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FINITE RELATIVE DETERMINATION AND THE ARTIN-REES LEMMA

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ABSTRACT. If we consider the 4-dimensional vector space of quasi-homogeneous maps in two variables of weights $(\frac{1}{3}, \frac{1}{6})$ and degree 1, we get two 1-parameter families $f_\epsilon(x,y) = x^3 + (\epsilon+1)x^2y^2 + \epsilon xy^4$ and $g_\epsilon(x,y) = x^3 + \epsilon x^2y^2 + xy^4 + \epsilon y^6$. We are interested in comparing the usual finite determination with a suitable group of diffeomorphisms. We prove analogous theorems of finite determination for such group. This work is a continuation of [**3**].

1. Introduction. Following our work [**3**], we consider the quasi-homogeneous maps in two variables of weights $(\frac{1}{3}, \frac{1}{6})$ and degree 1. The group of germs of diffeomorphisms considered there is of the form $h(x,y) = (\alpha x + \beta y^2, \delta y)$. We then study G, a subgroup of the group of diffeomorphisms of the form $h(x,y) = (\alpha x + \beta y^2 + h_1, \delta y + h_2)$ where $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$. We study the concepts of finite determination on the right and finite relative determination for two models, obtaining different numbers. We finish with a version of the Artin-Rees lemma in $\mathcal{E}(n)$, the algebra of smooth germs.

1. Finite determination and finite relative determination. Consider $\mathcal{E}(n)$ the algebra of smooth germs of functions from the n-dimensional Euclidean space to the real numbers.

Theorem 1 (Stefan). *A germ f is k-determined on the right if and only if for each germ $\mu \in m(n)^{k+1}$, we have that*

$$m(n)^{k+1} \subseteq m(n)\left\langle \frac{\partial(f+\mu)}{\partial x_i} \right\rangle + m(n)^{k+2}.$$

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Real case.

Proposition 2. *The germ* $f_\epsilon(x,y) = x^3 + (\epsilon+1)x^2y^2 + \epsilon xy^4$, *(for* $\epsilon \neq 0$, $\epsilon \neq 1$*) is* 6*-determined on the right and not less.*

Remark 3. The germ f_ϵ (for $\epsilon \neq 0$, $\epsilon \neq 1$) is not 5-determined on the right, since the following equality is impossible

$$y^6 = h_1(x,y)(3x^2 + 2(\epsilon+1)xy^2 + \epsilon y^4) + h_2(x,y)(2(\epsilon+1)x^2y + 4\epsilon xy^3)$$

with h_1 and h_2 in $m(2)$.

Definition 4. Let $\mathcal{A}$ be an $\mathbf{R}$-algebra and $\mathcal{B}$ a real vector subspace. If I is an ideal of $\mathcal{A}$, we say that $\mathcal{B}$ is an I-module if for some k,

$$I^k\mathcal{B} \subseteq \mathcal{B}.$$

Definition 5. Let I be an ideal of the $\mathbf{R}$-algebra $\mathcal{A}$. If $\mathcal{B}$ is a real vector subspace of $\mathcal{A}$, we say that it is finitely generated as an I-module if there exist $v_1, \dots, v_m$ in $\mathcal{B}$ such that any element of $\mathcal{B}$ can be written as a linear combination of the v_i with coefficients in I^k.

Next we give a version of Nakayama's lemma.

Lemma 6. *Let* $\mathcal{A}$ *be an* $\mathbf{R}$*-algebra, and let* $\mathcal{B}$ *be a subspace of* $\mathcal{A}$*. Suppose* I *is an ideal contained in the Jacobson radical of* $\mathcal{A}$ *and such that* $\mathcal{B}$ *is finitely generated as an* I*-module. If* $\mathcal{B} \subseteq I^k\mathcal{B}$*, hence* $\mathcal{B} = \{0\}$*.*

Proof. Let $v_1, \dots, v_m$ be a minimal set of generators of the I-module $\mathcal{B}$; hence, $v_1 = a_1v_1 + \cdots + a_nv_n$ with $a_i \in I^k$. Then $(1-a_1)v_1 = a_2v_2 + \cdots + a_nv_n$ and $v_2, \dots, v_n$ also generate $\mathcal{B}$ as an I-module. Therefore $B = \{0\}$. $\square$

As a generalization of the Lemma of Nakayama one has the following.

Corollary 7. *Let* $\mathcal{A}$ *be an* $\mathbf{R}$*-algebra,* $\mathcal{B}$ *and* $\mathcal{B}'$ I*-submodules of* $\mathcal{A}$*. Suppose* I *is an ideal contained in the Jacobian radical of* $\mathcal{A}$*. If*

$(\mathcal{B}+\mathcal{B}')/\mathcal{B}'$ is finitely generated as an I-module and $I^k\mathcal{B}' + I^{k+1}\mathcal{B} \supseteq I^k\mathcal{B}$, hence $I^k\mathcal{B}' \supseteq I^k\mathcal{B}$.

Proof. We have that

$$I\frac{I^k(\mathcal{B}+\mathcal{B}')}{I^k\mathcal{B}'} = \frac{I^{k+1}\mathcal{B}+I^k\mathcal{B}'}{I^k\mathcal{B}'} \supseteq \frac{I^k\mathcal{B}+I^k\mathcal{B}'}{I^k\mathcal{B}'}.$$

Hence, by the previous lemma, $(I^k\mathcal{B}+I^k\mathcal{B}')/(I^k\mathcal{B}')=0$ and therefore $I^k\mathcal{B} \subseteq I^k\mathcal{B}'$. $\square$

Notation. We denote by $\mathcal{G}$ the group of germs of diffeomorphisms of the form

$$h(x,y) = (\alpha x + \beta y^2 + h_1(x,y), \delta y + h_2(x,y))$$

where $\alpha\delta \neq 0$, $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$.

Definition 8. We say that the germ f is k-determined relative to $\mathcal{G}$ if, given a germ g such that $j^k f(0) = j^k g(0)$, there exists an $h \in \mathcal{G}$ such that $g = f \circ h$.

Theorem 9 (A). *If the germ f is k-determined on the right relative to $\mathcal{G}$, hence*

$$m(2)^{k+1} \subseteq \left\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\right\} + m(2)^{k+2},$$

where α, β and δ range in the real numbers, $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$.

Theorem 10 (B). *A germ f is $(k+2)$-determined on the right relative to $\mathcal{G}$ if*

$$m(2)^{k+1} \subseteq m(2)\left\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\right\} + m(2)^{k+2},$$

where α, β and δ range in the real numbers, $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$.

Corollary 11. *The germ $f_\epsilon(x,y) = x^3 + (\epsilon+1)x^2y^2 + \epsilon xy^4$ is 9-determined relative to $\mathcal{G}$.*

Proof. It is simple to prove that

$$m(2)^{7+1} \subseteq m(2)\{(3x^2 + (\epsilon+1)xy^2 + \epsilon y^4)(\alpha x + \beta y^2 + h_1) \\ + (2(\epsilon+1)x^2y + 4\epsilon xy^3)(\delta y + h_2)\} + m(2)^{7+2}.$$

Then using Theorem 10 (B) we obtain that f_ϵ is 9-determined. □

Proposition 12. *Let $\pi_n : m(2) \to \frac{m(2)}{m(2)^{n+1}}$ be the canonical projection, and let Q be the $\mathcal{G}$-orbit of z, where z is the n-jet of f. Then we get the equality*

$$T_z(\pi_n(Q)) = \pi_n\left\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\right\},$$

where α, β and δ range in the real numbers $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$.

The proof is a consequence of a classical theorem in Lie groups.

We now prove Theorem 9 (A).

Proof. Let z be the k-jet of f. Since f is k-determined relative to $\mathcal{G}$, we have that $z + m(2)^{k+1} \subseteq Q$, Q the $\mathcal{G}$-orbit of z. Therefore

$$\pi_{k+1}(z + m(2)^{k+1}) \subseteq \pi_{k+1}(Q).$$

Taking the tangent spaces at z, we get from our previous proposition that

$$\pi_{k+1}(m(2)^{k+1}) \subseteq \pi_{k+1}\left(\left\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\right\}\right),$$

and therefore

$$m(2)^{k+1} \subseteq \left\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\right\} + m(2)^{k+2}. \qquad \square$$

Let $\mathcal{A}$ be the $\mathbf{R}$-algebra of germs of functions around $(t_0, \overline{0})$ and $\pi : R^3 \to R^2$ the projection $\pi(t,(x,y)) = (x,y)$. Consider the induced monomorphism $\pi^* : \mathcal{E}(2) \to \mathcal{A}$; hence, $\mathcal{A}$ has the structure of an $\mathcal{E}(2)$-module.

Let f and g be germs; we define the homotopy

$$\phi(t,(x,y)) = tg(x,y) + (1-t)f(x,y).$$

Lemma 13. *If we have*

$$m(2)^{k+1} \subseteq m(2)\left\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\right\} + m(2)^{k+2};$$

hence, it is also true that

$$m(2)^{k+1}\mathcal{A} \subseteq m(2)\left\{\frac{\partial \phi}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial \phi}{\partial y}(\delta y + h_2)\right\} + m(2)^{k+2}\mathcal{A},$$

and by Nakayma's lemma we get

$$m(2)^{k+3}\mathcal{A} \subseteq m(2)^3\left\{\frac{\partial \phi}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial \phi}{\partial y}(\delta y + h_2)\right\}.$$

Proof. We have the following equalities

$$\frac{\partial \phi}{\partial x} = \frac{\partial f}{\partial x} + t\frac{\partial (g-f)}{\partial x}$$

and

$$\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y} + t\frac{\partial (g-f)}{\partial y}.$$

Clearly

$$(\alpha x + \beta y^2 + h_1)\frac{\partial f}{\partial x} \in (\alpha x + \beta y^2 + h_1)\frac{\partial \phi}{\partial x} + m(2)^{k+1}\mathcal{A},$$

and

$$(\delta y + h_2)\frac{\partial f}{\partial y} \in (\delta y + h_2)\frac{\partial \phi}{\partial y} + m(2)^{k+1}\mathcal{A}.$$

Therefore, our hypothesis implies that

$$m(2)^{k+1}\mathcal{A} \subseteq m(2)\left\{\frac{\partial\phi}{\partial x}(\alpha x+\beta y^2+h_1)+\frac{\partial\phi}{\partial y}(\delta y+h_2)\right\}+m(2)^{k+2}\mathcal{A}. \quad \square$$

The proof of Theorem 10 (B) follows by multiplying the last inclusion by $m(2)^2$ and our previous Lemma 13 in the usual way.

Complex case. We consider $g_\epsilon(x,y) = x^3 + \epsilon x^2y^2 + xy^4 + \epsilon y^6$. It is clear that g_ϵ is 6-determined on the right. The minimum k that satisfies Stefan's theorem is 6.

Remark 14. The germ g_ϵ is not 5-determined on the right, since the following equality is impossible

$$y^6 = h_1(x,y)\left(3x^2+2\epsilon xy^2+y^4+\frac{\partial\mu}{\partial x}\right)+h_2(x,y)\left(2\epsilon x^2y+4xy^3+6\epsilon y^5+\frac{\partial\mu}{\partial y}\right)$$

with h_1 and h_2 in $m(2)$.

The case $\epsilon^2 = 3$. It is not hard to show that

$$y^6 \notin m(2)\left\langle 3x^2+2\epsilon xy^2+y^4+\frac{\partial\mu}{\partial x}, 2\epsilon x^2y+4xy^3+6\epsilon y^5+\frac{\partial\mu}{\partial y}\right\rangle$$

for $\mu = (-8\epsilon/9)y^6$.

For our group $\mathcal{G}$ we get $m(2)^{7+1} \subseteq m(2)\{(3x^2+2\epsilon xy^2+y^4)(\alpha x+\beta y^2+h_1)+(2\epsilon x^2y+4xy^3+6\epsilon y^5)(\delta y+h_2)\}+m(2)^9$.

Using Theorem 10 (B) we get

Theorem 15. *Consider the germ $g_\epsilon(x,y) = x^3+\epsilon x^2y^2+xy^4+\epsilon y^6$. Then*

a) *If $\epsilon^2 \neq 3$, g_ϵ is 9-determined relative a $\mathcal{G}$.*

b) *If $\epsilon^2 = 3$, g_ϵ is 8-determined relative to $\mathcal{G}$.*

We state the following

Theorem 16. *A germ $f \in m(2)$ is finitely determined on the right if and only if there exists k such that $m(2)^k \subseteq m(2)\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$.*

We also have from Theorems 9 (A) and 10 (B).

Theorem 17. *A germ f is finitely determined relative to $\mathcal{G}$ if and only if there exists k such that*

$$m(2)^k \subseteq m(2)\left\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\right\} + m(2)^{k+1},$$

where α, β and δ range in the real numbers, $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$.

These two theorems give a relation between both concepts of determination.

Theorem 18. *A germ f is finitely determined on the right if and only if is finitely determined relative to $\mathcal{G}$.*

Proof. If $m(2)^k \subseteq m(2)\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$, then

$$m(2)^{k+3} \subseteq m(2)\left\{\frac{\partial f}{\partial x}\widetilde{h_1} + \frac{\partial f}{\partial y}\widetilde{h_2} \mid \widetilde{h_1}, \widetilde{h_2} \in m(2)^3\right\} + m\,(n)^{k+4}.$$

Hence,

$$m(2)^{(k+2)+1} \subseteq m(2)\left\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\right\} + m\,(n)^{k+4}$$

with $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$. Therefore f is $(k+4)$-determined relative to $\mathcal{G}$.

Now if f is k-determined relative to $\mathcal{G}$, then

$$m(2)^{k+1} \subseteq m(2)\left\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\right\}$$

and

$$m(2)^{k+1} \subseteq m(2)^2 \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle.$$

Therefore f is k-determined on the right. $\square$

2. Artin-Rees lemma (Following (1)). To do algebraic geometry in the real local ring $\mathcal{E}(n)$ it is necessary to introduce some concepts and results such as germ of real analytic set, coherence of real analytic sets, Malgrange's ideal, etc. In this direction one has the Artin-Rees lemma. It is well-known that in the completion process, the Artin-Rees lemma is used to simplify expressions in which Noetherian rings and their ideals are involved. In the case of determination of germs, it is also necessary to carry out "similar" simplifications in which are involved the real local algebra $\mathcal{E}(n)$ and its ideals. Since $\mathcal{E}(n)$ is not a Noetherian ring, it is not possible to use the Artin-Rees lemma directly, so it becomes necessary to establish a similar result for the case of this real ring.

We will denote by $\mathbf{R}[[x_1, \dots, x_n]]$ the algebra of formal power series in n variables over the real numbers.

Lemma 19 (Borel). *Consider the canonical map $\pi : \mathcal{E}(n) \to \mathbf{R}[[x_1, \dots, x_n]]$ given by the infinite Taylor series. Then π is onto and its kernel is $m(n)^\infty$, the ideal of germs f, such that f and all its derivatives vanish at the origin.*

Definition 20. a) Let X and Y be subsets of $\mathbf{R}^n$ containing the origin. We say that they are equivalent if there exists a neighborhood of the origin U such that $U \cap X = U \cap Y$. the germ $\mathcal{X}$ will denote the equivalent class of X.

b) A subset X of $\mathbf{R}^n$ is an analytic subset if there exist $f_1, \dots, f_m$ analytic maps such that their common zeros coincide with X.

Notation. If $f_1, \dots, f_n$ are analytic and I denotes the ideal generated by them, we will denote $X = V(I)$, the common zeros of I. Such ideal can be seen in $\mathcal{E}(n)$ or $\mathcal{O}_n$, the algebra of analytic germs.

Definition 21. Let $\mathcal{X}$ be a germ of an analytic set X, then $\mathcal{I}(\mathcal{X})$ is the ideal of germs vanishing at $\mathcal{X}$. This coincides with the ideal of germs f such that given f a representative, there exist U neighborhood of the origin such that f vanishes at $X \cap U$.

Definition 22. Let I be the ideal of $\mathcal{E}(n)$ generated by $f_1, \dots, f_m$. We denote by $\mathcal{V}(I)$ the germ of the common zeros of representatives $f_1, \dots, f_m$.

Definition 23. A germ $\mathcal{X}$ of an analytic set is irreducible if we have $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are germs of analytic subsets of $\mathcal{X}$, hence $\mathcal{X}_1 = \varnothing$ or $\mathcal{X}_2 = \varnothing$. A germ $\mathcal{X}$ of an analytic set is reducible if it is not irreducible.

Proposition 24. *A germ $\mathcal{X}$ of an analytic set is irreducible if and only if $I(\mathcal{X})$ is a prime ideal.*

Definition 25. a) Let I ideal of $\mathcal{E}(n)$, then the radical of I is $\mathcal{I}(\mathcal{V}(I))$, also denoted by $\mathrm{rad}(I)$.

b) An ideal I is radical if $I = \mathrm{rad}(I)$.

c) Let U be an open subset of $\mathbf{R}^n$, and let X be an analytic set in U. We say that X is coherent at a point x in X if there exist a W neighborhood of x and $f_1, \dots, f_m$ analytic functions in W such that for each $y \in W \cap X$ we have that $\mathcal{I}(W \cap X) = \langle f_1^y, \dots, f_m^y \rangle$, where f_i^y is the germ of f_i at the point y.

d) A germ of a set $\mathcal{X}$ is coherent if a representant X is coherent at $\overline{0}$.

Remark 26. In general, for an ideal I of $\mathcal{E}(n)$, it makes sense to define its real radical as $\mathbf{R}\text{-rad}(I) = \cap P$, where P are the real prime ideals containing I. We note that $m(n)$ and $m(n)^\infty$ are themselves real prime ideals of $\mathcal{E}(n)$.

Consider I to be a real radical ideal of $\mathcal{E}(n)$; hence, $\mathcal{I}(\mathcal{V}(I)) = I$. Also $\mathbf{R}\text{-rad}(I)$ is the smallest real ideal containing I, hence $\mathbf{R}\text{-rad}(I) = I$ and both concepts coincide. We also have $\mathbf{R}\text{-rad}(I) \subseteq \mathcal{I}(\mathcal{V}(I))$ for every ideal I.

In the case $I = m(n)^\infty, \mathcal{I}(\mathcal{V}(I)) = m(n)$ and $\mathbf{R}$-rad$(I) = m(n)^\infty$.

Theorem 27 (Malgrange-Tougeron). *Let $\mathcal{X}_0$ be a germ of an analytic set at the origin, $\mathcal{I}(\mathcal{X}_0)$ as above and $\mathcal{I}_*(\mathcal{X}_0)$ the ideal in $\mathcal{O}_n$ of analytic maps vanishing at $\mathcal{X}_0$. Then the following assertions are equivalent.*

i) $\mathcal{I}_*(\mathcal{X}_0)\mathcal{E}(n) = \mathcal{I}(\mathcal{X}_0)$.

ii) $\mathcal{X}_0$ *is coherent.*

Definition 28. An ideal I of $\mathcal{E}(n)$ is a Malgrange-ideal if

i) I is finitely generated by analytic germs.

ii) $\mathcal{V}(I)$ is coherent.

Proposition 29. *Consider the sets*

$$X = \{I \text{ is a Malgrange-ideal and also radical}\},$$
$$Y = \{\mathcal{V}(I) \mid \mathcal{V}(I) \text{ is a coherent germ}\}.$$

Hence $\mathcal{I} : Y \to X$ is well defined and $\mathcal{V} : X \to Y$ is its inverse.

Proof. Since I is a Malgrange ideal and also radical, $\mathcal{V}(I)$ is coherent and $\mathcal{I}(\mathcal{V}(I)) = I$. Also if $\mathcal{V}(I)$ is in Y, $\mathcal{I}(\mathcal{V}(I))$ is finitely generated and $\mathcal{V}(\mathcal{I}(\mathcal{V}(I))) = \mathcal{V}(I)$ is coherent. Moreover $\mathcal{I}(\mathcal{V}(\mathcal{I}(\mathcal{V}(I)))) = \mathcal{I}(\mathcal{V}(I))$. Therefore $\mathcal{I}$ and $\mathcal{V}$ are bijective and one is the inverse of the other. □

Definition 30. Let U be an open set of $\mathbf{R}^n$. We say that a smooth map f is flat in X closed subset of U if f and all its derivatives vanish at X. We denote the ideal of such maps by M_X^∞.

A very important theorem that assures a common divisor for a countable family of flat functions in a compact set is the following

Lemma 31 (Tougeron). *Let U be an open set of $\mathbf{R}^n$, X a compact subset in U and $(f_i)_{i\in\mathbf{N}}$ a family of functions in M_X^∞. Then there exist $g \in M_X^\infty$, $g(x) > 0$ for each $x \in U - X$ such that $f_i \in gM_X^\infty$ for each $i \in \mathbf{N}$.*

Proof. See [**5**, Lemma 2.4, page 93]. □

Proposition 32. *Let I be a radical ideal of $\mathcal{E}(n)$. Then we get $I \cap m(n)^\infty = Im(n)^\infty$.*

Proof. Let $f \in I \cap m(n)^\infty$, then $f \in m(n)^\infty$. Using Lemma 31 for a single element of the family we get $f = gh$, where g and h are in $m(n)^\infty$ and $h(x) > 0$, for each $x \neq 0$. Now since f vanishes at $V(I)$, then g too and $g \in \mathcal{I} = I$. Therefore $f \in m(n)^\infty I$. □

Remark 33. Since $\pi : \mathcal{E}(n) \to \mathbf{R}[[x_1, \dots, x_n]]$ is an epimorphism between local algebras, we get $\pi^{-1}(\mathcal{M}(n)) = m(n)$, where $\mathcal{M}(n)$ is the maximal ideal of $\mathbf{R}[[x_1, \dots, x_n]]$.

We state the Artin-Rees lemma for $\mathbf{R}[[x_1, \dots, x_n]]$.

Lemma 34 (Artin-Rees). *Let J be an ideal of $\mathbf{R}[[x_1, \dots, x_n]]$. Then there exists a natural number k_0 such that for each natural number k we get*

$$\mathcal{M}(n)^{k+k_0} \cap J = \mathcal{M}(n)^k(\mathcal{M}(n)^{k_0} \cap J).$$

The main purpose of this section is to give an interpretation of the Artin-Rees lemma for the algebra $\mathcal{E}(n)$ which is not Noetherian.

Theorem 35. *Let I be a radical ideal of $\mathcal{E}(n)$. Then there exists a natural number k_0 such that for each natural number k we get*

$$m(n)^{k+k_0} \cap I = m(n)^k(m(n)^{k_0} \cap I).$$

Proof. Consider the ideal $\pi(I)$ in $\mathbf{R}[[x_1, \dots, x_n]]$. Hence there exists a natural number k_0 such that for each natural number k we get

$$\mathcal{M}(n)^{k+k_0} \cap \pi(I) = \mathcal{M}(n)^k(\mathcal{M}(n)^{k_0} \cap \pi(I));$$

therefore,

$$m(n)^{k+k_0} \cap (I + m(n)^\infty) = m(n)^k(m(n)^{k_0} \cap (I + m(n)^\infty)).$$

Intersecting the equality with the ideal I and using the modular law

$$m(n)^{k+k_0} \cap I = m(n)^k(m(n)^{k_0} \cap I) + (m(n)^\infty \cap I).$$

Since I is radical, by Proposition 32, the result is obtained. $\square$

Lemma 36. *Let I be an ideal of $\mathcal{E}(n)$. Then there exist $g_1, \dots, g_m$ germs of smooth functions in $\mathcal{E}(n)$ such that $I = \langle g_1, \dots g_m \rangle + I \cap m(n)^\infty$.*

Proof. Let $h_1, \dots, h_m$ be generators of $\pi(I)$, and let $g_i \in I$ be such that $\pi(g_i) = h_i$, for $1 \leq i \leq m$; hence $I + m(n)^\infty = \langle g_1, \dots, g_m \rangle + m(n)^\infty$, if we intersect with I each side of the equality we get

$$I = \langle g_1, \dots, g_m \rangle + I \cap m(n)^\infty. \qquad \square$$

The following remark answers when an ideal J in $\mathcal{E}(n)$ is finitely generated.

Remark 37. With the above notation $I = \langle g_1, \dots, g_m \rangle$ if and only if $I \cap m(n)^\infty \subseteq \langle g_1, \dots, g_m \rangle$.

Proposition 38. *Let I be an ideal of $\mathcal{E}(n)$, $\mathcal{I}$ its radical, and suppose that $\mathcal{I}$ is finitely generated. If $\pi(I) = \pi(\mathcal{I})$, then $I = \mathcal{I}$.*

Proof. Since $\pi(I) = \pi(\mathcal{I})$ we get that $I + m(n)^\infty \cap \mathcal{I} = \mathcal{I}$ or $I + m(n)^\infty \mathcal{I} = \mathcal{I}$. Using Nakayama's lemma we get $I = \mathcal{I}$. $\square$

We know that in an arbitrary commutative ring $\mathcal{A}$ with unitary element the following assertion is not true in general: *Let I and J be finitely generated ideals in $\mathcal{A}$, hence $I \cap J$ is finitely generated. In our algebra $\mathcal{E}(n)$ we get the following result.*

Proposition 39. *Let I be a finitely generated ideal of $\mathcal{E}(n)$. Then for each k, $I \cap m(n)^{k+1}$ is also finitely generated.*

Proof. Let $f_1, \dots, f_s$ be generators of I and $f \in I$. Hence $f = h_1 f_1 + \cdots + h_s f_s$. We write $h_i = h_i^k + \widetilde{h}_i^k$, where h_i^k is the k-jet of h_i and $\widetilde{h}_i^k = h_i - h_i^k$. Therefore

$$f = h_1^k f_1 + \cdots + h_s^k f_s + \widetilde{h}_1 f_1 + \cdots + \widetilde{h}_s f_s.$$

Consider J to be the vector space generated by $\{x^K f_j\}$ where $1 \leq j \leq s$ and $|K| \leq k$. Hence as vector spaces we get $I = J + I\ m(n)^{k+1}$ and therefore $I \cap m(n)^{k+1} = J \cap m(n)^{k+1} + Im(n)^{k+1}$. Hence a set of generators for $I \cap m(n)^{k+1}$ can be formed by a basis for the vector space $J \cap m(n)^{k+1}$ and $\{x^K f_j\}$ where $1 \leq j \leq s$ and $|K| = k + 1$. $\square$

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