(x) + \sum_{k=1}^{M} li(x^{s_k}) + O(x^{3/4} \log^{-1} x),$$

where the implied constant depends solely on Γ.

Here s_k denotes the real zeros of Z_Γ larger than $1/2$.

This theorem is an improvement of the result obtained by Selberg and Huber [**2**, Theorem 6.19.], see also [**2**, Discussion 15.16, page 253; 319–320] and [**3, 4**].

4. Proof of the main result. Let us consider the following analogues of the well-known arithmetic functions

$$\psi(x) = \sum_{\substack{\{P\} \\ N(P)\leq x}} \Lambda(P) = \sum_{\substack{\{P\} \\ N(P)\leq x}} \frac{\log N(P_0)}{1-N(P)^{-1}}$$

and

$$\psi_1(x) = \int_1^x \psi(t)\,dt.$$

For $x \geq 1$ and $\sigma_0 > 1$, one has (see [**2**, Proposition 6.9, page 103])

$$\begin{aligned}\psi_1(x) &= \frac{1}{2\pi i}\int_{(\sigma_0)} \frac{x^{s+1}}{s(s+1)}\frac{Z_\Gamma'}{Z_\Gamma}(s)\,ds \\ &= \frac{1}{2\pi i}\lim_{L\to\infty}\int_{\sigma_0-iL}^{\sigma_0+iL} \frac{x^{s+1}}{s(s+1)}\frac{Z_\Gamma'}{Z_\Gamma}(s)\,ds.\end{aligned}$$

The main idea of our proof is to try to improve the error term $O(x^2\log x/T)$ in the explicit formula for $\psi_1(x)$ [**2**, Theorem 6.16, page 110] to the extent that it loses the role of the leading term in final estimations. This is done by moving the line of integration in the above integral expression for $\psi_1(x)$ to the left. Taking such a step at this point of argumentation constitutes a refinement of Hejhal's approach.

So, let $\varepsilon > 0$ be a number such that real zeros $s_0 = 1, s_1, \dots, s_M$ of Z_Γ belong to the segment $((1/2)+2\varepsilon, 1]$. For a fixed $x \geq 1$ application of the residue theorem on the rectangle with vertices $\sigma_0 - iL$, $\sigma_0 + iL$, $1/2+2\varepsilon+iL$, $1/2+2\varepsilon-iL$ yields

$$\begin{aligned}(1)\quad &\frac{1}{2\pi i}\int_{\sigma_0-iL}^{\sigma_0+iL} \frac{x^{s+1}}{s(s+1)}\frac{Z_\Gamma'}{Z_\Gamma}(s)\,ds \\ &= \sum_{n=0}^{M}\frac{x^{s_{n+1}}}{s_n(s_n+1)} + \frac{1}{2\pi i}\left[\int_{(1/2)+2\varepsilon-iL}^{(1/2)+2\varepsilon+iL} + \int_{\sigma_0-iL}^{(1/2)+2\varepsilon-iL} + \int_{(1/2)+2\varepsilon+iL}^{\sigma_0+iL}\right].\end{aligned}$$

Applying the bound for Z'_Γ/Z_Γ from the previous section, we obtain

$$\begin{aligned}\int_{(1/2)+2\varepsilon+iL}^{\sigma_0+iL} \frac{x^{s+1}}{s(s+1)}\frac{Z'_\Gamma}{Z_\Gamma}(s)\,ds &= O\biggl(\frac{x^{1+\sigma_0}}{\varepsilon L^2}\int_{(1/2)+2\varepsilon}^{\sigma_0}\biggl(\frac{L}{\log L}\biggr)^{2\max(0,1+\varepsilon-\sigma)}d\sigma\biggr)\\ &= O\biggl(\frac{x^{1+\sigma_0}}{\varepsilon L^2}\biggl(\frac{L}{\log L}\biggr)^{1-2\varepsilon}\biggr).\end{aligned}$$

Similarly,

$$\int_{\sigma_0-iL}^{(1/2)+2\varepsilon-iL} \frac{x^{s+1}}{s(s+1)}\frac{Z'_\Gamma}{Z_\Gamma}(s)\,ds = O\biggl(\frac{x^{1+\sigma_0}}{\varepsilon L^2}\biggl(\frac{L}{\log L}\biggr)^{1-2\varepsilon}\biggr).$$

One should notice that there exists a better estimate of the logarithmic derivative Z'_Γ/Z_Γ, given in [**1**, page 187]. However, due to the other terms appearing in the explicit formula for ψ_1, our bound on Z'_Γ/Z_Γ would yield the same error term in the prime geodesic theorem, as the one we shall obtain using Hejhal's estimate given above.

Passing to the limit $L \to \infty$ in (1), we get (for a fixed $x \geq 1$)

$$\psi_1(x) = \sum_{n=0}^{M} \frac{x^{s_n+1}}{s_n(s_n+1)} + \frac{1}{2\pi i}\int_{((1/2)+2\varepsilon)} \frac{x^{s+1}}{s(s+1)}\frac{Z'_\Gamma}{Z_\Gamma}(s)\,ds.$$

This new form of representing $\psi_1(x)$ is essential for our purpose.

Namely, for $T \geq 1000$,

$$\begin{aligned}\biggl|\biggl(\int_{((1/2)+2\varepsilon)} - \int_{(1/2)+2\varepsilon-iT}^{(1/2)+2\varepsilon+iT}\biggr)\biggl(\frac{x^{s+1}}{s(s+1)}\frac{Z'_\Gamma}{Z_\Gamma}(s)\,ds\biggr)\biggr| &= O\biggl(\int_T^\infty \frac{x^{(1/2)+2\varepsilon+1}}{\varepsilon t^2}\biggl(\frac{t}{\log t}\biggr)^{2(1+\varepsilon-(1/2)-2\varepsilon)}dt\biggr)\\ &= O\biggl(\frac{x^{(3/2)+2\varepsilon}}{\varepsilon\log T}\biggr).\end{aligned}$$

Hence,

$$\begin{aligned}\psi_1(x) &= \sum_{n=0}^{M} \frac{x^{s_n+1}}{s_n(s_n+1)} + \frac{1}{2\pi i}\int_{(1/2)+2\varepsilon-iT}^{(1/2)+2\varepsilon+iT} \frac{x^{s+1}}{s(s+1)}\frac{Z_\Gamma'}{Z_\Gamma}(s)\,ds \\ &\quad + O\bigg(\frac{x^{(3/2)+2\varepsilon}}{\varepsilon\log T}\bigg).\end{aligned} \tag{2}$$

The error term $O(x^{(3/2)+2\varepsilon}/(\varepsilon \log T))$ is the improvement we were looking for.

Now, the integral on the righthand side is treated in the same way as in [**2**], taking $A = N+(1/2)$, $N \in \mathbf{N}$ and applying the residue theorem on the rectangle with vertices $1/2+2\varepsilon-iT$, $1/2+2\varepsilon+iT$, $-A+iT$, $-A-iT$. That gives us

$$\begin{aligned}\frac{1}{2\pi i}&\int_{(1/2)+2\varepsilon-iT}^{(1/2)+2\varepsilon+iT} \frac{x^{s+1}}{s(s+1)}\frac{Z_\Gamma'}{Z_\Gamma}(s)\,ds \\ &= \frac{1}{2\pi i}\bigg[\int_{(1/2)+2\varepsilon-iT}^{(1/2)-2\varepsilon-iT} + \int_{(1/2)-2\varepsilon-iT}^{-1-iT} + \int_{-1-iT}^{-A-iT} + \int_{-A-iT}^{-A+iT} \\ &\qquad + \int_{-A+iT}^{-1+iT} + \int_{-1+iT}^{(1/2)-2\varepsilon+iT} + \int_{(1/2)-2\varepsilon+iT}^{(1/2)+2\varepsilon+iT}\bigg] \\ &\quad + a_0 x + \beta_0 x\log x + \alpha_1 + \beta_1 \log x \\ &\quad + (2g-2)\sum_{k=2}^{N} \frac{2k+1}{k(k-1)}x^{1-k} + \sum_{n=1}^{M} \frac{x^{\widetilde{s_n}+1}}{\widetilde{s_n}(\widetilde{s_n}+1)} \\ &\quad + \sum_{0\le r_n\le T}\bigg(\frac{x^{s_{n+1}}}{s_n(s_n+1)} + \frac{x^{\widetilde{s_n}+1}}{\widetilde{s_n}(\widetilde{s_n}+1)}\bigg).\end{aligned}$$

Here, α_0, β_0, α_1, β_1 are constants that depend solely on Γ. Seven integrals on the righthand side can be estimated as in [**2**, pages 105–107]. We obtain

$|I_1|$ and $|I_7|$ are $O(x^{(3/2)+2\varepsilon}/T)$,

$|I_2|$ and $|I_6|$ are $O((x^{(3/2)-2\varepsilon}/T)(1+1/(\varepsilon\log T)))$,

$|I_3|$ and $|I_5|$ are $O(1/(T\log x))$. Finally,

$|I_4|$ is $O(x^{1-A})$.

Passing to the limit $A \to \infty$ in the last sum, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{(1/2)+2\varepsilon-iT}^{(1/2)+2\varepsilon+iT} &\frac{x^{s+1}}{s(s+1)}\frac{Z_\Gamma'}{Z_\Gamma}(s)\,ds \\
&= a_0 x + \beta_0 x\log x + \alpha_1 + \beta_1 \log x \\
&\quad + (2g-2)\sum_{k=2}^{\infty}\frac{2k+1}{k(k-1)}x^{1-k} \\
&\quad + \sum_{n=1}^{M}\frac{x^{\widetilde{s_n}}+1}{\widetilde{s_n}(\widetilde{s_n}+1)} + \sum_{0\le r_n\le T}\left(\frac{x^{s_n+1}}{s_n(s_n+1)}+\frac{x^{\widetilde{s_n}+1}}{\widetilde{s_n}(\widetilde{s_n}+1)}\right) \\
&\quad + O\left(\frac{x^{(3/2)+2\varepsilon}}{T}\left(1+\frac{1}{\varepsilon\log T}\right)\right).
\end{aligned}$$

Equation (2) implies that

$$\begin{aligned}
\psi_1(x) = \alpha_0 x &+ \beta_0 x\log x + \alpha_1 + \beta_1\log x \\
&+ \sum_{0\le r_n\le T}\left(\frac{x^{s_n+1}}{s_n(s_n+1)}+\frac{x^{\widetilde{s_n}+1}}{\widetilde{s_n}(\widetilde{s_n}+1)}\right) \\
&+ (2g-2)\sum_{k=2}^{\infty}\frac{2k+1}{k(k-1)}x^{1-k} + \sum_{n=0}^{M}\frac{x^{s_n+1}}{s_n(s_n+1)} \\
&+ \sum_{n=1}^{M}\frac{x^{\widetilde{s_n}+1}}{\widetilde{s_n}(\widetilde{s_n}+1)} + O\left(\frac{x^{(3/2)+2\varepsilon}}{\varepsilon\log T}\right).
\end{aligned}$$

Now, we proceed as in the proof of [**2**, Theorem 6.18, pages 111–114] and obtain, for $x \ge 1000$ and $1 \le h \le x/2$,

$$\begin{aligned}
\int_x^{x+h} &\psi(t)\,dt \\
&= O(h\log x) + \sum_{n=0}^{M}\int_x^{x+h}\frac{t^{s_n}}{s_n}\,dt \\
&\quad + O(x^{3/2}) + O\left(\frac{x^{(3/2)+2\varepsilon}}{\varepsilon\log T}\right) \\
&\quad + \sum_{0\le r_n\le T}\left(\frac{(x+h)^{s_n+1}-x^{s_n+1}}{s_n(s_n+1)}+\frac{(x+h)^{\widetilde{s_n}+1}-x^{\widetilde{s_n}+1}}{\widetilde{s_n}(\widetilde{s_n}+1)}\right).
\end{aligned}$$

Hence,

$$\psi(x) = \sum_{n=0}^{M} \frac{x^{s_n}}{s_n} + O(\log x) + O(h)$$
$$+ O\left(\frac{x^{3/2}}{h}\right) + O\left(\frac{x^{(3/2)+2\varepsilon}}{h\varepsilon \log T}\right)$$
$$+ \frac{1}{h} \sum_{0 \le r_n \le T} \left(\frac{(x+h)^{s_n+1} - x^{s_n+1}}{s_n(s_n+1)} + \frac{(x+h)^{\widetilde{s_n}+1} - x^{\widetilde{s_n}+1}}{\widetilde{s_n}(\widetilde{s_n}+1)} \right).$$

Since

$$\left| \frac{1}{h} \sum_{0 \le r_n \le T} \frac{(x+h)^{s_n+1} - x^{s_n+1}}{s_n(s_n+1)} \right| = O\left(\frac{1}{h} \left(x^{3/2} + x^{3/2} \log\left(\frac{hT}{x}\right)\right)\right),$$

taking $T \sim x^{1/4}$, $h \sim x^{3/4}$, $\varepsilon \sim 1/\log x$, we obtain

$$\psi(x) = \sum_{n=0}^{M} \frac{x^{s_n}}{s_n} + O(x^{3/4}).$$

The theorem follows from this formula for $\psi(x)$, see [**2**].

Remark. Using the same method and applying results proved in [**2**, pages 240–252], it is possible to obtain better conditional estimates for $\psi(x)$. Under the hypothesis that $|S(t)| = O(t^{\alpha})$, we get

$$\psi(x) = \sum_{n=0}^{M} \frac{x^{s_n}}{s_n} + O(x^{(1/2)((1+2a)/(1+\alpha))}.$$

This estimate agrees with the one obtained in the case $\alpha = 1$.

REFERENCES

1. M. Avdispahić and L. Smajlović, *An explicit formula and its application to the Selberg trace formula*, Monatsh. Math. **147** (2006), 183–198.

2. D.A. Hejhal, *The Selberg trace formula for* $PSL(2,\mathbf{R})$, Vol. I, Lecture Notes Math. **548**, Springer Verlag, Berlin, 1976.

3. H. Huber, *Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen* II, Math. Ann. **142** (1961), 385–398.

4. ———, *Nachtrag zu* [3], Math. Ann. **143** (1961), 463–464.

5. H. Iwaniec and E. Kowalski, *Analytic number theory*, Amer. Math. Soc. Colloq. Publ. **53** (2004).

6. B. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse*, Mon. Not. Berlin Akad. (1859), 671–680.

7. A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Ind. Math. Soc. **20** (1956), 47–87.

University of Sarajevo, Department of Mathematics, Zmaja od Bosne 35, 71000 Sarajevo, Bosnia and Herzegovina
Email address: mavdispa@pmf.unsa.ba

University of Sarajevo, Department of Mathematics, Zmaja od Bosne 35, 71000 Sarajevo, Bosnia and Herzegovina
Email address: lejlas@pmf.unsa.ba